SPECTRAL DISCRETIZATION OF DARCY EQUATIONS COUPLED WITH NAVIER-STOKES EQUATIONS BY VORTICITY-VELOCITY-PRESSURE FORMULATION

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Abstract. We consider a model coupling the Darcy equations in a porous medium with the Navier-Stokes equations in the cracks, for which the coupling is provided by the pressure's continuity on the interface. We discretize the coupled problem by the spectral element method combined with a nonoverlapping domain decomposition method. We prove the existence of solution for the discrete problem and establish an error estimation. We conclude with some numerical tests confirming the results of our analysis.

Keywords: Navier-Stokes equation; Darcy equation; spectral element

MSC 2020: 65N55, 65N30, 65F08

1. Introduction

The filtration of fluids through porous media appears in several areas of medicine, engineering or geophysical systems. At high flow velocities in the pores, such phenomena can be modeled by the coupled Navier-Stokes-Darcy equations and have been extensively studied in the literature [5], [14], [17], [22]. Several types of coupling have been introduced, see [10], [21], [26], and the transmission conditions at the interface are yet the subject of controversy. It is usual to approximate the fluid motion near the true boundary by a condition for the tangential velocity and an ambiguity is implied by the notion of a true boundary for a permeable material.

In [6], the authors establish, empirically, the well-known Beavers-Joseph interface condition: The tangential component of the normal stress, that the free flow incurs along the interface, is proportional to the jump in the tangential velocity. In [26], Saffman proposes a simplification of the interface condition observed by Beavers and Joseph in [6], which has been obtained rigorously in [25] by the authors. In [13],

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the authors investigate the well-posedness of a coupled Stokes-Darcy model with Beavers-Joseph interface conditions.

In [7], the authors consider a model for a laminar flow of a river over a porous rock. The coupling is guaranteed by the continuity of the normal velocities and the normal stresses at the interface. They discretize each sub-domain by finite element method independently and use the mortar technics. In [10], the authors introduced new coupling conditions for a Darcy-Stokes problem. The continuity of the normal stress is approximated by the continuity of the pressure on the interface at high Reynolds number. They establish the existence and uniqueness of solution, propose a mixed finite element discretization and present a priori and a posteriori analysis.

In this paper, we consider the model introduced in [10] with the Navier-Stokes equations for the free fluid and the same interface conditions. Although in general the permeability exhibits strong variations and is usually not very smooth, we assume, as is frequent, that all physical parameters are constant. We introduce the vorticity as a new variable and rewrite the problem in terms of velocity, vorticity and pressure. Such a formulation motivated by the numerical simulations in computational fluid dynamics was firstly proposed in [18], [19]. One of its advantages is the possibility of considering the boundary conditions in a very general way. From the numerical point of view, the discretization can be seen as a conforming approximation of velocity field in the $H(\mathrm{div}, \Omega)$ Sobolev space and the vortex can be calculated accurately without subjecting it to the errors on the velocity's calculations. This can be more interesting for some industrial applications.

We write the weak associated formulation. We discretize by the spectral element method combined with a conform nonoverlapping domain decomposition method. We prove the existence of solution for the discrete problem, we then establish an error estimation. We linearize the discrete problem by a Newton method and conclude with some numerical tests.

An outline of the paper is as follows.

- ▷ In Section 2, we recall from [24] the variational formulation of the continuous problem and some of its properties.
- ▷ In Section 3, we describe the discrete problem and prove an existence and uniqueness result.
- ▷ In Section 4, we prove the optimal error estimates.
- ▷ Numerical results are presented in Section 5.

2. The velocity, vorticity and pressure formulation

We consider a flow problem of an incompressible viscous fluid in cracked porous medium. We are interested in a model that couples the Darcy equations in the middle with the Navier-Stokes equations in cracks and for which the coupling conditions involve the continuity of the pressure on the interface. The model we advocate is the one introduced in [10] with the Navier-Stokes equations instead of the Stokes ones. Keeping the notations of [10], we consider two bounded connected open domains Ω and Ω_F in $\mathbb{R}^d$, $d = 2$ or 3, with Lipschitz-continuous boundaries, such that $\overline{\Omega}_F$ is contained in Ω and Ω_F is simply connected and has a connected boundary. We set $\Omega_P = \Omega \setminus \overline{\Omega}_F$ and $\Gamma = \partial\Omega_F$ the interface between Ω_P and Ω_F (see Figure 1). Let $\boldsymbol{n}$ stand for the unit outward normal vector to Ω_P on its boundary $\partial\Omega_P$.

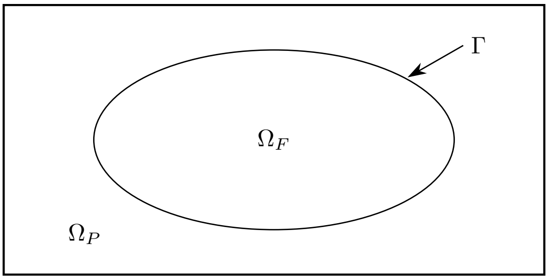

Figure 1. Bidimensional domain: Γ is the interface between Ω_F and Ω_P.

We recall from [24] the following system of equations:

$$
\begin{cases}
\mu \boldsymbol{u} + \nabla p = \boldsymbol{f} & \text{in } \Omega_P,\\
\nu\, \mathbf{curl}\, \boldsymbol{\omega} + \boldsymbol{\omega} \times \boldsymbol{u} + \nabla p = \boldsymbol{f} & \text{in } \Omega_F,\\
\operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega_P \text{ and } \Omega_F,\\
\boldsymbol{\omega} = \mathbf{curl}\ \boldsymbol{u} & \text{in } \Omega_F,\\
\boldsymbol{u} \cdot \boldsymbol{n} = 0 & \text{on } \partial\Omega,\\
(\boldsymbol{u}_{|\Omega_P} - \boldsymbol{u}_{|\Omega_F}) \cdot \boldsymbol{n} = 0 & \text{on } \Gamma,\\
p_{|\Omega_p} - p_{|\Omega_F} = 0 & \text{on } \Gamma,\\
\boldsymbol{\omega}_{|\Omega_F} \times n = 0 & \text{on } \Gamma.
\end{cases}
\tag{2.1}
$$

The unknowns are the velocity $\boldsymbol{u}$, the pressure p and the vorticity $\boldsymbol{\omega} = \mathbf{curl}\ \boldsymbol{u}$ as new unknown in Ω_F, $\boldsymbol{f}$ represents a density of body forces. The parameters μ and ν are positive constants and denote the ratio of the viscosity of the fluid to, respectively, the permeability of the medium and the viscosity of the fluid.

To write a variational formulation of (2.1), we introduce the Hilbert spaces

$$L_0^2(\Omega) = \left\{ q \in L^2(\Omega),\ \int_\Omega q(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 0 \right\}, \tag{2.2}$$

$$H(\mathrm{div}, \Omega) = \{\boldsymbol{v} \in L^2(\Omega)^d,\ \mathrm{div}\,\boldsymbol{v} \in L^2(\Omega)\}, \tag{2.3}$$

provided respectively with the norms

$$\|q\|_{L_0^2(\Omega)} = \|q\|_{L^2(\Omega)} = \left(\int_\Omega q^2(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \right)^{1/2}, \tag{2.4}$$

$$\|\boldsymbol{v}\|_{H(\mathrm{div},\Omega)} = (\|\boldsymbol{v}\|^2_{L^2(\Omega)^d} + \|\,\mathrm{div}\,\boldsymbol{v}\|^2_{L^2(\Omega)})^{1/2}, \tag{2.5}$$

and the closed subspace of $H(\mathrm{div}, \Omega)$

$$H_0(\mathrm{div}, \Omega) = \{\boldsymbol{v} \in H(\mathrm{div}, \Omega);\ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega\}. \tag{2.6}$$

We define the space of the **curl** operator

$$H(\mathbf{curl}\,, \Omega_F) = \{\boldsymbol{v} \in L^2(\Omega_F)^{d(d-1)/2};\ \mathbf{curl}\ \boldsymbol{v} \in L^2(\Omega_F)^d\} \tag{2.7}$$

and its subspace

$$H_0(\mathbf{curl}\,, \Omega_F) = \{\boldsymbol{\theta} \in H(\mathbf{curl}\,, \Omega_F);\ \gamma_t(\boldsymbol{\theta}) = \ \text{on } \partial\Omega_F\}, \tag{2.8}$$

provided with the graph norm

$$\|\boldsymbol{\theta}\|_{H(\mathbf{curl}\,,\Omega_F)} = (\|\boldsymbol{\theta}\|^2_{L^2(\Omega_F)^d} + \|\,\mathbf{curl}\ \boldsymbol{\theta}\|^2_{L^2(\Omega_F)^{d(d-1)/2}})^{1/2}. \tag{2.9}$$

For a data $\boldsymbol{f}$ in $H_0(\mathrm{div}, \Omega)'$, the variational problem associated with (2.1) writes:

Find $(\boldsymbol{\omega}, \boldsymbol{u}, p)$ in $H_0(\mathbf{curl}\,, \Omega_F) \times H_0(\mathrm{div}, \Omega) \times L_0^2(\Omega)$ such that

$$\begin{cases} a(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{v}) + K(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{v}) + b(\boldsymbol{v}, p) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle & \forall\, \boldsymbol{v} \in H_0(\mathrm{div}, \Omega), \\ b(\boldsymbol{u}, q) = 0 & \forall\, q \in L_0^2(\Omega), \\ c(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{\varphi}) = 0 & \forall\, \boldsymbol{\varphi} \in H_0(\mathbf{curl}\,, \Omega_F), \end{cases} \tag{2.10}$$

where $\langle \cdot, \cdot \rangle$ indicates the duality product between $H_0(\mathrm{div}, \Omega)$ and its dual space $H_0(\mathrm{div}, \Omega)'$ and the bilinear forms $a(\cdot, \cdot; \cdot)$, $b(\cdot, \cdot)$ and $c(\cdot, \cdot; \cdot)$ are defined by:

$$a(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{v}) = \mu \int_{\Omega_P} \boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \nu \int_{\Omega_F} (\mathbf{curl}\ \boldsymbol{\omega})(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x},$$

$$b(\boldsymbol{v}, q) = -\int_\Omega (\mathrm{div}\,\boldsymbol{v})(\boldsymbol{x}) q(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x},$$

and

$$c(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{\varphi}) = \int_{\Omega_F} \boldsymbol{\omega}(\boldsymbol{x}) \boldsymbol{\varphi}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \int_{\Omega_F} \boldsymbol{u}(\boldsymbol{x}) \cdot (\mathbf{curl}\ \boldsymbol{\varphi})(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.$$

In contrast, the trilinear form $K(\cdot,\cdot;\cdot)$ is defined by:

$$K(\boldsymbol{\omega},\boldsymbol{u};\boldsymbol{v}) = \int_{\Omega_F} (\boldsymbol{\omega}\times\boldsymbol{u})(\boldsymbol{x})\cdot\boldsymbol{v}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.$$

We note that $K(\cdot,\cdot;\cdot)$ is not necessarily defined for functions in the chosen spaces for both the solutions and the test functions. We will see later that additional assumptions are needed to ensure proper definition and continuity. Nevertheless, we have the equivalence, proved in [24], between problems (2.1) and (2.10). We recall from [24] the main arguments for proving the existence of a solution of problem (2.10). It is readily checked that the kernels

$$W = \{\boldsymbol{v}\in H_0(\mathrm{div},\Omega);\ \forall\, q\in L^2_0(\Omega),\ b(\boldsymbol{v},q)=0\}, \tag{2.11}$$

which coincides with the space of divergence-free functions in $H(\mathrm{div},\Omega)$ and

$$\mathcal{W} = \{(\boldsymbol{\theta},\boldsymbol{w})\in H_0(\mathbf{curl}\,,\Omega_F)\times W;\ \forall\,\boldsymbol{\varphi}\in H_0(\mathbf{curl}\,,\Omega_F),\ c(\boldsymbol{\theta},\boldsymbol{w};\boldsymbol{\varphi})=0\} \tag{2.12}$$

which coincides with the spaces of pairs $(\boldsymbol{\theta},\boldsymbol{w})$ in $H_0(\mathbf{curl}\,,\Omega)\times W$ such that $\boldsymbol{\theta}$ is equal to $\mathbf{curl}\ \boldsymbol{w}$ in the distribution sense.

We notice that for any solution $(\boldsymbol{\omega},\boldsymbol{u},p)$ of problem (2.10), $(\boldsymbol{\omega},\boldsymbol{u})$ is a solution of the following reduced problem:

Find $(\boldsymbol{\omega},\boldsymbol{u})$ in $\mathcal{W}$ such that

$$\forall\,\boldsymbol{v}\in W,\ a(\boldsymbol{\omega},\boldsymbol{u};\boldsymbol{v}) + K(\boldsymbol{\omega},\boldsymbol{u};\boldsymbol{v}) = \langle \boldsymbol{f},\boldsymbol{v}\rangle\,. \tag{2.13}$$

In [24] we showed the existence and the local uniqueness of the solution in a stationary regime using Brouwer's fixed-point theorem. We conclude with some regularity properties which can easily be derived from [1], Section 2, [15] and [16].

Proposition 2.1. *The mapping: $\boldsymbol{f}\mapsto(\boldsymbol{\omega},\boldsymbol{u},p)$, where $(\boldsymbol{\omega},\boldsymbol{u},p)$ is the solution of problem* (2.10) *with data $\boldsymbol{f}$, is continuous from $H^{\max\{0,s-1\}}(\Omega)^d$ into $H^s(\Omega_F)^{d(d-1)/2}\times H^{s-1}(\Omega)^d\times H^s(\Omega)$ for*

(i) *all $s\leqslant\frac{1}{2}$ in the general case,*
(ii) *all $s\leqslant 1$ when Ω is convex,*
(iii) *all $s<\pi/\alpha$ in dimension $d=2$ when Ω is a polygon with largest angle equal to α.*

Moreover, when the data $\boldsymbol{f}$ belong to $L^2(\Omega)^d$, the pressure p belongs to $H^1(\Omega)$, together with the vorticity ω in dimension $d=2$.

3.1. The discrete spaces. From now on, we assume that Ω admits a disjoint decomposition into a finite number of rectangles in dimension $d = 2$, rectangular parallelepiped in dimension $d = 3$:

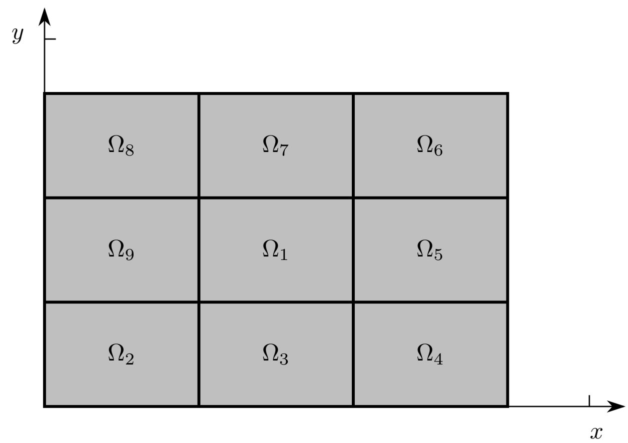

Figure 2. Example of decomposition of domain Ω.

$$\overline{\Omega} = \bigcup_{k=1}^{K} \overline{\Omega}_k, \quad \Omega_1 = \Omega_F, \quad \overline{\Omega}_P = \bigcup_{k=2}^{K} \overline{\Omega}_k, \quad \Omega_k \cap \Omega_{k'} = \emptyset, \quad 1 \leqslant k < k' \leqslant K. \tag{3.1}$$

We assume, as it is classical, that the intersection of two subdomains $\overline{\Omega}_k$ and $\overline{\Omega}_{k'}$, $1 \leqslant k < k' \leqslant K$, if not empty, is either a vertex or a whole edge or a whole face of both Ω_k and $\Omega_{k'}$, see Figure 2. We introduce the discretization parameter δ which is here a K-tuple of positive integers N_k, $1 \leqslant k \leqslant K$, $\delta = (N_1, N_2, \dots, N_k)$. For any non negative integer N_k, $\mathbb{P}_{N_k}(\Omega_k)$ stands for the space of polynomials on Ω_k with degree $\leqslant N_k$ with respect to each variable. For any triple (l, m, n) of non negative integers we introduce:

- ▷ in dimension $d = 2$, the space $\mathbb{P}_{l,m}(\Omega_k)$ of restrictions to Ω_k of polynomials with degree less than or equal to l with respect to x, less than or equal to m with respect to y,
- ▷ in dimension $d = 3$, the space $\mathbb{P}_{l,m,n}(\Omega_k)$ of restrictions to Ω_k of polynomials with degree less than or equal to l with respect to x, less than or equal to m with respect to y and less than or equal to n with respect to z.

When l and m are equal to n, these spaces are simply denoted by $\mathbb{P}_n(\Omega_k)$.

For each k, $1 \leqslant k \leqslant K$, and any integer $N_k \geqslant 2$, we define the spaces:

$$(3.2)\quad D^k_{N_k} = \begin{cases} \mathbb{P}_{N_k,N_k-1}(\Omega_k) \times \mathbb{P}_{N_k-1,N_k}(\Omega_k) & \text{if } d=2, \\ \mathbb{P}_{N_k,N_k-1,N_k-1}(\Omega_k) \times \mathbb{P}_{N_k-1,N_k,N_k-1}(\Omega_k) & \\ \qquad\qquad \times \mathbb{P}_{N_k-1,N_k-1,N_k}(\Omega_k) & \text{if } d=3, \end{cases}$$

$$C^1_{N_1} = \begin{cases} \mathbb{P}_{N_1}(\Omega_1) & \text{if } d=2, \\ \mathbb{P}_{N_1-1,N_1,N_1}(\Omega_1) \times \mathbb{P}_{N_1,N_1-1,N_1}(\Omega_1) \times \mathbb{P}_{N_1,N_1,N_1-1}(\Omega_1) & \text{if } d=3. \end{cases}$$

The broken norm on $H_0(\mathrm{div}, \Omega)$, making it a Hilbert space, is

$$\|\boldsymbol{v}\|^2_{H(\mathrm{div},\bigcup_{k=1}^K \Omega_k)} = \sum_{k=1}^K \|\boldsymbol{v}\|^2_{H(\mathrm{div},\Omega_k)}.$$

Inspired by [8], we define the discrete space of velocities $\mathbb{D}_\delta$, closed in $H_0(\mathrm{div}, \Omega)$ by

$$(3.3)\qquad \mathbb{D}_\delta = \{\boldsymbol{v}_\delta \in H_0(\mathrm{div}, \Omega),\ \boldsymbol{v}_{\delta|\Omega_k} \in D^k_{N_k},\ 1 \leqslant k \leqslant K\}.$$

The discrete space $\mathbb{C}_{N_1}$, which will approach $H_0(\mathbf{curl}, \Omega_F)$, is different according to the dimension and is defined by

$$(3.4)\qquad \mathbb{C}_{N_1} = C^1_{N_1} \cap H_0(\mathbf{curl}, \Omega_F).$$

Finally, for the approximation of $L^2_0(\Omega)$, we consider the space $\mathbb{M}_\delta$:

$$(3.5)\qquad \mathbb{M}_\delta = \{q_\delta \in L^2_0(\Omega);\ q_{\delta|\Omega_k} \in \mathbb{P}_{N_k-1}(\Omega_k),\ 1 \leqslant k \leqslant K\}.$$

We can note that the functions in $\mathbb{D}_\delta$ have continuous normal traces through the interfaces $\bigcup\limits_{1\leqslant k\neq k'\leqslant K} \partial\Omega_k \cap \partial\Omega_{k'}$ while the functions in $\mathbb{C}_{N_1}$ have continuous traces in dimension $d = 2$, continuous tangential traces in dimension $d = 3$. Therefore, the discretization that we propose is perfectly conforming.

We note that $\mathrm{div}\, \mathbb{D}_\delta \subset \mathbb{M}_\delta$, since for $\boldsymbol{v}_\delta \in \mathbb{D}_\delta$, $\mathrm{div}\, \boldsymbol{v}_{\delta|\Omega_k} \in \mathbb{P}_{N_{k-1}}(\Omega_k)$ and by Stokes formula we have

$$\int_\Omega \mathrm{div}\, \boldsymbol{v}_\delta(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = \sum_{k=1}^K \int_{\Omega_k} \mathrm{div}\, \boldsymbol{v}_\delta(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = \sum_{k=1}^K \int_{\partial\Omega_k} \boldsymbol{v}_\delta \cdot \boldsymbol{n}_k\, \mathrm{d}\tau = 0.$$

We introduce the kernel W_δ by

$$(3.6)\qquad W_\delta = \{\boldsymbol{v}_\delta \in \mathbb{D}_\delta;\ \forall q_\delta \in \mathbb{M}_\delta,\ b_\delta(\boldsymbol{v}_\delta, q_\delta) = 0\}.$$

It is readily checked, by taking q_δ equal to $\mathrm{div}\, \boldsymbol{v}_\delta$, that $W_\delta = \mathbb{D}_\delta \cap W$.

We now investigate some properties of the **curl** operator. We recall from [8], Lemmas 3.3, 3.4 and 3.5, [23] and readily adapting proofs, that:

(1) $\mathbf{curl}\ \mathbb{C}_{N_1} = W_{N_1} = \mathbb{D}_{N_1} \cap W$, where $\mathbb{D}_{N_1} = \{\boldsymbol{v} \in D^1_{N_1}, \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega_F\}$.

(2) The kernel of the **curl** operator in $\mathbb{C}_{N_1}$ is reduced to $\{0\}$ in dimension $d = 2$ and equal to the range of $H^1_0(\Omega_F) \cap \mathbb{P}_{N_1}(\Omega_F)$ by the gradient operator in dimension $d = 3$.

(3) There exists an operator A_{N_1} from W_{N_1} into $\mathbb{C}_{N_1}$ which satisfies:

▷ the fixed point property:

$$\forall\, \boldsymbol{v}_{N_1} \in W_{N_1},\ \mathbf{curl}\ A_{N_1}(\boldsymbol{v}_{N_1}) = \boldsymbol{v}_{N_1}; \tag{3.7}$$

▷ the orthogonality property in dimension $d = 3$:

$$\forall\, \mu_{N_1} \in H^1_0(\Omega_F) \cap \mathbb{P}_{N_1}(\Omega_1),\ \ (A_{N_1}(\boldsymbol{v}_{N_1}), \mathrm{grad}\ \mu_{N_1})_{N_1} = 0; \tag{3.8}$$

▷ the continuity: There exists a constant c independent of N_1 such that

$$\forall\, \boldsymbol{v}_{N_1} \in W_{N_1},\ \|A_{N_1}(\boldsymbol{v}_{N_1})\|_{H(\mathbf{curl},\Omega_F)} \leqslant c\|\boldsymbol{v}_{N_1}\|_{L^2(\Omega_F)^2}. \tag{3.9}$$

3.2. The quadrature formula and interpolation operators. Setting $\xi_0 = -1$ and $\xi_N = 1$, we introduce the $N-1$ nodes ξ_j, $1 \leqslant j \leqslant N-1$, and the $N+1$ weights ϱ_j, $0 \leqslant j \leqslant N$, of the Gauss-Lobatto quadrature formula on $[-1, 1]$.

Denoting by $\mathbb{P}_N(-1, 1)$ the space of restrictions to $[-1, 1]$ of polynomials with degree $\leqslant N$, we recall that the quadrature formula is exact for polynomials of degree $\leqslant 2N - 1$, so the following equality holds:

$$\forall\, \varphi \in \mathbb{P}_{2N-1}(-1, 1), \quad \int_{-1}^{1} \varphi(\zeta)\,\mathrm{d}\zeta = \sum_{j=0}^{N} \varphi(\xi_j)\varrho_j. \tag{3.10}$$

We also recall [11], form. (13.20) the following property:

$$\forall\, \varphi_N \in \mathbb{P}_N(-1, 1), \quad \|\varphi_N\|^2_{L^2(-1,1)} \leqslant \sum_{j=0}^{N} \varphi_N^2(\xi_j)\varrho_j \leqslant 3\|\varphi_N\|^2_{L^2(-1,1)}. \tag{3.11}$$

On each Ω_k, we take $N = N_k$ and by homothety and translation we construct from ξ^k_j and ϱ^k_j, $0 \leqslant j \leqslant N_k$:

▷ the nodes $\widetilde{\xi}^k_i$ and $\widetilde{\widetilde{\xi}}{}^k$ and the weights $\widetilde{\varrho}^k_i$ and $\widetilde{\widetilde{\varrho}}{}^k_i$, $0 \leqslant i \leqslant N_k$, for $d = 2$;

▷ the nodes $\widetilde{\xi}^k_i$, $\widetilde{\widetilde{\xi}}{}^k_i$ and $\widetilde{\widetilde{\widetilde{\xi}}}{}^k_i$, and the weights $\widetilde{\varrho}^k_i$, $\widetilde{\widetilde{\varrho}}{}^k_i$ and $\widetilde{\widetilde{\widetilde{\varrho}}}{}^k_i$, $0 \leqslant i \leqslant N_k$, for $d = 3$.

This leads to a discrete product on all functions φ and ψ which have continuous restrictions to all $\overline{\Omega}_k$, $1 \leqslant k \leqslant K$:

$$(3.12)\quad (\varphi, \psi)_{N_k} = \begin{cases} \displaystyle\sum_{i=0}^{N_k} \sum_{j=0}^{N_k} \varphi_{|\Omega_k}(\widetilde{\xi}_i^k, \widetilde{\widetilde{\xi}}_j^k)\psi_{|\Omega_k}(\widetilde{\xi}_i^k, \widetilde{\widetilde{\xi}}_j^k)\widetilde{\varrho}^k \widetilde{\widetilde{\varrho}}_j^k & \text{if } d = 2, \\ \displaystyle\sum_{i=0}^{N_k} \sum_{j=0}^{N_k} \sum_{l=0}^{N_k} \varphi_{|\Omega_k}(\widetilde{\xi}_i^k, \widetilde{\widetilde{\xi}}_j^k, \widetilde{\widetilde{\widetilde{\xi}}}_l^k)\psi_{|\Omega_k}(\widetilde{\xi}^k, \widetilde{\widetilde{\xi}}_j^k, \widetilde{\widetilde{\widetilde{\xi}}}_l^k)\widetilde{\varrho}_i^k \widetilde{\widetilde{\varrho}}_j^k \widetilde{\widetilde{\widetilde{\varrho}}}_l^k & \text{if } d = 3. \end{cases}$$

Also, we define the global scalar product on Ω by

$$(3.13)\qquad (\varphi, \psi)_\delta = \sum_{k=1}^{K} (\varphi, \psi)_{N_k}.$$

We define, for $1 \leqslant k \leqslant K$, $\mathcal{I}_\delta^k$ as the Lagrange interpolation operator on all nodes $(\widetilde{\xi}_i, \widetilde{\widetilde{\xi}}_j)$ in dimension $d = 2$ and $(\widetilde{\xi}_i, \widetilde{\widetilde{\xi}}_j, \widetilde{\widetilde{\widetilde{\xi}}}_k)$ in dimension $d = 3$, with values in $\mathbb{P}_{N_k}(\Omega_k)$ and the global interpolation operator $\mathcal{I}_\delta$, defined on continuous functions $\boldsymbol{f}$ by $(\mathcal{I}_\delta \boldsymbol{f})_{|\Omega_k} = \mathcal{I}_\delta^k \boldsymbol{f}_{|\Omega_k}$, $1 \leqslant k \leqslant K$. So from (3.11), we have for any $\boldsymbol{v}_\delta$:

$$(3.14)\qquad (\boldsymbol{f}, \boldsymbol{v}_\delta)_\delta = (\mathcal{I}_\delta \boldsymbol{f}, \boldsymbol{v}_\delta)_\delta \leqslant 3^d \|\mathcal{I}_\delta \boldsymbol{f}\|_{L^2(\Omega)^d} \|\boldsymbol{v}_\delta\|_{L^2(\Omega)^d}.$$

3.3. The discrete problem. From now on, we assume that the data $\boldsymbol{f}$ are continuous on $\overline{\Omega}$. The discrete problem is constructed from (2.10) by using the Galerkin method combined with numerical integration. It writes:

Find $(\boldsymbol{\omega}_{N_1}, \boldsymbol{u}_\delta, p_\delta)$ in $\mathbb{C}_{N_1} \times \mathbb{D}_\delta \times \mathbb{M}_\delta$ such that

(3.15)

$$\begin{cases} a_\delta(\boldsymbol{\omega}_{N_1}, \boldsymbol{u}_\delta; \boldsymbol{v}_\delta) + K_{N_1}(\boldsymbol{\omega}_{N_1}, \boldsymbol{u}_{N_1}; \boldsymbol{v}_{N_1}) + b_\delta(\boldsymbol{v}_\delta, p_\delta) = (\boldsymbol{f}, \boldsymbol{v}_\delta)_\delta & \forall\, \boldsymbol{v}_\delta \in \mathbb{D}_\delta, \\ b_\delta(\boldsymbol{u}_\delta, q_\delta) = 0 & \forall\, q_\delta \in \mathbb{M}_\delta, \\ c_{N_1}(\boldsymbol{\omega}_{N_1}, \boldsymbol{u}_{N_1}; \boldsymbol{\theta}_{N_1}) = 0 & \forall\, \boldsymbol{\theta}_{N_1} \in \mathbb{C}_{N_1}, \end{cases}$$

with $\boldsymbol{u}_{N_1} = \boldsymbol{u}_{\delta|\Omega_1}$, $\boldsymbol{v}_{N_1} = \boldsymbol{v}_{\delta|\Omega_1}$ and where the bilinear forms $a_\delta(\cdot, \cdot; \cdot)$, $b_\delta(\cdot, \cdot)$ and $c_{N_1}(\cdot, \cdot; \cdot)$ are defined, respectively, on $(\mathbb{C}_{N_1} \times \mathbb{D}_\delta) \times \mathbb{D}_\delta$, $\mathbb{D}_\delta \times \mathbb{M}_\delta$ and $(\mathbb{C}_{N_1} \times \mathbb{D}_\delta) \times \mathbb{C}_{N_1}$ by:

$$(3.16)\qquad a_\delta(\boldsymbol{\omega}_{N_1}, \boldsymbol{u}_\delta; \boldsymbol{v}_\delta) = \mu \sum_{k=2}^{K} (\boldsymbol{u}_{N_k}, \boldsymbol{v}_{N_k})_{N_k} + \nu\ (\mathbf{curl}\ \boldsymbol{\omega}_{N_1}, \boldsymbol{v}_{N_1})_{N_1},$$

$$b_\delta(\boldsymbol{u}_\delta, q_\delta) = -\sum_{k=1}^{K} (\operatorname{div} \boldsymbol{v}_{N_k}, q_{N_k})_{N_k},$$

$$c_{N_1}(\boldsymbol{\omega}_{N_1}, \boldsymbol{u}_{N_1}; \boldsymbol{\theta}_{N_1}) = (\boldsymbol{\omega}_{N_1}, \boldsymbol{\theta}_{N_1})_{N_1} - (\boldsymbol{u}_{N_1}, \mathbf{curl}\ \boldsymbol{\theta}_{N_1})_{N_1}.$$

The trilinear form $K_{N_1}(\cdot, \cdot; \cdot)$ is on the other hand defined by

$$(3.17)\qquad K_{N_1}(\boldsymbol{\omega}_{N_1}, \boldsymbol{u}_{N_1}; \boldsymbol{v}_{N_1}) = (\boldsymbol{\omega}_{N_1} \times \boldsymbol{u}_{N_1}, \boldsymbol{v}_{N_1})_{N_1}.$$

3.4. Existence and uniqueness for the discrete solution. We notice that if $(\boldsymbol{\omega}_{N_1}, \boldsymbol{u}_\delta, p_\delta)$ is a solution of problem (3.15), $(\boldsymbol{\omega}_{N_1}, \boldsymbol{u}_\delta)$ is a solution of the following reduced problem:

Find $(\boldsymbol{\omega}_{N_1}, \boldsymbol{u}_\delta)$ in $\mathcal{W}_\delta$ such that

$$\forall\, \boldsymbol{v}_\delta \in W_\delta,\ a_\delta(\boldsymbol{\omega}_{N_1}, \boldsymbol{u}_\delta; \boldsymbol{v}_\delta) + K_{N_1}(\boldsymbol{\omega}_{N_1}, \boldsymbol{u}_{N_1}; \boldsymbol{v}_{N_1}) = (\boldsymbol{f}, \boldsymbol{v}_\delta)_\delta, \tag{3.18}$$

where $\mathcal{W}_\delta$ is the kernel

$$\mathcal{W}_\delta = \{(\boldsymbol{\theta}_{N_1}, \boldsymbol{v}_\delta) \in \mathbb{C}_{N_1} \times W_\delta;\ \forall\, \varphi_{N_1} \in \mathbb{C}_{N_1},\ c_{N_1}(\boldsymbol{\theta}_{N_1}, \boldsymbol{v}_{N_1}; \boldsymbol{\varphi}_{N_1}) = 0\}. \tag{3.19}$$

As it is classical, we first show the existence of the solution for this reduced problem.

Proposition 3.1. *For each data $\boldsymbol{f}$ continuous on $\overline{\Omega}$, problem* (3.18) *admits a solution $(\boldsymbol{\omega}_{N_1}, \boldsymbol{u}_\delta)$ in $\mathcal{W}_\delta$. Moreover, this solution satisfies*

$$\|\boldsymbol{\omega}_{N_1}\|_{L^2(\Omega_F)^{d(d-1)/2}} + \|\boldsymbol{u}_\delta\|_{L^2(\Omega)^d} \leqslant c\nu^{-1}\|\mathcal{I}_\delta \boldsymbol{f}\|_{L^2(\Omega)^d} \tag{3.20}$$

with a constant c independent of δ.

Proof. We introduce the mapping φ_δ: $\mathcal{W}_\delta \mapsto (\mathcal{W}_\delta)'$ defined as:

$$\begin{gathered}\forall\,(\boldsymbol{\omega}_{N_1}, \boldsymbol{u}_\delta) \in \mathcal{W}_\delta \quad \forall\,(\boldsymbol{\theta}_{N_1}, \boldsymbol{w}_\delta) \in \mathcal{W}_\delta\\ \langle \varphi_\delta(\boldsymbol{\omega}_{N_1}, \boldsymbol{u}_\delta), (\boldsymbol{\theta}_{N_1}, \boldsymbol{w}_\delta)\rangle = a_\delta(\boldsymbol{\omega}_{N_1}, \boldsymbol{u}_\delta; \boldsymbol{w}_\delta) + K_{N_1}(\boldsymbol{\omega}_{N_1}, \boldsymbol{u}_{N_1}; \boldsymbol{w}_{N_1}) - (\boldsymbol{f}, \boldsymbol{w}_\delta)_\delta.\end{gathered}$$

We recall that $\mathcal{W}_\delta$ is a space of finite dimension and that it is provided with the norm

$$\begin{aligned}&(\|\boldsymbol{\omega}_{N_1}\|^2_{L^2(\Omega_F)^{d(d-1)/2}} + \|\boldsymbol{u}_\delta\|^2_{L^2(\Omega)^d})^{1/2}\\ &\qquad = \left(\|\boldsymbol{\omega}_{N_1}\|^2_{L^2(\Omega_F)^{d(d-1)/2}} + \sum_{k=1}^{K}\|\boldsymbol{u}_{N_k}\|^2_{L^2(\Omega_k)^d}\right)^{1/2}.\end{aligned}$$

The mapping φ_δ being linear on $\mathcal{W}_\delta$ is therefore continuous. According to [24], (3.5) we have

$$K_{N_1}(\boldsymbol{\omega}_{N_1}, \boldsymbol{u}_{N_1}; \boldsymbol{u}_{N_1}) = 0.$$

Then we have (following the notation in [23])

$$\begin{aligned}\langle \varphi_\delta(\boldsymbol{\omega}_{N_1}, \boldsymbol{u}_\delta), (\boldsymbol{\omega}_{N_1}, \boldsymbol{u}_\delta)\rangle &= a_\delta(\boldsymbol{\omega}_{N_1}, \boldsymbol{u}_\delta; \boldsymbol{u}_\delta) + K_{N_1}(\boldsymbol{\omega}_{N_1}, \boldsymbol{u}_{N_1}; \boldsymbol{u}_{N_1}) - (\boldsymbol{f}, \boldsymbol{u}_\delta)_\delta\\ &= \mu\sum_{k=2}^{K}(\boldsymbol{u}_{N_k}, \boldsymbol{u}_{N_k})_{N_k} + \nu(\mathbf{curl}\ \boldsymbol{\omega}_{N_1}, \boldsymbol{u}_{N_1})_{N_1} - (\mathcal{I}_\delta \boldsymbol{f}, \boldsymbol{u}_\delta)_\delta.\end{aligned} \tag{3.21}$$

According to the definition of $\mathcal{W}_\delta$, we have

$$c_{N_1}(\boldsymbol{\omega}_{N_1}, \boldsymbol{u}_{N_1}; \boldsymbol{\omega}_{N_1}) = 0,$$

which gives

$$(\boldsymbol{\omega}_{N_1}, \boldsymbol{\omega}_{N_1})_{N_1} = (\boldsymbol{u}_{N_1}, \mathbf{curl}\ \boldsymbol{\omega}_{N_1})_{N_1}. \tag{3.22}$$

Using (3.11) in dimension 2, or (3.10) and then (3.11) in dimension 3, we have

$$(\boldsymbol{\omega}_{N_1}, \boldsymbol{\omega}_{N_1})_{N_1} \geqslant \|\boldsymbol{\omega}_{N_1}\|^2_{L^2(\Omega_F)^{d(d-1)/2}} \tag{3.23}$$

and

$$\begin{aligned}(\mathcal{I}_\delta \boldsymbol{f}, \boldsymbol{u}_\delta)_\delta &= \left(\sum_{i=0}^{N_k}\sum_{j=0}^{N_k} \mathcal{I}_\delta \boldsymbol{f}(\widetilde{\xi}_i^k, \widetilde{\widetilde{\xi}}_j^k)\boldsymbol{u}_\delta(\widetilde{\xi}_i^k, \widetilde{\widetilde{\xi}}_j^k)\widetilde{\varrho}_i^k\widetilde{\widetilde{\varrho}}_j^k\right) \\ &\leqslant \left(\sum_{i=0}^{N_k}\sum_{j=0}^{N_k}(\mathcal{I}_\delta^2 \boldsymbol{f}(\widetilde{\xi}_i^k, \widetilde{\widetilde{\xi}}_j^k)\widetilde{\varrho}_i^k\widetilde{\widetilde{\varrho}}_j^k)\right)^{1/2}\left(\sum_{i=0}^{N_k}\sum_{j=0}^{N_k}(\boldsymbol{u}_\delta^2(\widetilde{\xi}_i^k, \widetilde{\widetilde{\xi}}_j^k)\widetilde{\varrho}_i^k\widetilde{\widetilde{\varrho}}_j^k)\right)^{1/2} \\ &\leqslant 3^{d/2}\|\mathcal{I}_\delta \boldsymbol{f}\|_{L^2(\Omega)}(\boldsymbol{u}_\delta, \boldsymbol{u}_\delta)_\delta^{1/2}.\end{aligned} \tag{3.24}$$

On the other hand, it follows from [8], Lemmas 3.3 and 3.5, that the rotational operator sends $\mathbb{C}_{N_1}$ to W_{N_1}, i.e.,

$$\forall\, \boldsymbol{u}_{N_1} \in W_{N_1} \text{ there exists } \boldsymbol{\psi}_{N_1} \in \mathbb{C}_{N_1} \text{ such that } \boldsymbol{u}_{N_1} = \mathbf{curl}\ \boldsymbol{\psi}_{N_1}.$$

In addition, we have for a constant c independent of δ,

$$\|\boldsymbol{\psi}_{N_1}\|_{L^2(\Omega_F)^{d(d-1)/2}} \leqslant c\|\boldsymbol{u}_{N_1}\|_{L^2(\Omega_F)^d}.$$

According to the definition of $\mathcal{W}_\delta$, we have

$$\begin{aligned}(\boldsymbol{u}_{N_1}, \boldsymbol{u}_{N_1})_{N_1} &= (\boldsymbol{u}_{N_1}, \mathbf{curl}\ \boldsymbol{\psi}_{N_1})_{N_1} = (\boldsymbol{\omega}_{N_1}, \boldsymbol{\psi}_{N_1})_{N_1} \\ &\leqslant 3^{d/2}\|\boldsymbol{\omega}_{N_1}\|_{L^2(\Omega_F)^{d(d-1)/2}} 3^{d/2}\|\boldsymbol{\psi}_{N_1}\|_{L^2(\Omega_F)^{d(d-1)/2}} \\ &\leqslant c3^d\|\boldsymbol{\omega}_{N_1}\|_{L^2(\Omega_F)^{d(d-1)/2}}\|\boldsymbol{u}_{N_1}\|_{L^2(\Omega_F)^d} \\ &\leqslant c3^d\|\boldsymbol{\omega}_{N_1}\|_{L^2(\Omega_F)^{d(d-1)/2}}(\boldsymbol{u}_{N_1}, \boldsymbol{u}_{N_1})_{N_1}^{1/2}.\end{aligned}$$

We deduce that

$$(\boldsymbol{u}_{N_1}, \boldsymbol{u}_{N_1})_{N_1}^{1/2} \leqslant 3^d c\|\boldsymbol{\omega}_{N_1}\|_{L^2(\Omega_F)^{d(d-1)/2}}. \tag{3.25}$$

Combining inequalities (3.21), (3.22), (3.23), (3.24) and (3.25), we obtain

$$
\begin{aligned}
(3.26)\quad & \langle \varphi_\delta(\boldsymbol{\omega}_{N_1}, \boldsymbol{u}_\delta), (\boldsymbol{\omega}_{N_1}, \boldsymbol{u}_\delta)\rangle \\
&\geqslant \mu \|\boldsymbol{u}_\delta\|^2_{L^2(\Omega_P)^d} + \nu \|\boldsymbol{\omega}_{N_1}\|^2_{L^2(\Omega_F)^{d(d-1)/2}} \\
&\quad - 3^{d/2} \|\mathcal{I}_\delta \boldsymbol{f}\|_{L^2(\Omega)^d} \|\boldsymbol{u}_\delta\|_{L^2(\Omega)^d} \\
&\geqslant \min\Big(\frac{\mu}{3^d}, \frac{\nu}{2c_1 3^{d/2}}\Big) \|\boldsymbol{u}_\delta\|^2_{L^2(\Omega)^d} + \frac{\nu}{2} \|\omega_{N_1}\|^2_{L^2(\Omega_F)^{d(d-1)/2}} \\
&\quad - 3^{d/2} \|\mathcal{I}_\delta \boldsymbol{f}\|_{L^2(\Omega)^d} \|\boldsymbol{u}_\delta\|_{L^2(\Omega)^d} \\
&\geqslant \min\Big(\frac{\mu}{3^d}, \frac{\nu}{2c_1 3^{d/2}}\Big) \|\boldsymbol{u}_\delta\|^2_{L^2(\Omega)^d} + \frac{\nu}{2} \|\omega_{N_1}\|^2_{L^2(\Omega_F)^{d(d-1)/2}} \\
&\quad - 3^{d/2} \|\mathcal{I}_\delta \boldsymbol{f}\|_{L^2(\Omega)^d} (\|\boldsymbol{u}_\delta\|^2_{L^2(\Omega)^d} + \|\omega_{N_1}\|^2_{L^2(\Omega_F)^{d(d-1)/2}})^{1/2} \\
&\geqslant \min\Big(\min\Big(\frac{\mu}{3^d}, \frac{\nu}{2c_1 3^{d/2}}\Big), \frac{\nu}{2}\Big) (\|\boldsymbol{u}_\delta\|^2_{L^2(\Omega)^d} + \|\omega_{N_1}\|^2_{L^2(\Omega_F)^{d(d-1)/2}}) \\
&\quad - 3^{d/2} \|\mathcal{I}_\delta \boldsymbol{f}\|_{L^2(\Omega)^d} (\|\boldsymbol{u}_\delta\|^2_{L^2(\Omega)^d} + \|\omega_{N_1}\|^2_{L^2(\Omega_F)^{d(d-1)/2}})^{1/2}
\end{aligned}
$$

with a constant c_1 independent of δ.

Let us fix

$$
\mu_\delta = \frac{3^{d/2} \|\mathcal{I}_\delta \boldsymbol{f}\|_{L^2(\Omega)^d}}{\min(\min(\mu/3^d, \nu/(2c_1 3^{d/2})), \nu/2)}
$$

and denote by S_{μ_δ} the sphere of radius μ_δ. We deduce from (3.26) that on the sphere S_{μ_δ} we have

$$
\langle \varphi_\delta(\boldsymbol{\omega}_{N_1}, \boldsymbol{u}_\delta), (\boldsymbol{\omega}_{N_1}, \boldsymbol{u}_\delta)\rangle \geqslant 0.
$$

We then have that $\varphi_\delta(\cdot,\cdot)$ is continuous on $\mathcal{W}_\delta$ and $\langle \varphi_\delta(\boldsymbol{\omega}_{N_1}, \boldsymbol{u}_\delta), (\boldsymbol{\omega}_{N_1}, \boldsymbol{u}_\delta)\rangle \geqslant 0$ for all $(\boldsymbol{\omega}_{N_1}, \boldsymbol{u}_\delta) \in \mathcal{W}_\delta \cap S_{\mu_\delta}$.

So applying Brouwer's fixed-point theorem (see [20], Chapter IV, Corollary 1.1), there exists $(\boldsymbol{\theta}_{N_1}, \boldsymbol{w}_\delta) \in \mathcal{W}_\delta \cap S_{\mu_\delta}$ such that

$$
\langle \varphi_\delta(\boldsymbol{\omega}_{N_1}, \boldsymbol{u}_\delta), (\boldsymbol{\theta}_{N_1}, \boldsymbol{w}_\delta)\rangle = 0,
$$

that is, $(\boldsymbol{\omega}_{N_1}, \boldsymbol{u}_\delta)$ is a solution of the reduced problem (3.18). This solution checks (3.20), because it is on the sphere S_{μ_δ}. □

We recall from [23] that there exists a constant β_* independent of δ such that

$$
(3.27)\qquad \forall\, q_\delta \in \mathbb{M}_\delta, \quad \sup_{\boldsymbol{v}_\delta \in \mathbb{D}_\delta} \frac{b_\delta(\boldsymbol{v}_\delta, q_\delta)}{\|\boldsymbol{v}_\delta\|_{H(\mathrm{div},\Omega)}} \geqslant \beta_* \|q_\delta\|_{L^2(\Omega)}.
$$

Theorem 3.2. *For any data function* $\boldsymbol{f}$ *continuous on* $\overline{\Omega}$*, the discrete problem* (3.15) *admits a solution* $(\boldsymbol{\omega}_{N_1}, \boldsymbol{u}_\delta, p_\delta) \in \mathbb{C}_{N_1} \times \mathbb{D}_\delta \times \mathbb{M}_\delta$*. Moreover,* $(\boldsymbol{\omega}_{N_1}, \boldsymbol{u}_\delta)$ *verifies* (3.20).

P r o o f. For a solution $(\boldsymbol{\omega}_{N_1}, \boldsymbol{u}_\delta)$ of (3.18), we get the pressure by writing:

$$\forall\, \boldsymbol{v}_\delta \in \mathbb{D}_\delta,\ b_\delta(\boldsymbol{v}_\delta, p_\delta) = (\boldsymbol{f}, \boldsymbol{v}_\delta)_\delta - a_\delta(\boldsymbol{\omega}_{N_1}, \boldsymbol{u}_\delta; \boldsymbol{v}_\delta) - K_{N_1}(\boldsymbol{\omega}_{N_1}, \boldsymbol{u}_{N_1}; \boldsymbol{v}_{N_1}). \tag{3.28}$$

We then have

$$\forall\, \boldsymbol{v}_\delta \in W_\delta, \quad b_\delta(\boldsymbol{v}_\delta, p_\delta) = 0.$$

Condition (3.27) leads to the existence of a pressure $p_\delta \in \mathbb{M}_\delta$ that verifies (3.28). □

R e m a r k 3.3. We note that Theorem 3.2 remains true when $K_{N_1}(\cdot,\cdot;\cdot)$ is replaced by $K(\cdot,\cdot;\cdot)$ in the discrete problem (3.15). In practice, we have to use a more accurate quadrature formula, exact on $\mathbb{P}_{3N_1-1}(\Omega_F)$, to evaluate the integrals appearing in the non-linear term.

The corresponding discrete problem writes:

Find $(\boldsymbol{\omega}_{N_1}, \boldsymbol{u}_\delta, p_\delta)$ in $\mathbb{C}_{N_1} \times \mathbb{D}_\delta \times \mathbb{M}_\delta$ such that

$$\begin{cases} a_\delta(\boldsymbol{\omega}_{N_1}, \boldsymbol{u}_\delta; \boldsymbol{v}_\delta) + K(\boldsymbol{\omega}_{N_1}, \boldsymbol{u}_{N_1}; \boldsymbol{v}_{N_1}) + b_\delta(\boldsymbol{v}_\delta, p_\delta) = \langle \boldsymbol{f}, \boldsymbol{v}_\delta \rangle & \forall\, \boldsymbol{v}_\delta \in \mathbb{D}_\delta, \\ b_\delta(\boldsymbol{u}_\delta, q_\delta) = 0 & \forall\, q_\delta \in \mathbb{M}_\delta, \\ c_{N_1}(\boldsymbol{\omega}_{N_1}, \boldsymbol{u}_{N_1}; \boldsymbol{\varphi}_{N_1}) = 0 & \forall\, \boldsymbol{\varphi}_{N_1} \in \mathbb{C}_{N_1}. \end{cases} \tag{3.29}$$

The associated reduced problem reads:

Find $(\boldsymbol{\omega}_{N_1}, \boldsymbol{u}_\delta)$ in $\mathcal{W}_\delta$ such that

$$\forall\, \boldsymbol{v}_\delta \in W_\delta, \quad a_\delta(\boldsymbol{\omega}_{N_1}, \boldsymbol{u}_\delta; \boldsymbol{v}_\delta) + K(\boldsymbol{\omega}_{N_1}, \boldsymbol{u}_{N_1}; \boldsymbol{v}_{N_1}) = (\boldsymbol{f}, \boldsymbol{v}_\delta)_\delta. \tag{3.30}$$

And it is precisely on this issue that we will study the convergence error.

4. Error estimate

We will now give an error estimate, in dimension $d = 2$, between the exact solution of (2.10) and the discrete solution of (3.29). We set $\mathcal{X} = H_0(\mathbf{curl}\,, \Omega_F) \times W$ and note by $\mathcal{SD}$ the Stokes-Darcy operator defined in the following way:

For any $\boldsymbol{f} \in H_0(\mathrm{div}, \Omega)'$, $\mathcal{SD}\boldsymbol{f}$ denotes the unique solution $(\omega, \boldsymbol{u})$ of the following problem:

$$\text{Find } (\omega, \boldsymbol{u}) \text{ in } \mathcal{W} \text{ such that for all } \boldsymbol{v} \in W,\ a(\omega, \boldsymbol{u}; \boldsymbol{v}) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle\,. \tag{4.1}$$

We also introduce the mapping G defined from $\mathcal{X}$ into the dual space of $H_0(\mathrm{div}, \Omega)$ by

$$\forall\, (\omega, \boldsymbol{u}) \in \mathcal{X} \quad \forall\, \boldsymbol{v} \in H_0(\mathrm{div}, \Omega),\ \langle G(\omega, \boldsymbol{u}), \boldsymbol{v} \rangle = K(\omega, \boldsymbol{u}; \boldsymbol{v}) - \langle \boldsymbol{f}, \boldsymbol{v} \rangle\,. \tag{4.2}$$

We recall that $\mathcal{W} \subset (H(\mathrm{div}, \Omega_F) \cap H_0(\mathbf{curl}\,, \Omega_F)) \times (H_0(\mathrm{div}, \Omega_F) \cap H(\mathbf{curl}\,, \Omega_F))$. For $d = 2$, $H_0(\mathbf{curl}\,, \Omega_F) = H_0^1(\Omega_F)$ is embedded compactly in $L^p(\Omega_F)$ for all $p < \infty$ and the space $H_0(\mathrm{div}, \Omega_F) \cap H(\mathbf{curl}\,, \Omega_F)$ is embedded into $H^{1/2}(\Omega_F)^2$, it is therefore embedded compactly in $L^3(\Omega_F)^2$ see [20]. The function G is then well defined and continuous.

The reduced problem (2.13) can be written in the equivalent form:

$$\begin{cases} \text{Find } (\omega, \boldsymbol{u}) \in \mathcal{X} \text{ such that} \\ (\omega, \boldsymbol{u}) + \mathcal{SD}G(\omega, \boldsymbol{u}) = 0. \end{cases} \tag{4.3}$$

Turning to the discrete problem, we set $\mathcal{X}_\delta = \mathbb{C}_{N_1} \times W_\delta$ and define the discrete Stokes-Darcy operator $\mathcal{SD}_\delta$ by:

For $\boldsymbol{f} \in H_0(\mathrm{div}, \Omega)'$, $\mathcal{SD}_\delta \boldsymbol{f}$ denotes the solution $(\omega_{N_1}, \boldsymbol{u}_\delta)$ of the reduced discrete problem:

$$\text{Find } (\omega_{N_1}, \boldsymbol{u}_\delta) \text{ in } \mathcal{W}_\delta \text{ such that } \forall\, \boldsymbol{v}_\delta \in W_\delta,\ a_\delta(\omega_{N_1}, \boldsymbol{u}_\delta; \boldsymbol{v}_\delta) = \langle \boldsymbol{f}, \boldsymbol{v}_\delta \rangle\,. \tag{4.4}$$

We recall from [23] the following results:

(1) The operator $\mathcal{SD}_\delta$ is well defined and satisfies the stability property

$$\|\mathcal{SD}_\delta \boldsymbol{f}\|_{\mathcal{X}} \leqslant c \sup_{\boldsymbol{v}_\delta \in W_\delta} \frac{\langle \boldsymbol{f}, \boldsymbol{v}_\delta \rangle}{\|\boldsymbol{v}_\delta\|_{L^2(\Omega)^d}}. \tag{4.5}$$

(2) For any data $\boldsymbol{f}$ such that $\mathcal{SD}\boldsymbol{f} \in H^{s+1}(\Omega_F) \times H^s(\Omega)^2$, $s \geqslant 1$, we have

$$\|(\mathcal{SD} - \mathcal{SD}_\delta)\boldsymbol{f}\|_{\mathcal{X}} \leqslant c N_\delta^{-s} \|\mathcal{SD}\boldsymbol{f}\|_{H^{s+1}(\Omega_F) \times H^s(\Omega)^2}, \tag{4.6}$$

where $N_\delta = \min N_k$, $1 \leqslant k \leqslant K$, and c is a constant independent of δ.

We consider also the mapping G_δ defined from $\mathcal{X}_\delta$ into the dual space of $\mathbb{D}_\delta$ by

$$\forall\, (\omega_{N_1}, \boldsymbol{u}_\delta) \in \mathcal{X}_\delta\ \forall\, \boldsymbol{v}_\delta \in \mathbb{D}_\delta, \quad \langle G_\delta(\omega_{N_1}, \boldsymbol{u}_\delta), \boldsymbol{v}_\delta \rangle = K(\omega_{N_1}, \boldsymbol{u}_{N_1}; \boldsymbol{v}_{N_1}) - (\boldsymbol{f}, \boldsymbol{v}_\delta)_\delta. \tag{4.7}$$

So, problem (3.30) can be written in the equivalent form:

$$\forall\, (\omega_{N_1}, \boldsymbol{u}_\delta) \in \mathcal{X}_\delta, \quad (\omega_{N_1}, \boldsymbol{u}_\delta) + \mathcal{SD}_\delta G_\delta(\omega_{N_1}, \boldsymbol{u}_\delta) = 0. \tag{4.8}$$

From now on and as in [4], this will be a hypothesis which will guarantee the local uniqueness by the theorem of local inversion.

Hypothesis 4.1. *The pair $(\omega, \boldsymbol{u})$ is a solution of the reduced problem* [24], (4.1) *such that the operator* $\mathrm{Id} + \mathcal{SD}DG(\omega, \boldsymbol{u})$ *is an isomorphism of $\mathcal{X}$ or equivalently* $\mathrm{Id} + \mathcal{SD}DG(\omega, \boldsymbol{u})$ *is an isomorphism of* $H_0(\mathbf{curl}\,, \Omega_F) \times H_0(\mathrm{div}, \Omega)$, *where D is the differential operator.*

Which translates by: For any function $\boldsymbol{g} \in H_0(\mathrm{div}, \Omega)'$, the following linearized problem:

Find $(\theta, \boldsymbol{w}, r)$ in $H_0(\mathbf{curl}\,, \Omega_F) \times H_0(\mathrm{div}, \Omega) \times L_0^2(\Omega)$ such that

$$
(4.9) \quad \begin{cases} a(\theta, \boldsymbol{w}; \boldsymbol{v}) + K(\omega, \boldsymbol{w}; \boldsymbol{v}) + K(\theta, \boldsymbol{u}; \boldsymbol{v}) + b(\boldsymbol{v}, r) = \langle \boldsymbol{g}, \boldsymbol{v} \rangle & \forall\, \boldsymbol{v} \in H_0(\mathrm{div}, \Omega), \\ b(\boldsymbol{w}, q) = 0 & \forall\, q \in L_0^2(\Omega), \\ c(\theta, \boldsymbol{w}; \varphi) = 0 & \forall\, \varphi \in H_0(\mathbf{curl}\,, \Omega_F), \end{cases}
$$

has a unique solution $(\theta, \boldsymbol{w}, r)$ and the mapping $\boldsymbol{g} \mapsto (\theta, \boldsymbol{w}, r)$ is continuous from $H_0(\mathrm{div}, \Omega)'$ into $H_0(\mathbf{curl}\,, \Omega_F) \times H_0(\mathrm{div}, \Omega) \times L_0^2(\Omega)$.

Lemma 4.2. *In dimension $d = 2$, we have for $N_1 \geqslant 5$:*

$$
(4.10) \qquad \forall\, \omega_{N_1} \in \mathbb{C}_{N_1},\ \forall\, \boldsymbol{u}_{N_1} \in \mathbb{D}_{N_1},\ \forall\, \boldsymbol{v}_{N_1} \in \mathbb{D}_{N_1},
$$
$$
|K(\omega_{N_1}, \boldsymbol{u}_{N_1}; \boldsymbol{v}_{N_1})| \leqslant c |\log\ N_1|^{1/2} \|\omega_{N_1}\|_{H_0(\mathbf{curl}\,, \Omega_F)} \|\boldsymbol{u}_{N_1}\|_{L^2(\Omega_F)^2} \|\boldsymbol{v}_{N_1}\|_{L^2(\Omega_F)^2}.
$$

P r o o f. The proof is based on Sobolev inequalities [27], and the inverse inequality, see [9], Proposition 3.1. It is the same as [4], Lemma 3.4, since the Navier-Stokes equations are considered here with a single domain. □

Lemma 4.3. *In dimension $d = 2$, if Hypothesis 4.1 is verified, there exists an integer n_0 such that for any $N_\delta = \min(N_k)_{1 \leqslant k \leqslant K} \geqslant n_0$, the operator $\mathrm{Id} + \mathcal{S}\mathcal{D}_\delta DG_\delta(\omega, \boldsymbol{u})$ is an isomorphism of $\mathcal{X}$. The norm of its inverse is bounded independently of δ.*

P r o o f. (1) We write the expansion

$$
(4.11) \quad \mathrm{Id} + \mathcal{S}\mathcal{D}_\delta DG_\delta(\omega, \boldsymbol{u}) = \mathrm{Id} + \mathcal{S}\mathcal{D} DG(\omega, \boldsymbol{u}) - (\mathcal{S}\mathcal{D} - \mathcal{S}\mathcal{D}_\delta) DG(\omega, \boldsymbol{u}) - \mathcal{S}\mathcal{D}_\delta (DG(\omega, \boldsymbol{u}) - DG_\delta(\omega, \boldsymbol{u})).
$$

Moreover, it follows from the definition of G and G_δ that for all $(\theta, \boldsymbol{w})$ in $\mathcal{X}$ and $\boldsymbol{v}_\delta \in W_\delta$

$$
\begin{aligned} \langle DG(\omega, \boldsymbol{u}) \cdot (\theta, \boldsymbol{w}), \boldsymbol{v}_\delta \rangle &= K(\omega, \boldsymbol{w}; \boldsymbol{v}_\delta) + K(\theta, \boldsymbol{u}; \boldsymbol{v}_\delta), \\ \langle DG_\delta(\omega, \boldsymbol{u}) \cdot (\theta, \boldsymbol{w}), \boldsymbol{v}_\delta \rangle &= K(\omega, \boldsymbol{w}; \boldsymbol{v}_\delta) + K(\theta, \boldsymbol{u}; \boldsymbol{v}_\delta), \end{aligned}
$$

so the last term in (4.11) vanishes. We now check that the second term tends to zero. We have, by differentiation, for all $(\theta, \boldsymbol{w})$ in $\mathcal{X}$,

$$
DG(\omega, \boldsymbol{u}) \cdot (\theta, \boldsymbol{w}) = \omega \times \boldsymbol{w} + \theta \times \boldsymbol{u}.
$$

Using the formula

$$
(4.12) \qquad \mathbf{curl}\,(\omega \times \boldsymbol{u}) = \boldsymbol{\nabla}\ \omega \cdot \boldsymbol{u} \quad \forall\, \omega \in H^1(\Omega_F),\ \forall\, \boldsymbol{u} \in W,
$$

we get

$$\begin{aligned}\mathbf{curl}\,(DG(\omega,\boldsymbol{u})\cdot(\theta,\boldsymbol{w})) &= \mathbf{curl}\,(\omega\times\boldsymbol{w}+\theta\times\boldsymbol{u})\\ &= \mathbf{curl}\,(\omega\times\boldsymbol{w})+\mathbf{curl}\,(\theta\times\boldsymbol{u})\\ &= \boldsymbol{\nabla}\,\omega\cdot\boldsymbol{w}+\boldsymbol{\nabla}\,\theta\cdot\boldsymbol{u},\end{aligned}$$

using the arguments of proof of Proposition 2.1, we get that $\mathcal{SD}DG(\omega,\boldsymbol{u})\cdot(\theta,\boldsymbol{w})\in H^2(\Omega_F)\times H^1(\Omega)^2$. In addition, we have

$$\begin{aligned}&\|\mathcal{SD}DG(\omega,\boldsymbol{u})(\theta,\boldsymbol{w})\|_{H^2(\Omega_F)\times H^1(\Omega)^2}\\ &\qquad\leqslant c(\|\omega\|_{H^{s_0+1}(\Omega_F)}+\|\boldsymbol{u}\|_{H^{s_0}(\Omega)^2})\|(\theta,\boldsymbol{w})\|_{\mathcal{X}}.\end{aligned}\tag{4.13}$$

Using (4.6) with $s=1$, we get from (4.13):

$$\|(\mathcal{SD}-\mathcal{SD}_\delta)DG(\omega,\boldsymbol{u})\|_{\mathcal{L}}\leqslant cN_\delta^{-1}\|\mathcal{SD}DG(\omega,\boldsymbol{u})(\theta,\boldsymbol{w})\|_{H^2(\Omega_F)\times H^1(\Omega)^2},$$

where $N_\delta=\min(N_k)_{1\leqslant k\leqslant K}$ and $\mathcal{L}(\mathcal{X})$ is the space of linear continuous operators from $\mathcal{X}$ into itself. So we have

$$\lim_{N_\delta\to\infty}\|(\mathcal{SD}-\mathcal{SD}_\delta)DG(\omega,\boldsymbol{u})\|_{\mathcal{L}}=0.\tag{4.14}$$

(2) If Hypothesis 4.1 is verified, we set

$$\|(\mathrm{Id}+\mathcal{SD}DG(\omega,\boldsymbol{u}))^{-1}\|_{\mathcal{L}}=\gamma.\tag{4.15}$$

So from (4.14) there exists $n_0\in\mathbb{N}$ such that

$$\forall\,N_\delta\geqslant n_0,\quad\|(\mathcal{SD}-\mathcal{SD}_\delta)DG(\omega,\boldsymbol{u})\|_{\mathcal{L}}\leqslant\frac{1}{2\gamma}$$

and from (4.15) we then conclude that

$$\|(\mathrm{Id}+\mathcal{SD}_\delta DG_\delta(\omega,\boldsymbol{u}))^{-1}\|_{\mathcal{L}}\leqslant 2\gamma.$$

□

Lemma 4.4. *In dimension $d=2$, the mapping $(\omega,\boldsymbol{u})\mapsto\mathrm{Id}+\mathcal{SD}_\delta DG_\delta(\omega,\boldsymbol{u})$ is Lipschitzian in $\mathcal{X}$ and verifies:*

$$\begin{aligned}&\|\mathcal{SD}_\delta(DG_\delta(\omega,\boldsymbol{u})-DG_\delta(\widetilde{\omega},\widetilde{\boldsymbol{u}}))\|_{\mathcal{L}}\\ &\qquad\leqslant c|\log N_1|^{1/2}(\|\omega-\widetilde{\omega}\|_{H(\mathbf{curl}\,,\Omega_F)}+\|\boldsymbol{u}-\widetilde{\boldsymbol{u}}\|_{L^2(\Omega)^2})\quad\forall\,(\widetilde{\omega},\widetilde{\boldsymbol{u}})\in\mathcal{X}.\end{aligned}\tag{4.16}$$

P r o o f. Let $(\theta,\boldsymbol{w})\in\mathcal{X}$ and $\boldsymbol{v}_\delta\in W_\delta$. We have from (4.7):

$$\langle(DG_\delta(\omega,\boldsymbol{u})-DG_\delta(\widetilde{\omega},\widetilde{\boldsymbol{u}}))\cdot(\theta,\boldsymbol{w}),\boldsymbol{v}_\delta\rangle=K(\omega-\widetilde{\omega},\boldsymbol{w};\boldsymbol{v}_{N_1})+K(\theta,\boldsymbol{u}-\widetilde{\boldsymbol{u}};\boldsymbol{v}_{N_1}).$$

By using arguments of Lemma 4.2 we have

$$\begin{aligned}\langle(DG_\delta(\omega,\boldsymbol{u})&-DG_\delta(\widetilde{\omega},\widetilde{\boldsymbol{u}}))(\theta,\boldsymbol{w}),\boldsymbol{v}_\delta\rangle\\&\leqslant c|\log\ N_1|^{1/2}\|\omega-\widetilde{\omega}\|_{H(\mathbf{curl},\Omega_F)}\|\boldsymbol{w}\|_{L^2(\Omega)^2}\|\boldsymbol{v}_{N_1}\|_{L^2(\Omega_F)^2}\\&\quad+c|\log\ N_1|^{1/2}\|\theta\|_{H(\mathbf{curl},\Omega_F)}\|\boldsymbol{u}-\widetilde{\boldsymbol{u}}\|_{L^2(\Omega)^2}\|\boldsymbol{v}_{N_1}\|_{L^2(\Omega_F)^2}\\&\leqslant c|\log\ N_1|^{1/2}(\|\omega-\widetilde{\omega}\|_{H(\mathbf{curl},\Omega_F)}+\|\boldsymbol{u}-\widetilde{\boldsymbol{u}}\|_{L^2(\Omega)^2})\\&\quad\times(\|\boldsymbol{w}\|_{L^2(\Omega)^2}+\|\theta\|_{H(\mathbf{curl},\Omega_F)})\|\boldsymbol{v}_{N_1}\|_{L^2(\Omega_F)^2},\end{aligned}$$

by the definition of the norm in $\mathcal{L}(\mathcal{X})$ and according to (4.5), we obtain the result.

□

Lemma 4.5. *In dimension $d=2$, if we suppose that $\boldsymbol{f}$ belongs to $H^\sigma(\Omega)^d$, $\sigma>1$, and that the solution $(\omega,\boldsymbol{u},p)$ of problem (2.10) is in $H^{s+1}(\Omega_F)\times H^s(\Omega)^2\times H^s(\Omega)$, $s>1$, we have the following estimation:*

(4.17)
$$\|(\omega,\boldsymbol{u})+\mathcal{SD}_\delta G_\delta(\omega,\boldsymbol{u})\|_{\mathcal{X}}\leqslant c(N_\delta^{-s}\|(\omega,\boldsymbol{u})\|_{H^{s+1}(\Omega_F)\times H^s(\Omega)^2}+N_\delta^{-\sigma}\|\boldsymbol{f}\|_{H^\sigma(\Omega)^2})$$

with $N_\delta=\min(N_k)_{1\leqslant k\leqslant K}$.

P r o o f. From (4.3), we have that $(\omega,\boldsymbol{u})$ is a solution of $(\omega,\boldsymbol{u})+\mathcal{SD}G(\omega,\boldsymbol{u})=0$, so we have

$$(\omega,\boldsymbol{u})+\mathcal{SD}_\delta G_\delta(\omega,\boldsymbol{u})=(\mathcal{SD}_\delta-\mathcal{SD})G(\omega,\boldsymbol{u})+\mathcal{SD}_\delta(G_\delta(\omega,\boldsymbol{u})-G(\omega,\boldsymbol{u})).$$

Consequently, we get

(4.18)
$$\|(\omega,\boldsymbol{u})+\mathcal{SD}_\delta G_\delta(\omega,\boldsymbol{u})\|_{\mathcal{X}}\leqslant\|(\mathcal{SD}-\mathcal{SD}_\delta)G(\omega,\boldsymbol{u})\|_{\mathcal{X}}+\|\mathcal{SD}_\delta(G_\delta(\omega,\boldsymbol{u})-G(\omega,\boldsymbol{u}))\|_{\mathcal{X}}.$$

From (4.6) we have

$$\begin{aligned}\|(\mathcal{SD}-\mathcal{SD}_\delta)G(\omega,\boldsymbol{u})\|_{\mathcal{X}}&\leqslant cN_\delta^{-s}\|\mathcal{SD}G(\omega,\boldsymbol{u})\|_{H^{s+1}(\Omega_F)\times H^s(\Omega)^2}\\&=cN_\delta^{-s}\|(\omega,\boldsymbol{u})\|_{H^{s+1}(\Omega_F)\times H^s(\Omega)^2}.\end{aligned}\tag{4.19}$$

On the other hand, we have that $\mathcal{SD}_\delta(G(\omega,\boldsymbol{u})-G_\delta(\omega,\boldsymbol{u}))$ represents the solution $(\omega^*_{N_1},\boldsymbol{u}^*_\delta)$ of problem

$$a_\delta(\omega^*_{N_1},\boldsymbol{u}^*_\delta;\boldsymbol{v}_\delta)=\langle G_\delta(\omega,\boldsymbol{u})-G(\omega,\boldsymbol{u}),\boldsymbol{v}_\delta\rangle=(\boldsymbol{f},\boldsymbol{v}_\delta)_\delta-\langle\boldsymbol{f},\boldsymbol{v}_\delta\rangle\,.$$

Denote by $\Pi^k_{N_k-1}$ the orthogonal projection operator from $L^2(\Omega_k)$ into $\mathbb{P}_{N_k-1}(\Omega_k)$. We have

$$a_\delta(\omega^*_{N_1}, \boldsymbol{u}^*_\delta; \boldsymbol{v}_\delta) = \sum_{k=1}^{K}((\boldsymbol{f} - \Pi^k_{N_k-1}\boldsymbol{f}, \boldsymbol{v}_{N_k})_{N_k} - \langle \boldsymbol{f} - \Pi^k_{N_k-1}\boldsymbol{f}, \boldsymbol{v}_{N_k}\rangle),$$

and then by using (4.5) we get
(4.20)

$$\|\mathcal{S}\mathcal{D}_\delta(G_\delta(\omega,\boldsymbol{u}) - G(\omega,\boldsymbol{u}))\|_{\mathcal{X}} \leqslant c' \sum_{k=1}^{K}(\|\boldsymbol{f} - \mathcal{I}^k_\delta \boldsymbol{f}\|_{L^2(\Omega)^2} + \|\boldsymbol{f} - \Pi^k_{N_k-1}\boldsymbol{f}\|_{L^2(\Omega_k)^2}).$$

By combining (4.18), (4.19), (4.20) and according to the approximation properties of the operators $\Pi^k_{N_k-1}$ and $\mathcal{I}^k_\delta$ [11], Theorems 7.1 and 14.2, we get result (4.17).

We are now able to estimate the error between the solution $(\omega, \boldsymbol{u}, p)$ of continuous problem (2.10) and the discrete solution $(\omega_{N_1}, \boldsymbol{u}_\delta, p_\delta)$ of discrete problem (3.29). □

Theorem 4.6. *In dimension $d = 2$, let $(\omega, \boldsymbol{u})$ be a solution of problem (4.3) which verifies Hypothesis 4.1. So there exists a neighborhood of $(\omega, \boldsymbol{u})$ such that for N_δ big enough, problem (4.8) has a unique solution $(\omega_{N_1}, \boldsymbol{u}_\delta)$ in this neighborhood. In addition, if we suppose that $\boldsymbol{f}$ is in $H^\sigma(\Omega)^2$, $\sigma > 1$, we have the error estimations*

$$\|\omega - \omega_{N_1}\|_{H(\mathbf{curl},\Omega_F)} + \|\boldsymbol{u} - \boldsymbol{u}_\delta\|_{H(\operatorname{div},\Omega)} \leqslant c_\star |\log\ N_1|^{-1/2} \tag{4.21}$$

and

$$\begin{aligned} \|\omega - \omega_{N_1}\|_{H(\mathbf{curl},\Omega_F)} &+ \|\boldsymbol{u} - \boldsymbol{u}_\delta\|_{H(\operatorname{div},\Omega)} \\ &\leqslant c(N_\delta^{-s}(\|\omega\|_{H^{s+1}(\Omega_F)} + \|\boldsymbol{u}\|_{H^s(\Omega)^2}) + N_\delta^{-\sigma}\|\boldsymbol{f}\|_{H^\sigma(\Omega)^2}), \end{aligned} \tag{4.22}$$

where $c_\star$ is a positive constant.

Proof. Combining (4.14), Lemma 4.3 and the Brezzi-Rappaz-Raviart theorem [12], we get that for N_δ big enough, problem (3.30) admits a unique solution $(\omega_{N_1}, \boldsymbol{u}_\delta)$.

We define the following constants:

$$\begin{aligned} \gamma_\delta &= \|(\mathrm{Id} + \mathcal{S}\mathcal{D}_\delta DG(\omega,\boldsymbol{u}))^{-1}\|_{\mathcal{L}}, \\ \eta_\delta &= \|(\omega,\boldsymbol{u}) + \mathcal{S}\mathcal{D}_\delta G_\delta(\omega,\boldsymbol{u})\|_{\mathcal{X}}, \\ B((\omega,\boldsymbol{u});\alpha) &= \{(\widetilde{\omega},\widetilde{\boldsymbol{u}}) \in \mathcal{X};\ \|(\widetilde{\omega},\widetilde{\boldsymbol{u}}) - (\omega,\boldsymbol{u})\|_{\mathcal{X}} \leqslant \alpha\}, \\ L_\delta(\alpha) &= \sup_{(\widetilde{\omega},\widetilde{\boldsymbol{u}}) \in B((\omega,\boldsymbol{u}),\alpha)} \|\mathcal{S}\mathcal{D}_\delta(DG_\delta(\widetilde{\omega},\widetilde{\boldsymbol{u}}) - DG_\delta(\omega,\boldsymbol{u}))\|_{\mathcal{L}}. \end{aligned}$$

According to Lemma 4.3 and Lemma 4.5, we have $\gamma_\delta \leqslant 2\gamma$, where γ is the constant given by (4.15) and $\lim_{N_\delta\to\infty} \eta_\delta = 0$.

According to (4.16), we have on the ball $B((\omega, \boldsymbol{u}), \alpha)$ with $\alpha = 2\gamma_\delta\eta_\delta$:

$$\gamma_\delta \times L_\delta(2\gamma_\delta\eta_\delta) < \frac{1}{2}. \tag{4.23}$$

From Lemma 4.3 and (4.23) we have according to Brezzi-Rappaz-Raviart theorem, that problem (4.3) admits a unique solution $(\omega_{N_1}, \boldsymbol{u}_\delta)$ in the ball $B((\omega_{N_1}, \boldsymbol{u}_\delta), \alpha)$ with a radius α which satisfies $\gamma_\delta \times L_\delta(\alpha) < 1$. This solution $(\omega_{N_1}, \boldsymbol{u}_\delta)$ verifies

$$(\omega_{N_1}, \boldsymbol{u}_\delta) + \mathcal{SD}_\delta G_\delta(\omega_{N_1}, \boldsymbol{u}_\delta) = 0,$$

and

$$\|\omega - \omega_{N_1}\|_{H(\mathbf{curl}, \Omega_F)} + \|\boldsymbol{u} - \boldsymbol{u}_\delta\|_{H(\mathrm{div}, \Omega)} \leqslant \alpha \leqslant c_\star |\log\ N_1|^{-1/2},$$

from which estimations (4.21) and (4.22) hold. □

Theorem 4.7. *In dimension $d = 2$, we suppose that $\boldsymbol{f}$ belongs to $H^\sigma(\Omega)^2$, $\sigma > 1$ and that the solution $(\omega, \boldsymbol{u}, p)$ of problem (2.10) belongs to $H^{s+1}(\Omega_F) \times H^s(\Omega)^2 \times H^s(\Omega)$, $s > 1$, and satisfies Hypothesis 4.1. So there exists an integer $N_\diamond$ such that for all $N_\delta \geqslant N_\diamond$, problem (3.29) admits a unique solution $(\omega_{N_1}, \boldsymbol{u}_\delta, p_\delta)$ such that*

$$\begin{aligned}\|\omega - \omega_{N_1}\|_{H(\mathbf{curl}, \Omega_F)} &+ \|\boldsymbol{u} - \boldsymbol{u}_\delta\|_{H(\mathrm{div}, \Omega)} + |\log N_1|^{-1/2} \|p - p_\delta\|_{L^2(\Omega)} \\ &\leqslant c(N_\delta^{-s}(\|\omega\|_{H^{s+1}(\Omega_F)} + \|\boldsymbol{u}\|_{H^s(\Omega)^2} + \|p\|_{H^s(\Omega)}) + N_\delta^{-\sigma} \|\boldsymbol{f}\|_{H^\sigma(\Omega)^2})\end{aligned} \tag{4.24}$$

with $N_\delta = \min N_k$, $1 \leqslant k \leqslant K$.

P r o o f. We consider the discrete problem: Find $p_\delta \in \mathbb{M}_\delta$ such that

$$b_\delta(\boldsymbol{v}_\delta, p_\delta) = (\boldsymbol{f}, \boldsymbol{v}_\delta)_\delta - a_\delta(\omega_{N_1}, \boldsymbol{u}_\delta; \boldsymbol{v}_\delta) - K(\omega_{N_1}, \boldsymbol{u}_{N_1}; \boldsymbol{v}_{N_1}) \quad \forall\, \boldsymbol{v}_\delta \in \mathbb{D}_\delta. \tag{4.25}$$

The inf-sup condition (3.27) ensures the existence and the uniqueness of the pressure p_δ.

Secondly, for any $q_\delta \in \mathbb{M}_\delta$ we have

$$\begin{aligned}b_\delta(\boldsymbol{v}_\delta, p_\delta - q_\delta) = b(\boldsymbol{v}_\delta, p - q_\delta) - \langle \boldsymbol{f}, \boldsymbol{v}_\delta \rangle + (\boldsymbol{f}, \boldsymbol{v}_\delta)_\delta + a(\omega - \omega_{N_1}, \boldsymbol{u} - \boldsymbol{u}_\delta; \boldsymbol{v}_\delta) \\ + (a - a_\delta)(\omega_{N_1}, \boldsymbol{u}_\delta; \boldsymbol{v}_\delta) + K(\omega, \boldsymbol{u}; \boldsymbol{v}_{N_1}) - K(\omega_{N_1}, \boldsymbol{u}_{N_1}; \boldsymbol{v}_{N_1}).\end{aligned}$$

According to (3.27), (4.10), and using the triangular inequality, we obtain

$$\begin{aligned}&|\log N_1|^{-1/2} \|p - p_\delta\|_{L^2(\Omega)} \\ &\quad \leqslant c_3 N_\delta^{-s}((\|\omega\|_{H^{s+1}(\Omega_F)} + \|\boldsymbol{u}\|_{H^s(\Omega)^2} + \|p\|_{H^s(\Omega)}) + N_\delta^{-\sigma} \|\boldsymbol{f}\|_{H^\sigma(\Omega)^2}).\end{aligned}$$

This last result and estimation (4.22) give estimation (4.24). □

5. Numerical studies

The unusual variational formulation of our problem requires a particular choice of nodes. The resulting matrices are not familiar and do not exist, to our knowledge, in the library of spectral codes. We linearize the discrete problem by a Newton method and use the code developed in [23] for each Newton iteration. We consider the domain Ω (Figure 3), broken into three subdomains. We solve locally Darcy's equations in Ω_2 and Ω_3, and Navier-Stokes equation in Ω_1.

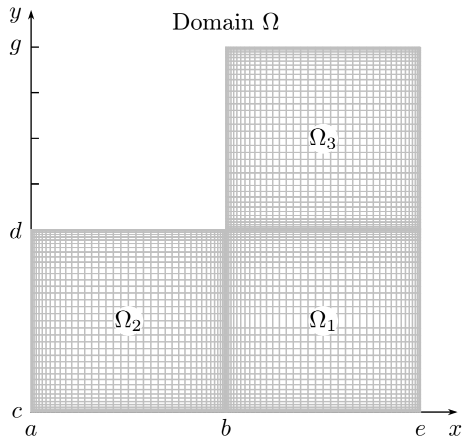

Figure 3. Domain of study.

For each iteration of the Newton method, we solve a problem of the form $\boldsymbol{A}\boldsymbol{X} = \boldsymbol{F}$, where $\boldsymbol{A}$ is represented as follows

$$\boldsymbol{A} = \begin{pmatrix} DA_I^1 & 0 & 0 & DA_\Gamma^1 \\ 0 & NSA_I^2 & 0 & NSA_\Gamma^2 \\ 0 & 0 & DA_I^3 & DA_\Gamma^3 \\ DL_\Gamma^1 & NSL_\Gamma^2 & DL_\Gamma^3 & C_\Gamma \end{pmatrix};$$

DA_I^1, DA_I^3 and NSA_I^2 present the matrices which act on internal nodes, in the problem of Darcy and Navier-Stokes, respectively. These matrices have the following forms: For $j \in \{1, 3\}$

$$DA_I^j = \begin{pmatrix} DA_{1,I}^j & 0 & DB_{1,I}^j \\ 0 & DA_{2,I}^j & DB_{2,I}^j \\ {}^tDB_{1,I}^j & {}^tDB_{2,I}^j & 0 \end{pmatrix},$$

$$NSA_I^2 = \begin{pmatrix} NSA_{1,I}^2 & 0 & 0 & -NSB_{1,I}^2 \\ -NSA_{2,I}^2 & 0 & 0 & -NSB_{2,I}^2 \\ 0 & {}^tNSB_{1,I}^2 & {}^tNSB_{2,I}^2 & 0 \\ C\omega & -{}^tNSA_{1,I}^2 & {}^tNSA_{2,I}^2 & 0 \end{pmatrix}.$$

Above,

- ▷ the matrices that start with D correspond to the problem of Darcy, and matrices that start with NS correspond to the Navier-Stokes problem;
- ▷ the matrices ending with the index I are internal nodes linked matrices, while those that end in an index Γ represent the matrices associated with nodes in interfaces;
- ▷ the matrices ending with the exponent 1, 2 or 3 are related to the domains Ω_1, Ω_2 or Ω_3, respectively;
- ▷ $DA^1_\Gamma, SA^2_\Gamma, DA^3_\Gamma$ correspond to the connections between the unknowns in Ω_i and which belong to the interfaces Γ_i, $i \in \{1, 2\}$;
- ▷ C_Γ represents the relations between the unknowns on the interfaces Γ_i, $i \in \{1, 2\}$;
- ▷ $\boldsymbol{X} = {}^t(X^1_I, X^2_I, X^3_I, X_\Gamma)$, where X^1_I, X^2_I, X^3_I are vectors having the values of the solution $(\boldsymbol{u}, p, \omega)$ on the internal nodes, and X_Γ represents the values of the solution on the interfaces Γ_i

$$X^1_I = {}^t(\boldsymbol{U}^1_I, P^1_I),\ X^2_I = {}^t(W, \boldsymbol{U}^2_I, P^2_I),\ X^3_I = {}^t(\boldsymbol{U}^3_I, P^3_I),\ X_\Gamma = {}^t(X_{\Gamma_1}, X_{\Gamma_2});$$

- ▷ $\boldsymbol{F} = {}^t(F^1, F^2, F^3, G_\Gamma)$ is the vector data on internal nodes with

$$F^j = {}^t(F^j_1, F^j_2, G^j_b) \quad \text{for } j = \{1, 3\}, \quad F^2 = {}^t(F^2_1, F^2_2, G^2_b, 0)$$

and G_Γ is the data vector on Γ_i.

R e m a r k 5.1. We do not have the same number of nodes on Ω_i, since the discrete spaces of the two problems, Stokes and Darcy, are different. Then matrices Q_i are used in the passage through the interfaces Γ_i.

In each iteration of Newton method, the linear problem $\boldsymbol{AX} = \boldsymbol{B}$ is resolved by an iterative method bicgstabl (in MATLAB software). We do not need more than 10 iterations ni of Newton method to obtain optimal convergence.

The number of iterations m in bicgstabl method (without preconditioning) increases in function of the nodal number, whereas if we use, for example, ILU preconditioner, m does not exceed 5 iterations.

5.1. First example. We begin with an example for which the analytical solution is known:

$$\boldsymbol{u}(x, y) = \begin{pmatrix} \pi \sin(\pi x)\cos(\pi y) \\ -\pi \sin(\pi y)\cos(\pi x) \end{pmatrix}, \quad p(x, y) = xy^3. \tag{5.1}$$

The domain is

$$\Omega = \,]-3, 1[\times]-1, 2[\setminus]-3, -1] \times [1, 2[, \quad \Omega_1 = \,]-1, 1[\times]-1, 1[,$$
$$\Omega_2 = \,]-3, -1[\times]-1, 1[, \quad \Omega_3 = \,]-1, 1[\times]1, 2[.$$

We present in Figure 4 the isovalues of the discrete solution corresponding to (5.1).

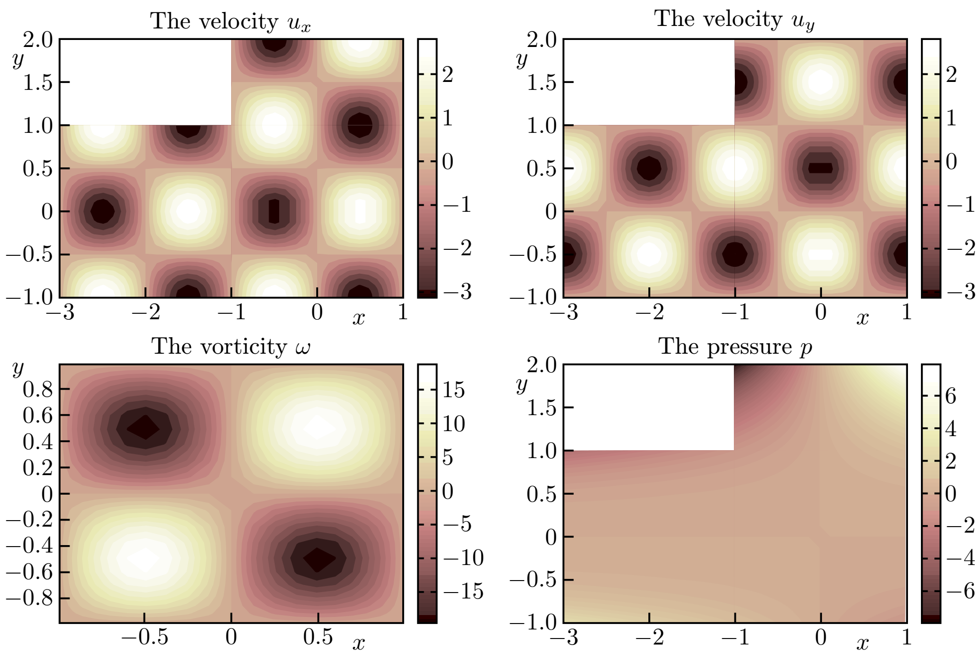

Figure 4. The discrete solution obtained from (5.1).

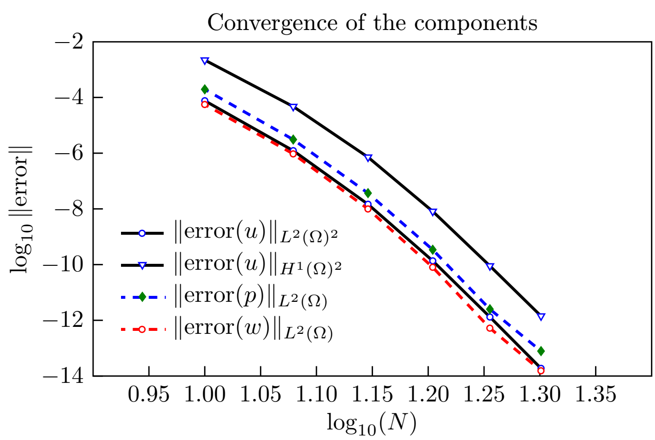

Figure 5. The discrete solution obtained from (5.1).

In Figure 5 we present the quantities

$$\log_{10}\|\boldsymbol{u}-\boldsymbol{u}_N\|_{L^2(\Omega)^2},\quad \log_{10}\|\boldsymbol{u}-\boldsymbol{u}_N\|_{H^1(\Omega)^2},$$
$$\log_{10}\|p-p_N\|_{L^2(\Omega)}\quad\text{and}\quad \log_{10}\|\omega-\omega_N\|_{L^2(\Omega)}$$

as function of $\log_{10}(N)$. We observe that the error between the exact solution and the discrete one decreases when N increases, and it reaches a good convergence for $N = 24$.

5.2. Second example. For the second experiment, we take

$$\Omega =]-2,1[\times]-1,2[\ \setminus\ ]-2,-1]\times[1,2[, \quad \Omega_1 =]-1,1[\times]-1,1[,$$
$$\Omega_2 =]-2,-1[\times]-1,1[, \quad \Omega_3 =]-1,1[\times]1,2[.$$

The exact solution is now given by (5.2):

$$\text{(5.2)} \qquad \boldsymbol{u}(x,y) = \begin{pmatrix} (x^2-1)^3(y^2-1)^2(x+2)^{9/2}(2-y)^{7/2}(-7y^2+8y+3) \\ -(x^2-1)^2(y^2-1)^3(x+2)^{7/2}(2-y)^{9/2}(7x^2+8x-3) \end{pmatrix},$$
$$p(x,y) = \cos(\pi x)\cos(\pi y).$$

In this example, the velocity components are taken with limited regularities in order to better assess the efficiency of our method.

Figure 6 presents the convergence of the relative errors in $\boldsymbol{u}$, ω and p in the $L^2(\Omega)^2, H^1(\Omega)^2$ and $L^2(\Omega)$ norm in logarithmic scales, as function of N, for N varying from 12 to 36. In the second Example the convergence is slower than in the first Example and this result is as expected and it has been proven in the theoretical part.

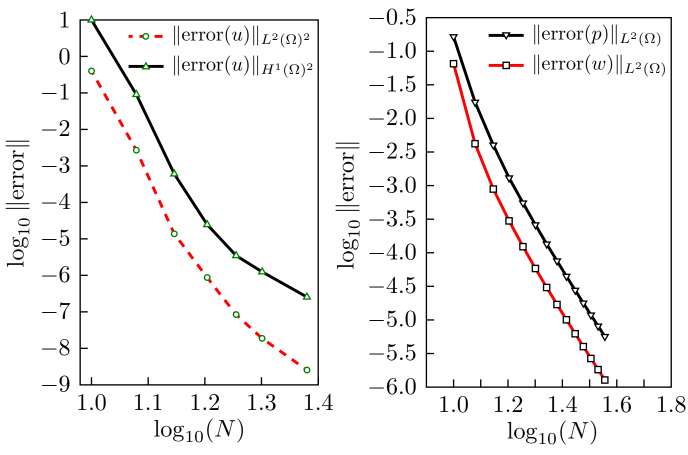

Figure 6. Estimations of error of the discrete solution corresponding to (5.2).

5.3. Third example. In this example we present the isovalues of an unknown solution, where

$$\text{(5.3)} \qquad \boldsymbol{f}(x,y) = \begin{cases} y - y(x^2-1)^3(y^2-1)^2 \\ x + x(x^2-1)^2(y^2-1)^3 \end{cases} \quad \text{in } \Omega \quad \text{and} \quad g \equiv 0 \quad \text{on } \Gamma,$$

Figures 7, 8 present the isovalues of u and ω for $N = 24$.

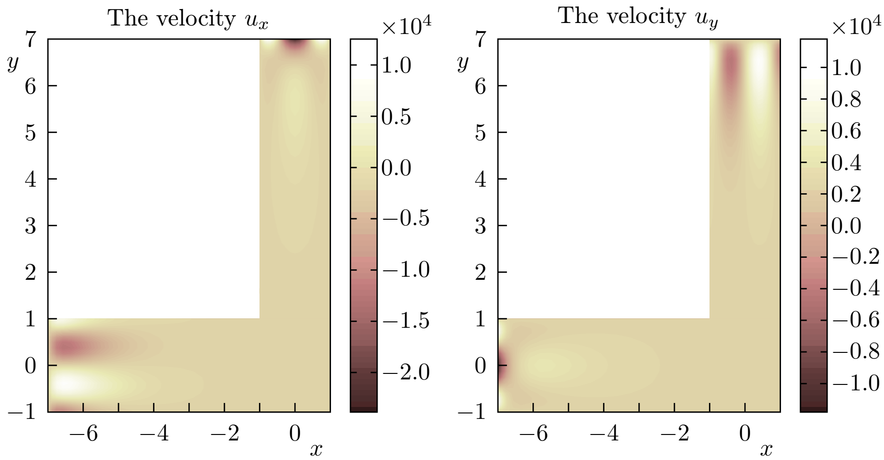

Figure 7. The discrete solution obtained from (5.3).

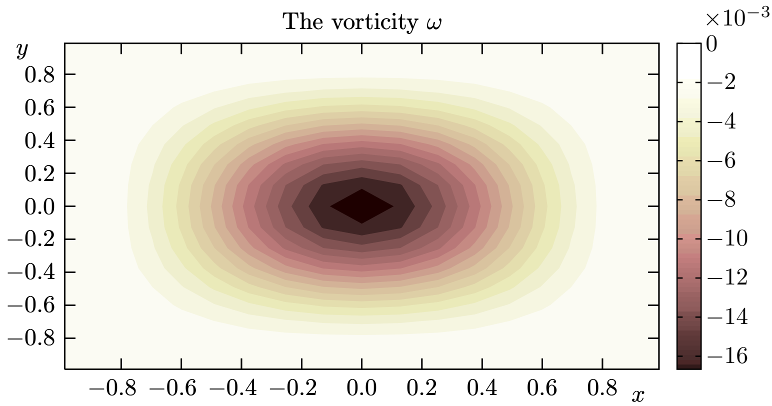

Figure 8. The discrete solution obtained from (5.3).

6. Conclusion

In this paper, we treated a Navier-Stokes-Darcy coupling problem. We described a discretization strategy based on the spectral method combined with a domain decomposition method. We made the numerical analysis of the resulting problem. We showed in particular the result of existence and uniqueness and established optimal error estimates. Some significative numerical results are presented in dimension 2. Much work associated with this coupling remains to be done.

We dealed here in Cartesian coordinates, but as suggested by several physical situations, the problem may be treated in cylindrical coordinates [2], [3].

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