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YETTER-DRINFELD-LONG BIMODULES ARE MODULES

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Abstract. Let H be a finite-dimensional bialgebra. In this paper, we prove that the category $\mathcal{LR}(H)$ of Yetter-Drinfeld-Long bimodules, introduced by F. Panaite, F. Van Oystaeyen (2008), is isomorphic to the Yetter-Drinfeld category ${}^{H\otimes H^*}_{H\otimes H^*}\mathcal{YD}$ over the tensor product bialgebra $H \otimes H^*$ as monoidal categories. Moreover if H is a finite-dimensional Hopf algebra with bijective antipode, the isomorphism is braided. Finally, as an application of this category isomorphism, we give two results.

Keywords: Hopf algebra; Yetter-Drinfeld-Long bimodule; braided monoidal category

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1. Introduction

Panaite and Oystaeyen in [5] introduced the notion of L-R smash biproduct, with the L-R smash product introduced in [4] (or in [7]) and L-R smash coproduct introduced in [5] as multiplication and comultiplication, respectively. When an object A, which is both an algebra and a coalgebra, and a bialgebra H form a L-R-admissible pair (H, A), $A\natural H$ becomes a bialgebra with the smash product and smash coproduct, and the Radford biproduct is a special case. It turns out that A is in fact a bialgebra in the category $\mathcal{LR}(H)$ of Yetter-Drinfeld-Long bimodules (introduced in [5]) with some compatible condition.

The aim of this paper is to show that the category $\mathcal{LR}(H)$ coincides with the Yetter-Drinfeld category over the bialgebra $H \otimes H^*$, in the case when H is finite-dimensional. Hence any object $M \in \mathcal{LR}(H)$ is just a module over the Drinfeld double $D(H \otimes H^*)$ (see [1]).

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The paper is organized as follows. In Section 2, we recall the category $\mathcal{LR}(H)$. In Section 3, we give the main result of this paper.

Throughout this article, all the vector spaces, tensor products and homomorphisms are over a fixed field k. For a coalgebra C, we will use the Heyneman-Sweedler notation $\Delta(c) = c_1 \otimes c_2$, for any $c \in C$ (summation omitted).

2. Preliminaries

Let H be a bialgebra. The category $\mathcal{LR}(H)$ is defined as follows. The objects of $\mathcal{LR}(H)$ are vector spaces M endowed with H-bimodule and H-bicomodule structures (denoted by $h \otimes m \mapsto h \cdot m$, $m \otimes h \mapsto m \cdot h$, $m \mapsto m_{(-1)} \otimes m_{(0)}$, $m \mapsto m_{\langle 0\rangle} \otimes m_{\langle 1\rangle}$ for all $h \in H$, $m \in M$), such that M is a left-left Yetter-Drinfeld module, a left-right Long module, a right-right Yetter-Drinfeld module and a right-left Long module, i.e.,

$$(h_1 \cdot m)_{(-1)} h_2 \otimes (h_1 \cdot m)_{(0)} = h_1 m_{(-1)} \otimes h_2 \cdot m_{(0)}, \tag{2.1}$$

$$(h \cdot m)_{\langle 0\rangle} \otimes (h \cdot m)_{\langle 1\rangle} = h \cdot m_{\langle 0\rangle} \otimes m_{\langle 1\rangle}, \tag{2.2}$$

$$(m \cdot h_2)_{\langle 0\rangle} \otimes h_1 (m \cdot h_2)_{\langle 1\rangle} = m_{\langle 0\rangle} \cdot h_1 \otimes m_{\langle 1\rangle} h_2, \tag{2.3}$$

$$(m \cdot h)_{(-1)} \otimes (m \cdot h)_{(0)} = m_{(-1)} \otimes m_{(0)} \cdot h. \tag{2.4}$$

The morphisms in $\mathcal{LR}(H)$ are H-bilinear and H-bicolinear maps.

If H has a bijective antipode S, $\mathcal{LR}(H)$ becomes a strict braided monoidal category with the following structures: for all $M, N \in \mathcal{LR}(H)$, and $m \in M$, $n \in N$, $h \in H$,

$$\begin{gathered} h \cdot (m \otimes n) = h_1 \cdot m \otimes h_2 \cdot n, \\ (m \otimes n)_{(-1)} \otimes (m \otimes n)_{(0)} = m_{(-1)} n_{(-1)} \otimes m_{(0)} \otimes n_{(0)}, \\ (m \otimes n) \cdot h = m \cdot h_1 \otimes n \cdot h_2, \\ (m \otimes n)_{\langle 0\rangle} \otimes (m \otimes n)_{\langle 1\rangle} = m_{\langle 0\rangle} \otimes n_{\langle 0\rangle} \otimes m_{\langle 1\rangle} n_{\langle 1\rangle}, \end{gathered}$$

the braiding

$$c_{M,N} : M \otimes N \mapsto N \otimes M, \quad m \otimes n \mapsto m_{(-1)} \cdot n_{\langle 0\rangle} \otimes m_{(0)} \cdot n_{\langle 1\rangle},$$

and the inverse

$$c^{-1}_{M,N} : N \otimes M \mapsto M \otimes N, \quad n \otimes m \mapsto m_{(0)} \cdot S^{-1}(n_{\langle 1\rangle}) \otimes S^{-1}(m_{(-1)}) \cdot n_{\langle 0\rangle}.$$

3. Main result

In this section, we will give the main result of this paper.

Lemma 3.1. *Let H be a finite-dimensional bialgebra. Then we have a functor $F\colon \mathcal{LR}(H) \to {}^{H\otimes H^*}_{H\otimes H^*}\mathcal{YD}$ given for any object $M \in \mathcal{LR}(H)$ and any morphism ϑ by*

$$F(M) = M \quad \text{and} \quad F(\vartheta) = \vartheta,$$

where $H \otimes H^*$ is a bialgebra with the tensor product and tensor coproduct.

Proof. For all $M \in \mathcal{LR}(H)$, first of all, define the left action of $H \otimes H^*$ on M by

$$(h \otimes f) \cdot m = \langle f, m_{\langle 1\rangle}\rangle h \cdot m_{\langle 0\rangle}, \tag{3.1}$$

for all $h \in H$, $f \in H^*$ and $m \in M$. Then M is a left $H \otimes H^*$-module. Indeed, for all $h, h' \in H$, $f, f' \in H^*$ and $m \in M$,

$$\begin{aligned}
(h \otimes f)(h' \otimes f') \cdot m &= (hh' \otimes ff') \cdot m \\
&= \langle ff', m_{\langle 1\rangle}\rangle hh' \cdot m_{\langle 0\rangle} \\
&= \langle f, m_{\langle 1\rangle 1}\rangle\langle f', m_{\langle 1\rangle 2}\rangle h \cdot (h' \cdot m_{\langle 0\rangle}) \\
&= \langle f, m_{\langle 0\rangle\langle 1\rangle}\rangle\langle f', m_{\langle 1\rangle}\rangle h \cdot (h' \cdot m_{\langle 0\rangle\langle 0\rangle}) \\
&\overset{(2.2)}{=} \langle f, (h' \cdot m_{\langle 0\rangle})_{\langle 1\rangle}\rangle\langle f', m_{\langle 1\rangle}\rangle h \cdot (h' \cdot m_{\langle 0\rangle})_{\langle 0\rangle} \\
&= \langle f', m_{\langle 1\rangle}\rangle (h \otimes f) \cdot (h' \cdot m_{\langle 0\rangle}) \\
&= (h \otimes f) \cdot ((h' \otimes f') \cdot m).
\end{aligned}$$

And

$$(1 \otimes \varepsilon) \cdot m = \langle \varepsilon, m_{\langle 1\rangle}\rangle m_{\langle 0\rangle} = m,$$

as claimed. Next, for all $m \in M$, define the left coaction of $H \otimes H^*$ on M by

$$\varrho(m) = m_{[-1]} \otimes m_{[0]} = \sum m_{(-1)} \otimes h^i \otimes m_{(0)} \cdot h_i, \tag{3.2}$$

where $\{h_i\}_i$ and $\{h^i\}_i$ are dual bases in H and H^*. Then on the one hand,

$$(\Delta_{H\otimes H^*} \otimes \mathrm{id})\varrho(m) = \sum m_{(-1)1} \otimes h^i_1 \otimes m_{(-1)2} \otimes h^i_2 \otimes m_{(0)} \cdot h_i.$$

Evaluating the right-hand side of the equation on $\mathrm{id} \otimes g \otimes \mathrm{id} \otimes h \otimes \mathrm{id}$, we obtain

$$m_{(-1)1} \otimes m_{(-1)2} \otimes m_{(0)} \cdot gh.$$

On the other hand,

$$\begin{aligned}(\mathrm{id}\otimes\varrho)\varrho(m) &= \sum m_{(-1)}\otimes h^i\otimes(m_{(0)}\cdot h_i)_{(-1)}\otimes h^j\otimes(m_{(0)}\cdot h_i)_{(0)}\cdot h_j\\ &\overset{(2.4)}{=}\sum m_{(-1)}\otimes h^i\otimes m_{(0)(-1)}\otimes h^j\otimes(m_{(0)(0)}\cdot h_i)\cdot h_j\\ &= \sum m_{(-1)1}\otimes h^i\otimes m_{(-1)2}\otimes h^j\otimes m_{(0)}\cdot h_ih_j.\end{aligned}$$

Evaluating the right-hand side of the equation on $\mathrm{id}\otimes g\otimes\mathrm{id}\otimes h\otimes\mathrm{id}$, we obtain

$$m_{(-1)1}\otimes m_{(-1)2}\otimes m_{(0)}\cdot gh.$$

Since $g,h\in H$ were arbitrary, we have

$$(\Delta_{H\otimes H^*}\otimes\mathrm{id})\varrho=(\mathrm{id}\otimes\varrho)\varrho.$$

And since

$$(\varepsilon_{H\otimes H^*}\otimes\mathrm{id})(\varrho(m))=\varepsilon(m_{(-1)})m_{(0)}=m,$$

M is a left $H\otimes H^*$-comodule.

Finally,

$$\begin{aligned}&[(h\otimes f)_1\cdot m]_{[-1]}(h\otimes f)_2\otimes[(h\otimes f)_1\cdot m]_{[0]}\\ &\quad= (h_1\cdot m_{\langle 0\rangle})_{[-1]}\langle f_1,m_{\langle 1\rangle}\rangle(h_2\otimes f_2)\otimes(h_1\cdot m_{\langle 0\rangle})_{[0]}\\ &\quad= \sum\langle f_1,m_{\langle 1\rangle}\rangle((h_1\cdot m_{\langle 0\rangle})_{(-1)}h_2\otimes h^if_2)\otimes(h_1\cdot m_{\langle 0\rangle})_{(0)}\cdot h_i\\ &\quad\overset{(2.1)}{=}\sum\langle f_1,m_{\langle 1\rangle}\rangle h_1m_{\langle 0\rangle(-1)}\otimes h^if_2\otimes h_2\cdot m_{\langle 0\rangle(0)}\cdot h_i.\end{aligned}$$

Evaluating the right-hand side of the equation on $\mathrm{id}\otimes g\otimes\mathrm{id}$, we obtain

$$\langle f,m_{\langle 1\rangle}g_2\rangle h_1m_{\langle 0\rangle(-1)}\otimes h_2\cdot m_{\langle 0\rangle(0)}\cdot g_1.$$

And

$$\begin{aligned}&(h\otimes f)_1m_{[-1]}\otimes(h\otimes f)_2\cdot m_{[0]}\\ &\quad=\sum(h_1\otimes f_1)(m_{(-1)}\otimes h^i)\otimes(h_2\otimes f_2)\cdot(m_{(0)}\cdot h_i)\\ &\quad=\sum h_1m_{(-1)}\otimes f_1h^i\otimes\langle f_2,(m_{(0)}\cdot h_i)_{\langle 1\rangle}\rangle h_2\cdot(m_{(0)}\cdot h_i)_{\langle 0\rangle}.\end{aligned}$$

Evaluating the right-hand side of the equation on $\mathrm{id}\otimes g\otimes\mathrm{id}$, we obtain

$$\begin{aligned}&h_1m_{(-1)}\otimes\langle f,g_1(m_{(0)}\cdot g_2)_{\langle 1\rangle}\rangle h_2\cdot(m_{(0)}\cdot g_2)_{\langle 0\rangle}\\ &\qquad\overset{(2.3)}{=}h_1m_{(-1)}\otimes\langle f,m_{(0)\langle 1\rangle}g_2\rangle h_2\cdot m_{(0)\langle 0\rangle}\cdot g_1\\ &\qquad=\langle f,m_{\langle 1\rangle}g_2\rangle h_1m_{\langle 0\rangle(-1)}\otimes h_2\cdot m_{\langle 0\rangle(0)}\cdot g_1.\end{aligned}$$

Therefore M is a left-left Yetter-Drinfeld module over $H \otimes H^*$. It is straightforward to verify that any morphism in $\mathcal{LR}(H)$ is also a morphism in ${}^{H\otimes H^*}_{H\otimes H^*}\mathcal{YD}$. The proof is completed. $\square$

Lemma 3.2. *Let H be a finite-dimensional bialgebra. Then we have a functor $G\colon {}^{H\otimes H^*}_{H\otimes H^*}\mathcal{YD} \to \mathcal{LR}(H)$ given for any object $M \in {}^{H\otimes H^*}_{H\otimes H^*}\mathcal{YD}$ and any morphism θ by*

$$G(M) = M \quad \text{and} \quad G(\theta) = \theta.$$

Proof. We denote by ε^* the map ε_{H^*} defined by $\varepsilon_{H^*}(f) = f(1)$ for all $f \in H^*$. For any $M \in {}^{H\otimes H^*}_{H\otimes H^*}\mathcal{YD}$, denote the left $H \otimes H^*$-coaction on M by

$$m \mapsto m_{[-1]} \otimes m_{[0]},$$

for all $m \in M$. Define the H-bimodule and H-bicomodule structures as follows:

$$h \cdot m = (h \otimes \varepsilon) \cdot m, \tag{3.3}$$

$$\varrho_L(m) = m_{(-1)} \otimes m_{(0)} = (\mathrm{id} \otimes \varepsilon^*)(m_{[-1]}) \otimes m_{[0]},$$

$$m \cdot h = \langle (\varepsilon \otimes \mathrm{id}) m_{[-1]}, h \rangle m_{[0]}, \tag{3.4}$$

$$\varrho_R(m) = m_{\langle 0\rangle} \otimes m_{\langle 1\rangle} = \sum (1 \otimes h^i) \cdot m \otimes h_i,$$

for all $h \in H$.

Obviously M is a left H-module. And

$$\begin{aligned}
(\Delta \otimes \mathrm{id})\varrho_L(m) &= \Delta((\mathrm{id} \otimes \varepsilon^*)(m_{[-1]})) \otimes m_{[0]} \\
&= (\mathrm{id} \otimes \varepsilon^*)(m_{[-1]1})(\mathrm{id} \otimes \varepsilon^*)(m_{[-1]2}) \otimes m_{[0]} \\
&= (\mathrm{id} \otimes \varepsilon^*)(m_{[-1]})(\mathrm{id} \otimes \varepsilon^*)(m_{[0][-1]}) \otimes m_{[0][0]} \\
&= (\mathrm{id} \otimes \varrho_L)\varrho_L(m).
\end{aligned}$$

The counit is straightforward. Thus M is a left H-comodule. For all $h, h' \in M$,

$$\begin{aligned}
m \cdot hh' &= \langle (\varepsilon \otimes \mathrm{id}) m_{[-1]}, hh' \rangle m_{[0]} \\
&= \langle (\varepsilon \otimes \mathrm{id}) m_{[-1]1}, h \rangle \langle (\varepsilon \otimes \mathrm{id}) m_{[-1]2}, h' \rangle m_{[0]} \\
&= \langle (\varepsilon \otimes \mathrm{id}) m_{[-1]}, h \rangle \langle (\varepsilon \otimes \mathrm{id}) m_{[0][-1]}, h' \rangle m_{[0][0]} \\
&= \langle (\varepsilon \otimes \mathrm{id}) m_{[-1]}, h \rangle m \cdot h' \\
&= (m \cdot h) \cdot h'.
\end{aligned}$$

The unit is obvious. Thus M is a right H-module. Since

$$\begin{aligned}(\mathrm{id}\otimes\Delta)\varrho_R(m) &= \sum(1\otimes h^i)\cdot m\otimes h_{i1}\otimes h_{i2}\\ &= \sum(1\otimes h^i h^j)\cdot m\otimes h^j\otimes h^i\\ &= (\varrho_R\otimes\mathrm{id})\varrho_R(m),\end{aligned}$$

it follows that M is a right H-comodule. Moreover,

$$\begin{aligned}(h\cdot m)\cdot h' &= ((h\otimes\varepsilon)\cdot m)\cdot h'\\ &= \langle(\varepsilon\otimes\mathrm{id})((h\otimes\varepsilon)\cdot m)_{[-1]}, h'\rangle((h\otimes\varepsilon)\cdot m)_{[0]}\\ &= \langle(\varepsilon\otimes\mathrm{id})[((h_1\otimes\varepsilon)\cdot m)_{[-1]}(h_2\otimes\varepsilon)], h'\rangle((h_1\otimes\varepsilon)\cdot m)_{[0]}\\ &\overset{(2.1)}{=} \langle(\varepsilon\otimes\mathrm{id})((h_1\otimes\varepsilon)m_{[-1]}), h'\rangle(h_2\otimes\varepsilon)\cdot m_{[0]}\\ &= \langle(\varepsilon\otimes\mathrm{id})m_{[-1]}, h'\rangle(h\otimes\varepsilon)\cdot m_{[0]}\\ &= h\cdot(m\cdot h').\end{aligned}$$

Thus M is an H-bimodule. And

$$\begin{aligned}(\varrho_L\otimes\mathrm{id})\varrho_R(m) &= \sum(\mathrm{id}\otimes\varepsilon^*)((1\otimes h^i)\cdot m)_{[-1]}\otimes((1\otimes h^i)\cdot m)_{[0]}\otimes h_i\\ &= \sum(\mathrm{id}\otimes\varepsilon^*)[((1\otimes h^i_1)\cdot m)_{[-1]}(1\otimes h^i_2)]\otimes((1\otimes h^i_1)\cdot m)_{[0]}\otimes h_i\\ &\overset{(2.1)}{=} \sum(\mathrm{id}\otimes\varepsilon^*)((1\otimes h^i_1)m_{[-1]})\otimes(1\otimes h^i_2)\cdot m_{[0]}\otimes h_i\\ &= (\mathrm{id}\otimes\varrho_R)\varrho_L(m).\end{aligned}$$

Thus M is an H-bicomodule.

We now prove (2.1). For all $h\in H$, $m\in M$,

$$\begin{aligned}&(h_1\cdot m)_{(-1)}h_2\otimes(h_1\cdot m)_{(0)}\\ &\quad= ((h_1\otimes\varepsilon)\cdot m)_{(-1)}h_2\otimes((h_1\otimes\varepsilon)\cdot m)_{(0)}\\ &\quad= (\mathrm{id}\otimes\varepsilon^*)(((h_1\otimes\varepsilon)\cdot m)_{[-1]}(h_2\otimes\varepsilon))\otimes((h_1\otimes\varepsilon)\cdot m)_{[0]}\\ &\quad\overset{(2.1)}{=} (\mathrm{id}\otimes\varepsilon^*)((h_1\otimes\varepsilon)m_{[-1]})\otimes(h_2\otimes\varepsilon)\cdot m_{[0]}\\ &\quad= h_1 m_{(-1)}\otimes h_2\cdot m_{(0)}.\end{aligned}$$

We now prove (2.2):

$$\begin{aligned}(h\cdot m)_{\langle 0\rangle}\otimes(h\cdot m)_{\langle 1\rangle} &= ((h\otimes\varepsilon)\cdot m)_{\langle 0\rangle}\otimes((h\otimes\varepsilon)\cdot m)_{\langle 1\rangle}\\ &= \sum(1\otimes h^i)(h\otimes\varepsilon)\cdot m\otimes h_i\\ &= \sum(h\otimes\varepsilon)(1\otimes h^i)\cdot m\otimes h_i\\ &= h\cdot m_{\langle 0\rangle}\otimes m_{\langle 1\rangle}.\end{aligned}$$

We now prove (2.3): On the one hand,

$$\begin{aligned}(m\cdot h_2)_{\langle 0\rangle}\otimes h_1(m\cdot h_2)_{\langle 1\rangle} &= \langle(\varepsilon\otimes \mathrm{id})m_{[-1]},h_2\rangle m_{[0]\langle 0\rangle}\otimes h_1 m_{[0]\langle 1\rangle}\\ &= \sum\langle(\varepsilon\otimes \mathrm{id})m_{[-1]},h_2\rangle(1\otimes h^i)\cdot m_{[0]}\otimes h_1h_i.\end{aligned}$$

Evaluating the right-hand side on $\mathrm{id}\otimes f$ for all $f\in H^*$, we have

$$\begin{aligned}&\langle(\varepsilon\otimes \mathrm{id})m_{[-1]},h_2\rangle(1\otimes f_2)\cdot m_{[0]}f_1(h_1)\\ &\quad= \langle(\varepsilon\otimes \mathrm{id})(1\otimes f_1)m_{[-1]},h\rangle(1\otimes f_2)\cdot m_{[0]}\\ &\quad\overset{(2.1)}{=} \langle(\varepsilon\otimes \mathrm{id})(((1\otimes f_1)\cdot m)_{[-1]}(1\otimes f_2)),h\rangle((1\otimes f_1)\cdot m)_{[0]}.\end{aligned}$$

On the other hand,

$$\begin{aligned}m_{\langle 0\rangle}\cdot h_1\otimes m_{\langle 1\rangle}h_2 &= \sum((1\otimes h^i)\cdot m)\cdot h_1\otimes h_ih_2\\ &= \sum\langle(\varepsilon\otimes \mathrm{id})((1\otimes h^i)\cdot m)_{[-1]},h_1\rangle((1\otimes h^i)\cdot m)_{[0]}\otimes h_ih_2.\end{aligned}$$

Evaluating the right-hand side on $\mathrm{id}\otimes f$, we have

$$\begin{aligned}&\langle(\varepsilon\otimes \mathrm{id})((1\otimes f_1)\cdot m)_{[-1]},h_1\rangle((1\otimes f_1)\cdot m)_{[0]}f_2(h_2)\\ &\quad= \langle(\varepsilon\otimes \mathrm{id})(((1\otimes f_1)\cdot m)_{[-1]}(1\otimes f_2)),h\rangle((1\otimes f_1)\cdot m)_{[0]}.\end{aligned}$$

Hence $(m\cdot h_2)_{\langle 0\rangle}\otimes h_1(m\cdot h_2)_{\langle 1\rangle}=m_{\langle 0\rangle}\cdot h_1\otimes m_{\langle 1\rangle}h_2$, since f was arbitrary.

We now prove (2.4):

$$\begin{aligned}&(m\cdot h)_{(-1)}\otimes(m\cdot h)_{(0)}\\ &\quad= \langle(\varepsilon\otimes \mathrm{id})m_{[-1]},h\rangle(\mathrm{id}\otimes\varepsilon^*)(m_{[0][-1]})\otimes m_{[0][0]}\\ &\quad= \langle(\varepsilon\otimes \mathrm{id})m_{[-1]1},h\rangle(\mathrm{id}\otimes\varepsilon^*)(m_{[-1]2})\otimes m_{[0]}\\ &\quad= (\mathrm{id}\otimes h)m_{[-1]}\otimes m_{[0]}\\ &\quad= \langle(\varepsilon\otimes \mathrm{id})m_{[-1]2},h\rangle(\mathrm{id}\otimes\varepsilon^*)(m_{[-1]1})\otimes m_{[0]}\\ &\quad= m_{(-1)}\otimes m_{(0)}\cdot h,\end{aligned}$$

where in the third equality, $(\mathrm{id}\otimes h)m_{[-1]}$ means that the second factor of $m_{[-1]}$ acts on h.

Therefore $M\in\mathcal{LR}(H)$. It is straightforward to verify that any morphism in ${}^{H\otimes H^*}_{H\otimes H^*}\mathcal{YD}$ is also a morphism in $\mathcal{LR}(H)$. The proof is completed. □

Theorem 3.3. *Let H be a finite-dimensional bialgebra. Then we have a monoidal category isomorphism*

$$\mathcal{LR}(H) \cong {}^{H\otimes H^*}_{H\otimes H^*}\mathcal{YD}.$$

Moreover, if H is a Hopf algebra with bijective antipode S, they are isomorphic as braided monoidal categories. Consequently

$$\mathcal{LR}(H) \cong {}_{D(H\otimes H^*)}\mathcal{M},$$

where $D(H\otimes H^*)$ is the Drinfeld double of $H\otimes H^*$.

P r o o f. It is easy to see that the functor $F\colon \mathcal{LR}(H) \to {}^{H\otimes H^*}_{H\otimes H^*}\mathcal{YD}$ is monoidal and that $F\circ G = \mathrm{id}$ and $G\circ F = \mathrm{id}$. And for all $M, N \in \mathcal{LR}(H)$, and $m \in M$, $n \in N$,

$$\begin{aligned} m_{[-1]}\cdot n \otimes m_{[0]} &\overset{(3.2)}{=} \sum (m_{(-1)}\otimes h^i)\cdot n \otimes m_{(0)}\cdot h_i \\ &\overset{(3.1)}{=} \sum m_{(-1)}\cdot n_{\langle 0\rangle} \otimes m_{(0)}\cdot n_{\langle 1\rangle}. \end{aligned}$$

The proof is completed. □

Corollary 3.4. *(A, H) is an* L-R-*admissible pair if and only if $(A, H\otimes H^*)$ is an admissible pair (introduced in* [6]*) satisfying the condition* (1.14) *in* [5].

By the isomorphism in Theorem 3.3, we can obtain the following result of [2] directly.

Proposition 3.5. *Let H be a finite-dimensional Hopf algebra. The canonical braiding of $\mathcal{LR}(H)$ is pseudosymmetric if and only if H is commutative and cocommutative.*

P r o o f. From [3], the canonical braiding of ${}^{H\otimes H^*}_{H\otimes H^*}\mathcal{YD}$ is pseudosymmetric if and only if $H\otimes H^*$ is commutative and cocommutative. By the bialgebra structure of $H\otimes H^*$, the proof is completed. □

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