GLOBAL ATTRACTORS FOR A TROPICAL CLIMATE MODEL

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Abstract. This paper is devoted to the global attractors of the tropical climate model. We first establish the global well-posedness of the system. Then by studying the existence of bounded absorbing sets, the global attractor is constructed. The estimates of the Hausdorff dimension and of the fractal dimension of the global attractor are obtained in the end.

Keywords: tropical climate model; global attractor; Hausdorff dimension; fractal dimension

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1. Introduction

In the present paper, we consider the following two-dimensional (2D) tropical climate model in a bounded domain $\Omega \subset \mathbb{R}^2$:

$$\begin{cases} \partial_t u + (u\cdot\nabla)u - \mu\Delta u + \nabla p + \nabla\cdot(v\otimes v) = f^1, \\ \partial_t v + (u\cdot\nabla)v - \nu\Delta v + \nabla\theta + (v\cdot\nabla)u = f^2, \\ \partial_t\theta + (u\cdot\nabla)\theta - \eta\Delta\theta + \nabla\cdot v = f^3, \\ \nabla\cdot u = 0, \end{cases} \tag{1.1}$$

where $u = (u^1(x,t), u^2(x,t))$, $v = (v^1(x,t), v^2(x,t))$ are the barotropic mode and the first baroclinic mode of the velocity, respectively, $\theta = \theta(x,t)$ and $p = p(x,t)$ represent, respectively, the scalar temperature and the scalar pressure, $f = (f^1, f^2, f^3)$ is the external volume force. Here $v\otimes v$ is the standard tensor notation, i.e.,

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$v\otimes v = (v^i v^j)_{1\leqslant i,j\leqslant 2}$, μ, ν, η are nonnegative constants, where μ, ν are the viscosities and η is the thermal diffusivity. In the present paper, we consider $\mu = \nu = \eta = 1$.

When the region Ω is a smooth bounded domain, we supplement (1.1) with the initial conditions

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad \theta(x,0) = \theta_0(x) \quad \text{in } \Omega,$$

and the (non-slip) boundary conditions

$$u = 0, \quad v = 0, \quad \theta = 0 \quad \text{on } \partial\Omega.$$

We shall also treat the space-periodic case, where $\Omega = (0, L_1) \times (0, L_2)$ and the boundary conditions are replaced by

$$\begin{aligned} u((x_1,0),t) &= u((x_1,L_2),t), \quad & u((0,x_2),t) &= u((L_1,x_2),t),\\ v((x_1,0),t) &= v((x_1,L_2),t), \quad & v((0,x_2),t) &= v((L_1,x_2),t),\\ \theta((x_1,0),t) &= \theta((x_1,L_2),t), \quad & \theta((0,x_2),t) &= \theta((L_1,x_2),t) \end{aligned}$$

for all $0 \leqslant x_1 \leqslant L_1$, $0 \leqslant x_2 \leqslant L_2$.

By performing a Galerkin truncation to the hydrostatic Boussinesq equations, Frierson, Majda, and Pauluis [10] derived the tropical climate model without any dissipation terms ($\mu = \nu = \eta = 0$). The first baroclinic mode of (1.1) was used in some studies of large-scale dynamics of precipitation fronts in the tropical atmosphere. The tropical climate model is related to other equations in fluid mechanics. If the temperature θ is a constant, it is similar to the magnetohydrodynamics (MHD) equations. If $v = 0$, it is analogous to the Boussinesq equations. The tropical climate model has attracted a lot of attentions recently. For tropical climate model with fractional dissipation, $\alpha, \mu, \nu, \eta > 0$, Ye [20] studied the global regularity:

$$\|(u,v,\theta)\|^2_{H^s(\mathbb{R}^2)} + \int_0^t [\|u\|^2_{H^{s+\alpha}(\mathbb{R}^2)} + \|v\|^2_{H^{s+1}(\mathbb{R}^2)} + \|\theta\|^2_{H^{s+1}(\mathbb{R}^2)}]\,\mathrm{d}\tau \leqslant C.$$

If $(u_0, v_0) \in H^s(\mathbb{R}^2)$, $s > 2$ and $\theta_0 \in \dot{H}^{-1}(\mathbb{R}^2) \cap H^{s+1-\beta}(\mathbb{R}^2)$, Dong, Wang, Wu, and Zhang [7] established the global existence and regularity of solutions to the system with fractional dissipative terms $(-\Delta)^\alpha u$, $(-\Delta)^\beta v$, $\alpha + \beta = 2$. For $(u_0, v_0, \theta_0) \in H^s(\mathbb{R}^2)$, $s > 2$, Dong, Wu, and Ye [9] examined the global existence and regularity of weak solutions with fractional dissipation. For $(u_0, v_0, \theta_0) \in H^s(\mathbb{R}^2)$, $s > 1$, μ, ν, $\eta \geqslant \beta$, for some $\beta > 0$, Dong, Li, Xu, and Ye [8] showed that there is a unique global smooth solution of (1.1) in the Sobolev class $H^{s'}$ for any $s' > 0$.

It is important to study the long-time behavior of solutions to evolution partial differential equations as $t \to \infty$. If the external force term decays to zero at some rate, then the solutions will decay to zero either. Schonbek [18] first obtained the optimal decay rates of weak solutions to the incompressible Navier-Stokes equations. He and Zhou [15] showed the temporal decay for strong solutions of an incompressible Newtonian flow with intrinsic degree of freedom. The spatial decay for the Navier-Stokes equations in $\mathbb{R}^n$ was obtained by He and Xin [14]. Brandolese [4] studied the space-time decay of left invariant Navier-Stokes flows. For the Navier-Stokes equations in $\mathbb{R}^n$, Bae and Jin [2], [3] obtained the upper and lower bounds for temporal-spatial decays.

If the external force does not decay to zero, then solutions may not decay to zero. But for evolution partial differential equations, the long-time behavior of solutions can also be described in terms of attractors. The maximal attractor is able to describe all the flows corresponding to all initial data [12]. Therefore, it is also called the *universal attractor* or the *global attractor*. Theory of attractors for dynamical systems has been widely studied. The existence of attractors for the 2D Navier-Stokes equations was first proved in the works of Ladyžhenskaya [16] for bounded domains. Caraballo, Łukaszewicz and Real [5] proved the existence of pullback attractors for the non-autonomous 2D Navier-Stokes model in an unbounded domain. Lu, Wu, Zhong [17] and Gong, Song, Zhong [13] proved the existence of the uniform attractor for the non-autonomous 2D Navier-Stokes equations in a bounded domain and in a periodic domain. Sermange and Temam [19] proved the existence of invariant sets for the MHD equations in bounded domains with non-slip boundary conditions and with periodic conditions.

The geometrical properties of an attractor can be very complicated. An attractor can be a fractal, like a Cantor set or the product of a Cantor set and an interval. Global attractors can be complicated objects consisting of stationary points, time periodic orbits, quasiperiodic orbits or even trajectories with chaotic behaviors (see [6]). Essentially, the concept of dimension is one of the few pieces of information which is related to attractors. The global attractor is very *thin* in some directions, and may possess finite dimension, because the semigroup on the attractor is contracting in some directions and expanding in other directions. It is well-known that the dimension of the global attractor for the 2D Navier-Stokes equations (and for the 2D MHD equations) is finite. In the present paper, we shall prove similar properties for the 2D tropical climate model. We understand that the number of degrees of freedom of turbulent phenomenon is the dimension of the attractor, which represents it in here. For the frameworks of general dimension estimations, see [6].

The nonlinear terms of the tropical climate model are subtler than those of the Navier-stokes equations and those of the MHD equations. This gives some difficulties on the uniqueness of weak solutions with L^2-initial data.

This paper is organized as follows. In Section 2, we shall introduce some function spaces, some operators, and the weak formulation of (1.1). In Section 3, we will prove the existence and uniqueness of weak solutions. Moreover, we shall obtain higher regularity. In the fourth section, we shall study bounded absorbing sets and obtain the global attractor. The estimates of the dimension of the global attractor will be given in the last section.

2. Functional settings

The spaces we shall use are combinations of those used for the Navier-Stokes equations and the usual Sobolev spaces. For a Hilbert space X (e.g., $L^2(\Omega)$ or $H^1(\Omega)$), we do not distinguish the inner products on X and on $[X]^2 := X \times X$, which will be denoted by $(\cdot,\cdot)_X$.

In the non-slip case, we set $\widetilde{L}^2(\Omega) = L^2(\Omega)$, $\widetilde{H}^1(\Omega) = H^1_0(\Omega)$ and

$$\mathscr{V} = \{\varphi \in [C_c^\infty(\Omega)]^2;\ \nabla\cdot\varphi = 0\}.$$

In the periodic case, we define

$$\begin{aligned}\mathrm{Per}(\Omega) &= \left\{\theta = \Theta|_\Omega;\ \Theta \in C^\infty(\mathbb{R}^2) \text{ is periodic and } \frac{1}{|\Omega|}\int_\Omega \Theta(x)\,\mathrm{d}x = 0\right\},\\ \mathscr{V} &= \{\varphi \in [\mathrm{Per}(\Omega)]^2;\ \nabla\cdot\varphi = 0\},\end{aligned}$$

and set $\widetilde{L}^2(\Omega), \widetilde{H}^1(\Omega)$ the closure of $\mathrm{Per}(\Omega)$ in $L^2(\Omega)$ and in $H^1(\Omega)$, respectively. In both cases, we equip $\widetilde{H}^1(\Omega)$ with the inner product

$$(v_1,v_2)_{\widetilde{H}^1} = \sum_{i=1}^{2}\Big(\frac{\partial v_1}{\partial x_i},\frac{\partial v_2}{\partial x_i}\Big)_{L^2} \quad \forall\, v_1,v_2 \in \widetilde{H}^1(\Omega),$$

and set H, V the closure of $\mathscr{V}$ in $[\widetilde{L}^2(\Omega)]^2$ and in $[\widetilde{H}^1(\Omega)]^2$, respectively. Now we introduce

$$\mathbb{H} = H \times [\widetilde{L}^2(\Omega)]^2 \times \widetilde{L}^2(\Omega), \quad \mathbb{V} = V \times [\widetilde{H}^1(\Omega)]^2 \times \widetilde{H}^1(\Omega).$$

We equip $\mathbb{H}$ and $\mathbb{V}$ with inner products

$$\begin{aligned}(\varphi_1,\varphi_2)_{\mathbb{H}} &= (u_1,u_2)_H + (v_1,v_2)_{L^2} + (\theta_1,\theta_2)_{L^2} \quad \forall\,\varphi_i = (u_i,v_i,\theta_i) \in \mathbb{H},\\ (\varphi_1,\varphi_2)_{\mathbb{V}} &= (u_1,u_2)_V + (v_1,v_2)_{\widetilde{H}^1} + (\theta_1,\theta_2)_{\widetilde{H}^1} \quad \forall\,\varphi_i = (u_i,v_i,\theta_i) \in \mathbb{V}.\end{aligned}$$

Then $\mathbb{H}, \mathbb{V}$ are Hilbert spaces. If we identify $\mathbb{H}$ with its dual $\mathbb{H}'$, then

$$\mathbb{V} \subset \mathbb{H} \equiv \mathbb{H}' \subset \mathbb{V}',$$

where each space is dense and can be continuously embedded into the following one.

We define two linear bounded operators $A \in \mathcal{L}(V, V')$ and $\mathbb{A} \in \mathcal{L}(\mathbb{V}, \mathbb{V}')$ by setting

$$\begin{aligned} \langle Au_1, u_2 \rangle_{V',V} &= (u_1, u_2)_V \quad \forall\, u_1, u_2 \in V, \\ \langle \mathbb{A}\varphi_1, \varphi_2 \rangle_{\mathbb{V}',\mathbb{V}} &= (\varphi_1, \varphi_2)_{\mathbb{V}} \quad \forall\, \varphi_1, \varphi_2 \in \mathbb{V}. \end{aligned}$$

Obviously,

$$\|Au\|_{V'} \leqslant \|u\|_V, \quad \|\mathbb{A}\varphi\|_{\mathbb{V}'} \leqslant \|\varphi\|_{\mathbb{V}}.$$

From now on, we denote $\langle \cdot, \cdot \rangle_{\mathbb{V}',\mathbb{V}}$ by $\langle \cdot, \cdot \rangle$ for simplicity. We can also consider A, $\mathbb{A}$ as unbounded operators on H, $\mathbb{H}$, respectively, whose domains (see [11]) are

$$\begin{aligned} D(A) &= \{u \in V;\ Au \in H\} = V \cap [H^2(\Omega)]^2, \\ D(\mathbb{A}) &= \{\Phi \in \mathbb{V};\ \mathbb{A}\Phi \in \mathbb{H}\} = \mathbb{V} \cap ([H^2(\Omega)]^2 \times [H^2(\Omega)]^2 \times H^2(\Omega)). \end{aligned}$$

The operator $\mathbb{A}$ is a self-adjoint positive linear operator in $\mathbb{H}$, and $\mathbb{A}^{-1}$ is a self-adjoint positive compact linear operator on $\mathbb{H}$. Therefore, we can consider the eigenvalue problem $\mathbb{A}\gamma = \lambda\gamma \in \mathbb{H}$, $\lambda \in \mathbb{R}$, and all eigenvalues of $\mathbb{A}$ can be sorted into an increasing positive sequence

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \to \infty.$$

Moreover, $\{\lambda_j\}_{j=1}^{\infty} = \{\lambda_i(A)\}_{i=1}^{\infty} \cup \{\lambda_i(-\Delta)\}_{i=1}^{\infty}$ and each $\lambda_i(-\Delta)$ is a triple eigenvalue of $\mathbb{A}$, where $\lambda_i(A)$ is the ith eigenvalue of A and $\lambda_i(-\Delta)$ is the ith eigenvalue of $-\Delta$. Specially in the periodic case, the eigenvectors of $-\Delta$ are $\mathrm{e}^{2\pi\mathrm{i}(k_1x_1/L_1+k_2x_2/L_2)}$ corresponding to the eigenvalues $4\pi^2(k_1^2/L_1^2 + k_2^2/L_2^2)$ for all $(k_1, k_2) \in \mathbb{Z}^2 \setminus \{0\}$. If $u \in D(A)$, then $Au = -\Delta u$. Therefore, the eigenvectors of A are $\mathrm{e}^{2\pi\mathrm{i}(k_1x_1/L_1+k_2x_2/L_2)}(-k_2/L_2, k_1/L_1)$ w.r.t. the eigenvalues $4\pi^2(k_1^2/L_1^2 + k_2^2/L_2^2)$. In other words, $\lambda_i(A) = \lambda_i(-\Delta)$ for all $i = 1, 2, \ldots$

The principle eigenvalue of $\mathbb{A}$ is $\lambda_1 = \min\{\lambda_1(A), \lambda_1(-\Delta)\}$, which gives the Poincaré inequality

$$\lambda_1\|\varphi\|_{\mathbb{H}}^2 \leqslant \|\varphi\|_{\mathbb{V}}^2.$$

In addition, $\|f\|_{\mathbb{V}'} \leqslant \lambda_1^{-1/2}\|f\|_{\mathbb{H}}$. For each $\varphi \in D(\mathbb{A})$, $\mathbb{A}\varphi \in \mathbb{H}$,

$$\|\varphi\|_{\mathbb{V}}^2 = (\varphi, \varphi)_{\mathbb{V}} = \langle \mathbb{A}\varphi, \varphi \rangle = (\mathbb{A}\varphi, \varphi)_{\mathbb{H}} \leqslant \|\mathbb{A}\varphi\|_{\mathbb{H}}\|\varphi\|_{\mathbb{H}}.$$

This together with the Poincaré inequality gives

$$\|\varphi\|_{\mathbb{V}} \leqslant \lambda_1^{-1/2}\|\mathbb{A}\varphi\|_{\mathbb{H}}.$$

Standard regularity theories show that

$$\|\nabla^2\varphi\|_{L^2} \leqslant C\|\mathbb{A}\varphi\|_{\mathbb{H}} \leqslant C\|\nabla^2\varphi\|_{L^2}.$$

Therefore, for $\varphi \in D(\mathbb{A})$,

$$c(\|\varphi\|_{L^2}^2 + \|\nabla\varphi\|_{L^2}^2 + \|\nabla^2\varphi\|_{L^2}^2) \leqslant \|\mathbb{A}\varphi\|_{\mathbb{H}}^2 \leqslant C\|\nabla^2\varphi\|_{L^2}^2,$$

where c and C depend on Ω only. Moreover, $D(\mathbb{A})$ is a closed linear subspace of $[H^2(\Omega)]^2 \times [H^2(\Omega)]^2 \times H^2(\Omega)$ with the equivalent norm $\|\varphi\|_{D(\mathbb{A})} = \|\mathbb{A}\varphi\|_{\mathbb{H}}$.

Now we define a trilinear form b_1 on $[H^1(\Omega)]^2 \times [H^1(\Omega)]^2 \times [H^1(\Omega)]^2$ by

$$b_1(v_1, v_2, v_3) = \sum_{j,k=1}^{2} \int_\Omega v_1^k \frac{\partial v_2^j}{\partial x_k} v_3^j \, \mathrm{d}x, \quad v_i = (v_i^1, v_i^2),\ i = 1, 2, 3,$$

and a trilinear form b_2 on $[H^1(\Omega)]^2 \times H^1(\Omega) \times H^1(\Omega)$ by

$$b_2(v, \theta_2, \theta_3) = \sum_{i=1}^{2} \int_\Omega v^i \frac{\partial \theta_2}{\partial x_i} \theta_3 \, \mathrm{d}x, \quad v = (v^1, v^2).$$

A continuous trilinear form b can be defined on $\mathbb{V} \times \mathbb{V} \times \mathbb{V}$ as

$$\begin{aligned} b(\varphi_1, \varphi_2, \varphi_3) = b_1(u_1, u_2, u_3) - b_1(v_1, u_3, v_2) + b_1(u_1, v_2, v_3) \\ + b_1(v_1, u_2, v_3) + b_2(u_1, \theta_2, \theta_3) \end{aligned}$$

for all $\varphi_i = (u_i, v_i, \theta_i) \in \mathbb{V}$, $i = 1, 2, 3$. We can check that

$$\begin{aligned} b(\varphi_1, \varphi_2, \varphi_3) &= -b(\varphi_1, \varphi_3, \varphi_2) \quad && \forall\, \varphi_1, \varphi_2, \varphi_3 \in \mathbb{V}, \\ b(\varphi, \psi, \psi) &= 0 && \forall\, \varphi, \psi \in \mathbb{V}. \end{aligned}$$

So for all $\varphi_i \in \mathbb{V}$, $i = 1, 2, 3$,

$$|b(\varphi_1, \varphi_2, \varphi_3)| \leqslant C(\|\varphi_1\|_{L^4}\|\varphi_2\|_{\mathbb{V}}\|\varphi_3\|_{L^4} + \|\varphi_1\|_{\mathbb{V}}\|\varphi_2\|_{L^4}\|\varphi_3\|_{L^4}), \tag{2.1}$$

$$\begin{aligned} |b(\varphi_1, \varphi_2, \varphi_3)| \leqslant C(&\|\varphi_1\|_{\mathbb{H}}^{1/2}\|\varphi_1\|_{\mathbb{V}}^{1/2}\|\varphi_2\|_{\mathbb{V}}\|\varphi_3\|_{\mathbb{H}}^{1/2}\|\varphi_3\|_{\mathbb{V}}^{1/2} \\ &+ \|\varphi_2\|_{\mathbb{H}}^{1/2}\|\varphi_2\|_{\mathbb{V}}^{1/2}\|\varphi_1\|_{\mathbb{V}}\|\varphi_3\|_{\mathbb{H}}^{1/2}\|\varphi_3\|_{\mathbb{V}}^{1/2}). \end{aligned} \tag{2.2}$$

Moreover, for all $\varphi_1, \varphi_2 \in D(\mathbb{A})$, $\varphi_3 \in \mathbb{H}$,

$$(2.3) \quad |b(\varphi_1,\varphi_2,\varphi_3)| \leqslant C(\|\varphi_1\|_{\mathbb{H}}^{1/2}\|\varphi_1\|_{\mathbb{V}}^{1/2}\|\varphi_2\|_{\mathbb{V}}^{1/2}\|\mathbb{A}\varphi_2\|_{\mathbb{H}}^{1/2}\|\varphi_3\|_{\mathbb{H}} + \|\varphi_2\|_{\mathbb{H}}^{1/2}\|\varphi_2\|_{\mathbb{V}}^{1/2}\|\varphi_1\|_{\mathbb{V}}^{1/2}\|\mathbb{A}\varphi_1\|_{\mathbb{H}}^{1/2}\|\varphi_3\|_{\mathbb{H}}),$$

$$(2.4) \quad |b(\varphi_1,\varphi_2,\varphi_3)| + |b(\varphi_2,\varphi_1,\varphi_3)| \leqslant C(\|\varphi_1\|_{\mathbb{H}}^{1/2}\|\varphi_1\|_{\mathbb{V}}^{1/2}\|\varphi_2\|_{\mathbb{V}}^{1/2}\|\mathbb{A}\varphi_2\|_{\mathbb{H}}^{1/2}\|\varphi_3\|_{\mathbb{H}} + \|\varphi_2\|_{\mathbb{H}}^{1/2}\|\mathbb{A}\varphi_2\|_{\mathbb{H}}^{1/2}\|\varphi_1\|_{\mathbb{V}}\|\varphi_3\|_{\mathbb{H}}),$$

where C depends on Ω only.

The Poincaré inequality together with (2.2) shows that we can define a continuous bilinear operator $\mathbb{B}\colon \mathbb{V}\times\mathbb{V}\to\mathbb{V}'$ by setting

$$\langle \mathbb{B}(\varphi_1,\varphi_2),\varphi_3\rangle = b(\varphi_1,\varphi_2,\varphi_3) \quad \forall\,\varphi_1,\varphi_2,\varphi_3\in\mathbb{V}.$$

Inequality (2.3) gives

$$(2.5) \quad \|\mathbb{B}(\varphi_1,\varphi_2)\|_{\mathbb{H}} \leqslant C(\|\varphi_1\|_{\mathbb{H}}^{1/2}\|\varphi_1\|_{\mathbb{V}}^{1/2}\|\varphi_2\|_{\mathbb{V}}^{1/2}\|\mathbb{A}\varphi_2\|_{\mathbb{H}}^{1/2} + \|\varphi_2\|_{\mathbb{H}}^{1/2}\|\varphi_2\|_{\mathbb{V}}^{1/2}\|\varphi_1\|_{\mathbb{V}}^{1/2}\|\mathbb{A}\varphi_1\|_{\mathbb{H}}^{1/2}).$$

We can also define a continuous linear operator $\mathbb{C}\colon \mathbb{V}\to\mathbb{H}\subset\mathbb{V}'$ by setting

$$\mathbb{C}\varphi = (0,\nabla\theta,\nabla\cdot v) \quad \forall\,\varphi=(u,v,\theta)\in\mathbb{V}.$$

Especially, $\langle\mathbb{C}\varphi,\psi\rangle = (\mathbb{C}\varphi,\psi)_{\mathbb{H}}$ and $\langle\mathbb{C}\varphi,\varphi\rangle = (\mathbb{C}\varphi,\varphi)_{\mathbb{H}} = 0$ for all $\varphi,\psi\in\mathbb{V}$. Direct calculation gives

$$(2.6) \quad (\mathbb{C}\varphi,\psi)_{\mathbb{H}} \leqslant \|\varphi\|_{\mathbb{V}}\|\psi\|_{\mathbb{H}},$$

$$(2.7) \quad \langle\mathbb{C}\varphi,\psi\rangle \leqslant \lambda_1^{-1/2}\|\varphi\|_{\mathbb{V}}\|\psi\|_{\mathbb{V}}.$$

Definition 2.1. Let $f\in\mathbb{V}'$, $\Phi_0\in\mathbb{H}$. Then $\Phi\in L^\infty(0,T;\mathbb{H})\cap L^2(0,T;\mathbb{V})$ is called a weak solution to (1.1) if for all $\psi\in\mathbb{V}$,

$$(2.8) \quad (\partial_t\Phi,\psi)_{\mathbb{H}} + (\Phi,\psi)_{\mathbb{V}} + b(\Phi,\Phi,\psi) + (\mathbb{C}\Phi,\psi)_{\mathbb{H}} = \langle f,\psi\rangle, \quad 0<t<T,$$

$$(2.9) \quad \Phi(t)\to\Phi_0 \quad \text{in } \mathbb{H} \text{ as } t\to 0^+.$$

When $f\in\mathbb{H}$, $\Phi_0\in\mathbb{V}$, if a weak solution Φ satisfies $\Phi\in L^\infty(0,T;\mathbb{V})\cap L^2(0,T;D(\mathbb{A}))$, we call Φ a strong solution.

Remark 2.2. (i) Equation (2.8) is equivalent to the following operator equation:

$$\partial_t\Phi + \mathbb{A}\Phi + \mathbb{B}(\Phi,\Phi) + \mathbb{C}\Phi = f \quad \text{in } \mathbb{V}'.$$

(ii) If Φ is a strong solution, then we deduce from (2.8) that $\partial_t\Phi\in L^2(0,T;\mathbb{H})$ and $\Phi\in C([0,T];\mathbb{V})$. More regularity will be obtained in the next section.

3. Well-posedness and regularity

In this section, we shall prove the existence and uniqueness results for problem (2.8). Higher regularity will be obtained. At last, we will define a family of operators $\{L_{(\Phi_0,f)}(t)\}_{t\geqslant 0}$.

Theorem 3.1 (Uniqueness). *Let $f\in\mathbb{V}'$, $\Phi_0\in\mathbb{H}$, and Φ,Ψ be two weak solutions to problem* (2.8) *with the same initial data Φ_0. Then $\Phi\equiv\Psi\in C([0,T];\mathbb{H})\cap L^2(0,T;\mathbb{V})$, and*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Phi\|_{\mathbb{H}}^2+\|\Phi\|_{\mathbb{V}}^2=\langle f,\Phi\rangle, \tag{3.1}$$

$$\|\Phi(t)\|_{\mathbb{H}}^2+2\int_0^t\|\Phi(s)\|_{\mathbb{V}}^2\,\mathrm{d}s=\|\Phi_0\|_{\mathbb{H}}^2+2\int_0^t\langle f,\Phi(s)\rangle\,\mathrm{d}s. \tag{3.2}$$

Proof. Estimates (2.1) and (2.2) give

$$\|\mathbb{B}(\Phi,\Phi)\|_{\mathbb{V}'}+\|\mathbb{B}(\Phi,\Phi)\|_{L^{4/3}}\leqslant C\|\Phi\|_{\mathbb{H}}^{1/2}\|\Phi\|_{\mathbb{V}}^{3/2}.$$

Because $\Phi\in L^\infty(0,T;\mathbb{H})\cap L^2(0,T;\mathbb{V})$, we deduce from (2.8) that

$$\|\partial_t\Phi\|_{L^{4/3}(0,T;\mathbb{V}')}\leqslant C, \tag{3.3}$$

which is not sufficient to obtain the uniqueness for weak solutions directly.

Notice that

$$\|\Phi\|_{L^4(0,T;L^4)}\leqslant\|\Phi\|_{L^\infty(0,T;\mathbb{H})}^{1/2}\|\Phi\|_{L^2(0,T;\mathbb{V})}^{1/2}<\infty. \tag{3.4}$$

Thanks to (2.7) and (2.8), if we set

$$\Phi_1'=-\mathbb{A}\Phi-\mathbb{C}\Phi+f,\quad \Phi_2'=-\mathbb{B}(\Phi,\Phi),$$

then

$$\Phi_1'\in L^2(0,T;\mathbb{V}'),\quad \Phi_2'\in L^{4/3}(0,T;L^{4/3}),$$

and

$$\partial_t\Phi=\Phi_1'+\Phi_2'\in L^2(0,T;\mathbb{V}')+L^{4/3}(0,T;L^{4/3}).$$

By standard extension and mollification, one can derive

$$\frac{\mathrm{d}}{\mathrm{d}t}(\Phi,\Psi)_{\mathbb{H}}=\langle\Phi_1',\Psi\rangle_{\mathbb{V}',\mathbb{V}}+\langle\Phi_2',\Psi\rangle_{L^{4/3},L^4}+\langle\Psi_1',\Phi\rangle_{\mathbb{V}',\mathbb{V}}+\langle\Psi_2',\Phi\rangle_{L^{4/3},L^4}. \tag{3.5}$$

Then (3.1) follows from (3.5).

Noticing that $(\mathbb{C}\Psi, \Phi)_{\mathbb{H}} = -(\mathbb{C}\Phi, \Psi)_{\mathbb{H}}$, we integrate (3.5) on $[\varepsilon, t] \subset (0, T)$ and deduce

$$(\Psi(t), \Phi(t))_{\mathbb{H}} - (\Psi(\varepsilon), \Phi(\varepsilon))_{\mathbb{H}} + 2\int_{\varepsilon}^{t} (\Psi, \Phi)_{\mathbb{V}}\, \mathrm{d}s$$
$$= -\int_{\varepsilon}^{t} [b(\Psi, \Psi, \Phi) + b(\Phi, \Phi, \Psi)]\, \mathrm{d}s + \int_{\varepsilon}^{t} \langle f, \Phi + \Psi\rangle\, \mathrm{d}s.$$

Letting $\varepsilon \to 0^+$ and using (2.9), we obtain

$$(\Psi(t), \Phi(t))_{\mathbb{H}} - \|\Phi_0\|_{\mathbb{H}}^2 + 2\int_{0}^{t} (\Psi, \Phi)_{\mathbb{V}}\, \mathrm{d}s \tag{3.6}$$
$$= -\int_{0}^{t} [b(\Psi, \Psi, \Phi) + b(\Phi, \Phi, \Psi)]\, \mathrm{d}s + \int_{0}^{t} \langle f, \Phi + \Psi\rangle\, \mathrm{d}s.$$

Then (3.2) follows from (3.6).

Letting $Z = \Psi - \Phi$, we deduce from (3.2) and (3.6) that

$$\begin{aligned}\|Z(t)\|_{\mathbb{H}}^2 &= \|\Psi(t)\|_{\mathbb{H}}^2 + \|\Phi(t)\|_{\mathbb{H}}^2 - 2(\Psi(t), \Phi(t))_{\mathbb{H}} \\ &= -2\int_0^t \|Z(s)\|_{\mathbb{V}}^2\, \mathrm{d}s + 2\int_0^t [b(\Psi, \Psi, \Phi) + b(\Phi, \Phi, \Psi)]\, \mathrm{d}s \\ &= -2\int_0^t \|Z(s)\|_{\mathbb{V}}^2\, \mathrm{d}s + 2\int_0^t b(Z, Z, \Phi)\, \mathrm{d}s.\end{aligned}$$

Here we use

$$b(\Psi, \Psi, \Phi) = b(Z, \Psi, \Phi) + b(\Phi, \Psi, \Phi) = b(Z, Z, \Phi) + b(\Phi, \Psi, \Phi),$$

and

$$b(\Phi, \Psi, \Phi) + b(\Phi, \Phi, \Psi) = b(\Phi, \Psi, \Phi) - b(\Phi, \Psi, \Phi) = 0.$$

Recalling (2.1) and (3.4),

$$\begin{aligned}\int_0^t b(Z, Z, \Phi)\, \mathrm{d}s &\leqslant C\int_0^t \|Z\|_{\mathbb{H}}^{1/2}\|Z\|_{\mathbb{V}}^{3/2}\|\Phi\|_{L^4}\, \mathrm{d}s \\ &\leqslant \int_0^t \|Z\|_{\mathbb{V}}^2\, \mathrm{d}s + C\int_0^t \|Z\|_{\mathbb{H}}^2\|\Phi\|_{L^4}^4\, \mathrm{d}s.\end{aligned}$$

Thus,

$$\|Z(t)\|_{\mathbb{H}}^2 \leqslant C\int_0^t \|Z(s)\|_{\mathbb{H}}^2\|\Phi(s)\|_{L^4}^4\, \mathrm{d}s. \tag{3.7}$$

Application of the Gronwall inequality gives

$$\int_0^t \|Z(s)\|_{\mathbb{H}}^2\|\Phi(s)\|_{L^4}^4\, \mathrm{d}s \equiv 0,$$

and therefore, $\|Z(t)\|_{\mathbb{H}}^2 \equiv 0$, i.e., $\Psi = \Phi$. □

Theorem 3.2 (Existence). *For $f \in \mathbb{V}'$, $\Phi_0 \in \mathbb{H}$ there exists a unique weak solution Φ to problem* (2.8) *with the initial data Φ_0. Moreover, if $f \in \mathbb{H}$, $\Phi_0 \in \mathbb{V}$, then Φ is a strong solution.*

P r o o f. We obtain from (3.2) that

$$\|\Phi(t)\|_{\mathbb{H}}^2 + \int_0^t \|\Phi(s)\|_{\mathbb{V}}^2 \,\mathrm{d}s \leqslant \|\Phi_0\|_{\mathbb{H}}^2 + t\|f\|_{\mathbb{V}'}^2. \tag{3.8}$$

This gives that

$$\sup_{t\in[0,T]} \|\Phi(t)\|_{\mathbb{H}}^2 + \int_0^T \|\Phi(t)\|_{\mathbb{V}}^2 \,\mathrm{d}t \leqslant 2\|\Phi_0\|_{\mathbb{H}}^2 + 2T\|f\|_{\mathbb{V}'}^2. \tag{3.9}$$

Estimate (3.9) is sufficient for us to construct a weak solution Φ to (2.8) by approximation. Theorem 3.1 says that Φ is unique.

Next, taking $\psi = \mathbb{A}\Phi$ in (2.8), we obtain from (2.5) and (2.6) that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Phi\|_{\mathbb{V}}^2 + \|\mathbb{A}\Phi\|_{\mathbb{H}}^2 &= -b(\Phi,\Phi,\mathbb{A}\Phi) - (\mathbb{C}\Phi,\mathbb{A}\Phi) + \langle f,\mathbb{A}\Phi\rangle \\ &\leqslant \frac{1}{2}\|\mathbb{A}\Phi\|_{\mathbb{H}}^2 + C(1+\|\Phi\|_{\mathbb{H}}^2\|\Phi\|_{\mathbb{V}}^2)\|\Phi\|_{\mathbb{V}}^2 + C\|f\|_{\mathbb{H}}^2.\end{aligned}$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Phi\|_{\mathbb{V}}^2 + \|\mathbb{A}\Phi\|_{\mathbb{H}}^2 \leqslant C(1+\|\Phi\|_{\mathbb{H}}^2\|\Phi\|_{\mathbb{V}}^2)\|\Phi\|_{\mathbb{V}}^2 + C\|f\|_{\mathbb{H}}^2. \tag{3.10}$$

From (3.9) we have

$$\int_0^T \|\Phi(t)\|_{\mathbb{H}}^2\|\Phi(t)\|_{\mathbb{V}}^2 \,\mathrm{d}t \leqslant C.$$

By the Gronwall inequality,

$$\sup_{t\in[0,T]} \|\Phi(t)\|_{\mathbb{V}}^2 + \int_0^T \|\mathbb{A}\Phi(t)\|_{\mathbb{H}}^2 \,\mathrm{d}t \leqslant C, \tag{3.11}$$

where $C = C(T, \|f\|_{\mathbb{H}}, \|\Phi_0\|_{\mathbb{V}})$. Substituting (3.11) into (2.5), we derive from (2.8) that

$$\int_0^T \|\partial_t\Phi(t)\|_{\mathbb{H}}^2 \,\mathrm{d}t \leqslant C. \tag{3.12}$$

□

By Theorem 3.2, for $f \in \mathbb{V}'$ we can define an operator from $\mathbb{H}$ into $\mathbb{V}$, denoted by $S(t)\colon \Phi_0 \mapsto \Phi(t)$, where Φ is the unique weak solution to (2.8) with the initial data $\Phi_0 \in \mathbb{H}$ and the external force f.

R e m a r k 3.3. Let $f \in \mathbb{V}'$, $S(t)\colon \Phi_0 \mapsto \Phi(t)$. Then

$$\begin{cases} S(t) = S(t-s)S(s) \quad \forall\, t \geqslant s \geqslant 0, \\ S(0) = \mathrm{Id}_{\mathbb{H}}. \end{cases}$$

Before studying higher regularity, we first recall the uniform Gronwall inequality.

Lemma 3.4. *Let g, h, y be three positive locally integrable functions on $(0,\infty)$ such that y' is locally integrable on $(0,\infty)$, and which satisfy*

$$y' \leqslant gy + h \quad \forall\, t \geqslant 0,$$

and

$$\int_t^{t+\tau} g(s)\,\mathrm{d}s \leqslant a_1, \quad \int_t^{t+\tau} h(s)\,\mathrm{d}s \leqslant a_2, \quad \int_t^{t+\tau} y(s)\,\mathrm{d}s \leqslant a_3 \quad \forall\, t \geqslant 0,$$

where τ, a_1, a_2, a_3 are positive constants. Then

$$y(t+\tau) \leqslant \Big(\frac{a_3}{\tau} + a_2\Big)\mathrm{e}^{a_1} \quad \forall\, t \geqslant 0.$$

Theorem 3.5. *For $f \in \mathbb{H}$, $\Phi_0 \in \mathbb{H}$, let Φ be the weak solution obtained in Theorem 3.2. Then $\Phi \in C((0,T]; D(\mathbb{A}))$.*

P r o o f. For any $\Phi_0 \in \mathbb{H}$, we can deduce from (3.9) and (3.10) by Lemma 3.4 that

$$\sup_{t\in[\tau,T]} \|\Phi(t)\|_{\mathbb{V}}^2 + \int_\tau^T \|\mathbb{A}\Phi(t)\|_{\mathbb{H}}^2\,\mathrm{d}t + \int_\tau^T \|\partial_t\Phi(t)\|_{\mathbb{H}}^2\,\mathrm{d}t \leqslant C_\tau \tag{3.13}$$

for any $0 < \tau < T$, where $C_\tau = C(\tau, T, \|f\|_{\mathbb{H}}, \|\Phi_0\|_{\mathbb{H}})$.

Differentiate (2.8) w.r.t. time t and then obtain

$$\partial_t\Phi_t + \mathbb{A}\Phi_t + \mathbb{B}(\Phi_t,\Phi) + \mathbb{B}(\Phi,\Phi_t) + \mathbb{C}\Phi_t = 0, \tag{3.14}$$

where $\Phi_t = \partial_t\Phi$. Multiplying (3.14) by Φ_t, we deduce that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Phi_t\|_{\mathbb{H}}^2 + \|\Phi_t\|_{\mathbb{V}}^2 = -b(\Phi_t,\Phi,\Phi_t) = b(\Phi_t,\Phi_t,\Phi) \leqslant C\|\Phi\|_{\mathbb{H}}^{1/2}\|\Phi\|_{\mathbb{V}}^{1/2}\|\Phi_t\|_{\mathbb{H}}^{1/2}\|\Phi_t\|_{\mathbb{V}}^{3/2}.$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Phi_t\|_{\mathbb{H}}^2 + \|\Phi_t\|_{\mathbb{V}}^2 \leqslant C\|\Phi\|_{\mathbb{H}}^2\|\Phi\|_{\mathbb{V}}^2\|\Phi_t\|_{\mathbb{H}}^2. \tag{3.15}$$

Using Lemma 3.4 again,

$$\sup_{t\in[\tau,T]}\|\Phi_t(t)\|_{\mathbb{H}}^2+\int_\tau^T\|\Phi_t(t)\|_{\mathbb{V}}^2\,\mathrm{d}t\leqslant C_\tau. \tag{3.16}$$

Now, taking the $\mathbb{H}$-inner product of (3.14) with $\mathbb{A}\Phi_t$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Phi_t\|_{\mathbb{V}}^2+\|\mathbb{A}\Phi_t\|_{\mathbb{H}}^2=-b(\Phi_t,\Phi,\mathbb{A}\Phi_t)-b(\Phi,\Phi_t,\mathbb{A}\Phi_t)-(\mathbb{C}\Phi_t,\mathbb{A}\Phi_t).$$

Using Hölder's inequality, Young's inequality, (2.3), and (2.6), we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Phi_t\|_{\mathbb{V}}^2+\|\mathbb{A}\Phi_t\|_{\mathbb{H}}^2\leqslant C[1+(\|\Phi\|_{\mathbb{H}}^2+\|\Phi_t\|_{\mathbb{H}}^2)\|\Phi\|_{\mathbb{V}}^2]\|\Phi_t\|_{\mathbb{V}}^2+C\|\mathbb{A}\Phi\|_{\mathbb{H}}^2.$$

Lemma 3.4 together with (3.9), (3.13), (3.16) yields that

$$\sup_{t\in[\tau,T]}\|\Phi_t(t)\|_{\mathbb{V}}^2+\int_\tau^T\|\mathbb{A}\Phi_t(t)\|_{\mathbb{H}}^2\,\mathrm{d}t\leqslant C_\tau. \tag{3.17}$$

Combining (3.13) and (3.17), that is, $\Phi,\partial_t\Phi\in L^2(\tau,T;D(\mathbb{A}))$, we find that $\Phi\in C([\tau,T];D(\mathbb{A}))$. □

Remark 3.6. By induction, we can prove $\Phi\in C^\infty((0,T];D(\mathbb{A}))$. Furthermore, if $f\in C^\infty(\overline{\Omega})$ additionally, then $\Phi\in C^\infty(\overline{\Omega}\times(0,T])$.

Theorem 3.7. *If $f\in\mathbb{H}$, then $S(t)\colon\ \mathbb{H}\to\mathbb{V}$ is locally Lipschitz for $t>0$.*

Proof. For any $t>0$ we take $T>t$. Let Ψ,Φ be two weak solutions to (2.8) with $\Psi_0,\Phi_0\in\mathbb{H}$ and $Z=\Psi-\Phi$. Then $Z\in C([0,T];\mathbb{H})\cap L^2(0,T;\mathbb{V})$ satisfies

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}t}Z+\mathbb{A}Z+\mathbb{B}(\Psi,Z)+\mathbb{B}(Z,\Phi)+\mathbb{C}Z=0, & 0<t<T,\\ Z(0)=\Psi_0-\Phi_0.\end{cases} \tag{3.18}$$

By similar procedures to (3.7), we derive

$$\|Z(t)\|_{\mathbb{H}}^2\leqslant\|Z(0)\|_{\mathbb{H}}^2+C\int_0^t\|Z(s)\|_{\mathbb{H}}^2\|\Phi(s)\|_{L^4}^4\,\mathrm{d}s.$$

Using the Gronwall inequality, we deduce that

$$\sup_{t\in[0,T]}\|Z(t)\|_{\mathbb{H}}^2+\int_0^T\|Z(t)\|_{\mathbb{V}}^2\,\mathrm{d}t\leqslant C\|Z(0)\|_{\mathbb{H}}^2=C\|\Psi_0-\Phi_0\|_{\mathbb{H}}^2, \tag{3.19}$$

where $C=C(T,\|f\|_{\mathbb{V}'},\|\Phi_0\|_{\mathbb{H}})$.

Multiplying (3.10) by t, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(t\|\Phi\|_{\mathbb{V}}^2) + t\|\mathbb{A}\Phi\|_{\mathbb{H}}^2 \leqslant C(1 + \|\Phi\|_{\mathbb{H}}^2\|\Phi\|_{\mathbb{V}}^2)(t\|\Phi\|_{\mathbb{V}}^2) + \|\Phi\|_{\mathbb{V}}^2 + C\|f\|_{\mathbb{H}}^2.$$

Using the Gronwall inequality again, we deduce that

$$\sup_{t\in[0,T]}(t\|\Phi(t)\|_{\mathbb{V}}^2) \leqslant C, \tag{3.20}$$

where $C = C(T, \|f\|_{\mathbb{H}}, \|\Phi_0\|_{\mathbb{H}})$.

Multiplying $(3.18)_1$ by $\mathbb{A}Z$ and using (2.4), we obtain that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|Z\|_{\mathbb{V}}^2 + \|\mathbb{A}Z\|_{\mathbb{H}}^2 =& -b(\Psi, Z, \mathbb{A}Z) - b(Z, \Phi, \mathbb{A}Z) - (\mathbb{C}Z, \mathbb{A}Z)_{\mathbb{H}}\\ \leqslant& \frac{1}{2}\|\mathbb{A}Z\|_{\mathbb{H}}^2 + C(\|\Psi\|_{\mathbb{H}}^2\|\Psi\|_{\mathbb{V}}^2 + \|\Phi\|_{\mathbb{H}}^2\|\Phi\|_{\mathbb{V}}^2)\|Z\|_{\mathbb{V}}^2\\ &+ C(\|\Psi\|_{\mathbb{V}}^4 + \|\Phi\|_{\mathbb{V}}^4)\|Z\|_{\mathbb{H}}^2 + C\|Z\|_{\mathbb{V}}^2.\end{aligned}$$

Thus,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(t\|Z\|_{\mathbb{V}}^2) + t\|\mathbb{A}Z\|_{\mathbb{H}}^2 \leqslant& C(\|\Psi\|_{\mathbb{H}}^2\|\Psi\|_{\mathbb{V}}^2 + \|\Phi\|_{\mathbb{H}}^2\|\Phi\|_{\mathbb{V}}^2)(t\|Z\|_{\mathbb{V}}^2)\\ &+ C(t\|\Psi\|_{\mathbb{V}}^4 + t\|\Phi\|_{\mathbb{V}}^4)\|Z\|_{\mathbb{H}}^2 + C\|Z\|_{\mathbb{V}}^2.\end{aligned} \tag{3.21}$$

Estimates (3.9), (3.19), (3.20) imply that

$$\int_0^T (t\|\Psi(t)\|_{\mathbb{V}}^4 + t\|\Phi(t)\|_{\mathbb{V}}^4)\|Z(t)\|_{\mathbb{H}}^2\,\mathrm{d}t \leqslant C\|\Psi_0 - \Phi_0\|_{\mathbb{H}}^2.$$

Application of the Gronwall inequality to (3.21) gives

$$\sup_{t\in[0,T]}(t\|Z(t)\|_{\mathbb{V}}^2) \leqslant C\|\Psi_0 - \Phi_0\|_{\mathbb{H}}^2.$$

Therefore, for any fixed $t \in (0, T]$,

$$\|\Psi(t) - \Phi(t)\|_{\mathbb{V}}^2 = \|Z(t)\|_{\mathbb{V}}^2 \leqslant \frac{C}{t}\|\Psi_0 - \Phi_0\|_{\mathbb{H}}^2, \tag{3.22}$$

where $C = C(T, \|f\|_{\mathbb{H}}, \|\Psi_0\|_{\mathbb{H}}, \|\Phi_0\|_{\mathbb{H}})$. □

Now we study the linearized problem to (2.8). Let $f \in \mathbb{V}'$, $\Phi_0 \in \mathbb{H}$, $\Phi(t) = S(t)\Phi_0$, define $F'_{(\Phi_0,f)}(t)\colon D(\mathbb{A}) \to \mathbb{H}$,

$$F'_{(\Phi_0,f)}(t)\colon w \mapsto -\mathbb{A}w - \mathbb{B}(w,\Phi(t)) - \mathbb{B}(\Phi(t),w) - \mathbb{C}w.$$

Theorem 3.8. *If $f \in \mathbb{V}'$, $\Phi_0 \in \mathbb{H}$, then there exists a family of linear operators $\{L_{(\Phi_0,f)}(t);\ t \geqslant 0\}$ such that:*

(i) *For any $W_0 \in \mathbb{H}$, let $\Phi(t) = S(t)\Phi_0$ and $W(t) = L_{(\Phi_0,f)}(t)W_0$. Then for all $T > 0$, $W \in C([0,T];\mathbb{H}) \cap L^2(0,T;\mathbb{V})$ is the unique weak solution to*

$$\begin{cases} \partial_t W + \mathbb{A}W + \mathbb{B}(W,\Phi) + \mathbb{B}(\Phi,W) + \mathbb{C}W = 0, \quad t > 0, \\ W(0) = W_0. \end{cases} \tag{3.23}$$

(ii)

$$\begin{cases} L_{(\Phi_0,f)}(t) = L_{(\Phi(s),f)}(t-s)L_{(\Phi_0,f)}(s) \quad \forall\, t \geqslant s \geqslant 0, \\ L_{(\Phi_0,f)}(0) = \mathrm{Id}_{\mathbb{H}}. \end{cases}$$

(iii) *If $f \in \mathbb{H}$, then $L_{(\Phi_0,f)}(t) \in \mathcal{L}(\mathbb{H},\mathbb{V})$ for $t > 0$.*

Proof. The proof is similar to those of Theorems 3.2, 3.5, and 3.7. Multiplying $(3.23)_1$ by W yields

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|W\|_{\mathbb{H}}^2 + \|W\|_{\mathbb{V}}^2 &= -b(W,\Phi,W) = b(W,W,\Phi) \\ &\leqslant \frac{1}{2}\|W\|_{\mathbb{V}}^2 + C\|\Phi\|_{\mathbb{H}}^2\|\Phi\|_{\mathbb{V}}^2\|W\|_{\mathbb{H}}^2. \end{aligned}$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|W\|_{\mathbb{H}}^2 + \|W\|_{\mathbb{V}}^2 \leqslant C\|\Phi\|_{\mathbb{H}}^2\|\Phi\|_{\mathbb{V}}^2\|W\|_{\mathbb{H}}^2.$$

Using the Gronwall inequality, we deduce that

$$\sup_{t\in[0,T]} \|W(t)\|_{\mathbb{H}}^2 + \int_0^T \|W(t)\|_{\mathbb{V}}^2 \,\mathrm{d}t \leqslant C\|W_0\|_{\mathbb{H}}^2, \tag{3.24}$$

where $C = C(T, \|f\|_{\mathbb{V}'}, \|\Phi_0\|_{\mathbb{H}})$. Let $L_{(\Phi_0,f)}(t)\colon W_0 \mapsto W(t)$. Then (i) follows from (3.24) and (ii) is derived from (i).

Taking the $\mathbb{H}$-inner product of $(3.23)_1$ w.r.t. $\mathbb{A}W$, we obtain

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|W\|_{\mathbb{V}}^2 + \|\mathbb{A}W\|_{\mathbb{H}}^2 &= -b(W,\Phi,\mathbb{A}W) - b(\Phi,W,\mathbb{A}W) - (\mathbb{C}W,\mathbb{A}W) \\ &\leqslant \frac{1}{2}\|\mathbb{A}W\|_{\mathbb{H}}^2 + C(1 + \|\Phi\|_{\mathbb{H}}^2\|\Phi\|_{\mathbb{V}}^2)\|W\|_{\mathbb{V}}^2 + C\|\Phi\|_{\mathbb{V}}^4\|W\|_{\mathbb{H}}^2. \end{aligned}$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\|W\|_{\mathbb{V}}^2 + \|\mathbb{A}W\|_{\mathbb{H}}^2 \leqslant C(1+\|\Phi\|_{\mathbb{H}}^2\|\Phi\|_{\mathbb{V}}^2)\|W\|_{\mathbb{V}}^2 + C\|\Phi\|_{\mathbb{V}}^4\|W\|_{\mathbb{H}}^2.$$

Multiplying it by t, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(t\|W\|_{\mathbb{V}}^2)+t\|\mathbb{A}W\|_{\mathbb{H}}^2 \leqslant C(1+\|\Phi\|_{\mathbb{H}}^2\|\Phi\|_{\mathbb{V}}^2)(t\|W\|_{\mathbb{V}}^2)+Ct\|\Phi\|_{\mathbb{V}}^4\|W\|_{\mathbb{H}}^2+\|W\|_{\mathbb{V}}^2. \tag{3.25}$$

Estimates (3.9), (3.20), (3.24) give

$$\int_0^T t\|\Phi(t)\|_{\mathbb{V}}^4\|W(t)\|_{\mathbb{H}}^2\,\mathrm{d}t \leqslant C\|W_0\|_{\mathbb{H}}^2.$$

Putting it into (3.25), we obtain from the Gronwall inequality that

$$\sup_{t\in[0,T]}(t\|W(t)\|_{\mathbb{V}}^2) \leqslant C\|W_0\|_{\mathbb{H}}^2.$$

Therefore, for any fixed $t \in (0,T]$,

$$\|W(t)\|_{\mathbb{V}}^2 \leqslant \frac{C}{t}\|W_0\|_{\mathbb{H}}^2,$$

where $C = C(T, \|f\|_{\mathbb{H}}, \|\Phi_0\|_{\mathbb{H}})$. □

4. Semigroups and attractors

For dynamic systems whose states can be described by elements of a complete metric space (X, d_X), we often want to find a family of continuous operators $S(t)\colon X \to X$, $t \geqslant 0$, satisfying the semigroup properties

$$\begin{cases} S(t) = S(t-s)S(s) \quad \forall\, t \geqslant s \geqslant 0, \\ S(0) = \mathrm{Id}_{\mathbb{H}}. \end{cases}$$

If φ is the state of the system at time s, then $S(t)\varphi$ is the state of the system at time $t+s$. For problem (2.8), $\{S(t)\}_{t\geqslant 0}$ defined in Section 3 is a continuous operator semigroup from X to X, $X = \mathbb{H}$ or $\mathbb{V}$.

Definition 4.1. Let (X, d_X) be a complete metric space and $\{S(t)\}_{t\geqslant 0}$ be a continuous operator semigroup defined on X.

(i) Operators $\{S(t)\}_{t\geqslant 0}$ are said to be uniformly compact for t large if for every bounded set $K \subset X$ there exists $t_K > 0$ such that $\bigcup_{t\geqslant t_K} S(t)K$ is relatively compact in X.

(ii) For any nonempty set $K \subset X$ we define the ω-limit set of K by

$$\omega(K; X) = \bigcap_{s \geqslant 0} \overline{\bigcup_{t \geqslant s} S(t)K},$$

where the closures are taken in X.

(iii) A nonempty set $K \subset X$ is said to be invariant for $\{S(t)\}_{t \geqslant 0}$ if

$$K = S(t)K \quad \forall\, t \geqslant 0.$$

Speaking informally, the ω-limit set $\omega(K; X)$ describes the limit behaviors of all trajectories starting at $\Phi_0 \in K$. We can check that if for some $t_0 > 0$, $\bigcup_{t \geqslant t_0} S(t)K$ is nonempty and relatively compact in X, then $\omega(K; X)$ is nonempty, compact, and invariant.

Definition 4.2. Let (X, d_X) and $\{S(t)\}_{t \geqslant 0}$ be as in Definition 4.1.

(i) An invariant set $\mathcal{A} \subset X$ is called an attractor if it possesses an open neighborhood $\mathcal{O}$ such that for all $\varphi \in \mathcal{O}$,

$$\inf_{\psi \in \mathcal{A}} d_X(S(t)\varphi, \psi) \to 0 \quad \text{as } t \to \infty. \tag{4.1}$$

(ii) Let $\mathcal{A}$ be an attractor. The largest open set $\mathcal{O}$ such that (4.1) is satisfied for all $\varphi \in \mathcal{O}$ is called the basin of attraction of $\mathcal{A}$.

(iii) Let $\mathcal{A}$ be an attractor and $\mathcal{O}$ be its basin of attraction, $\mathcal{B} \subset \mathcal{O}$. If

$$\sup_{\varphi \in \mathcal{B}} \inf_{\psi \in \mathcal{A}} d_X(S(t)\varphi, \psi) \to 0 \quad \text{as } t \to \infty, \tag{4.2}$$

we say that $\mathcal{A}$ uniformly attracts $\mathcal{B}$.

(iv) Let $\mathcal{A}$ be an attractor. If $\mathcal{A}$ is compact and uniformly attracts all bounded sets in X, then $\mathcal{A}$ is said to be a global attractor for $\{S(t)\}_{t \geqslant 0}$.

Obviously, the basin of attraction of a global attractor is X. For a global attractor $\mathcal{A}$ and a bounded invariant set $K \subset X$, for any $\varepsilon > 0$ we derive from (4.2) that there exists $t_{\varepsilon,K} > 0$ such that for any $t \geqslant t_{\varepsilon,K}$,

$$K = S(t)K \subset \bigcup_{\psi \in \mathcal{A}} B_X(\psi, \varepsilon), \tag{4.3}$$

where $B_X(\psi, \varepsilon)$ is the ball centered at ψ with radius ε in space X. By the compactness of $\mathcal{A}$ and the arbitrariness of ε, we have $K \subset \mathcal{A}$. Thus, we say that a global attractor $\mathcal{A}$ is *maximal* for the inclusion relation among the bounded invariant sets. The maximal property also gives that the global attractor (if exists) is unique.

Definition 4.3. Let (X, d_X) and $\{S(t)\}_{t\geqslant 0}$ be as in Definition 4.1. For an open subset $\mathcal{O}$ of X and $\mathcal{B} \subset \mathcal{O}$, if for any bounded set $K \subset \mathcal{O}$ there exists $t_{K,\mathcal{B}} > 0$ such that for any $t \geqslant t_{K,\mathcal{B}}$, $S(t)K \subset \mathcal{B}$, then we say that $\mathcal{B}$ is absorbing in $\mathcal{O}$.

The bounded absorbing sets are highly related to the global attractor. We deduce from (4.3) that any ε-neighborhood of the global attractor is absorbing in X. Conversely, we can construct a global attractor by bounded absorbing sets. To be specific, the following theorem is valid (see [6]).

Theorem 4.4. *Let (X, d_X) and $\{S(t)\}_{t\geqslant 0}$ be as in Definition 4.1. If $\{S(t)\}_{t\geqslant 0}$ are uniformly compact and there exists a bounded set $\mathcal{B}$ absorbing in X, then the ω-limit set $\mathcal{A} = \omega(\mathcal{B}; X)$ is a global attractor. Furthermore, if X is a Banach space and for each $\varphi \in X$, the mapping $t \mapsto S(t)\varphi$ is continuous from $[0, \infty)$ into X, then $\mathcal{A}$ is connected.*

5. Existence of the global attractor

Now we go back to problem (2.8).

Lemma 5.1. *Let $f \in \mathbb{H}$, $\Phi_0 \in \mathbb{H}$. There exists $t_* = t_*(\lambda_1, \|f\|_{\mathbb{H}}, \|\Phi_0\|_{\mathbb{H}})$, s.t.*

$$\|\Phi(t)\|_{\mathbb{H}}^2 + \int_t^{t+1} \|\Phi(s)\|_{\mathbb{V}}^2 \, \mathrm{d}s \leqslant C\|f\|_{\mathbb{H}}^2 \quad \forall\, t \geqslant t_*.$$

P r o o f. Theorem 3.1 together with the Poincaré inequality gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Phi\|_{\mathbb{H}}^2 + \lambda_1\|\Phi\|_{\mathbb{H}}^2 \leqslant \langle f, \Phi\rangle.$$

Using Hölder's inequality together with Young's inequality, we deduce that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Phi\|_{\mathbb{H}}^2 + \lambda_1\|\Phi\|_{\mathbb{H}}^2 \leqslant \|f\|_{\mathbb{H}}\|\Phi\|_{\mathbb{H}} \leqslant \frac{1}{2\lambda_1}\|f\|_{\mathbb{H}}^2 + \frac{\lambda_1}{2}\|\Phi\|_{\mathbb{H}}^2.$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Phi\|_{\mathbb{H}}^2 + \lambda_1\|\Phi\|_{\mathbb{H}}^2 \leqslant \frac{1}{\lambda_1}\|f\|_{\mathbb{H}}^2.$$

Multiplying this by $\mathrm{e}^{\lambda_1 t}$, we derive

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{\lambda_1 t}\|\Phi\|_{\mathbb{H}}^2) \leqslant \frac{1}{\lambda_1}\mathrm{e}^{\lambda_1 t}\|f\|_{\mathbb{H}}^2.$$

Integrating from 0 to t, we deduce that

$$\|\Phi\|_{\mathbb{H}}^2 \leqslant \mathrm{e}^{-\lambda_1 t}\|\Phi_0\|_{\mathbb{H}}^2 + \frac{1}{\lambda_1^2}\|f\|_{\mathbb{H}}^2(1-\mathrm{e}^{-\lambda_1 t}).$$

Let

$$t_* = \max\Big\{-\frac{1}{\lambda_1}\ln\frac{\|f\|_{\mathbb{H}}^2}{\lambda_1^2\|\Phi_0\|_{\mathbb{H}}^2}, 0\Big\}.$$

Then for any $t \geqslant t_*$ we have

$$\|\Phi(t)\|_{\mathbb{H}}^2 \leqslant \frac{2}{\lambda_1^2}\|f\|_{\mathbb{H}}^2.$$

Estimate (3.8) gives

$$\int_t^{t+1}\|\Phi(s)\|_{\mathbb{V}}^2\,\mathrm{d}s \leqslant \|\Phi(t)\|_{\mathbb{H}}^2 + \|f\|_{\mathbb{V}'}^2 \leqslant C\|f\|_{\mathbb{H}}^2.$$

□

Lemma 5.2. *Let $f \in \mathbb{H}$, $\Phi_0 \in \mathbb{H}$. Then*

$$\|\Phi(t)\|_{\mathbb{V}}^2 + \int_t^{t+1}\|\mathbb{A}\Phi(s)\|_{\mathbb{H}}^2 \leqslant C\|f\|_{\mathbb{H}}^2\mathrm{e}^{C\|f\|_{\mathbb{H}}^4} \quad \forall\, t \geqslant t_* + 1.$$

Proof. We rewrite (3.10) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Phi\|_{\mathbb{V}}^2 + \|\mathbb{A}\Phi\|_{\mathbb{H}}^2 \leqslant C\|\Phi\|_{\mathbb{H}}^2\|\Phi\|_{\mathbb{V}}^4 + C(\|\Phi\|_{\mathbb{V}}^2 + \|f\|_{\mathbb{H}}^2). \tag{5.1}$$

By Lemma 3.4 and Lemma 5.1, we deduce

$$\|\Phi(t)\|_{\mathbb{V}}^2 \leqslant C\|f\|_{\mathbb{H}}^2\mathrm{e}^{C\|f\|_{\mathbb{H}}^4} \quad \forall\, t \geqslant t_* + 1.$$

Integrating (5.1) on $[t, t+1]$, we obtain that

$$\begin{aligned}\int_t^{t+1}\|\mathbb{A}\Phi(s)\|_{\mathbb{H}}^2\,\mathrm{d}s &\leqslant \|\Phi(t)\|_{\mathbb{V}}^2 + C\int_t^{t+1}(\|\Phi(s)\|_{\mathbb{H}}^2\|\Phi(s)\|_{\mathbb{V}}^4 + \|\Phi(s)\|_{\mathbb{V}}^2 + \|f\|_{\mathbb{H}}^2)\,\mathrm{d}s \\ &\leqslant C\|f\|_{\mathbb{H}}^2\mathrm{e}^{C\|f\|_{\mathbb{H}}^4} + C\|f\|_{\mathbb{H}}^6\mathrm{e}^{C\|f\|_{\mathbb{H}}^4} + C\|f\|_{\mathbb{H}}^2.\end{aligned}$$

Using $\|f\|_{\mathbb{H}}^4 \leqslant \|f\|_{\mathbb{H}}^4 + 1 \leqslant \mathrm{e}^{\|f\|_{\mathbb{H}}^4}$, we derive

$$\int_t^{t+1}\|\mathbb{A}\Phi(s)\|_{\mathbb{H}}^2\,\mathrm{d}s \leqslant C\|f\|_{\mathbb{H}}^2\mathrm{e}^{C\|f\|_{\mathbb{H}}^4}.$$

□

Lemma 5.3. *Let $f \in \mathbb{H}$, $\Phi_0 \in \mathbb{H}$. Then*

$$\|\mathbb{A}\Phi(t)\|_{\mathbb{H}}^2 \leqslant C\|f\|_{\mathbb{H}}^2\mathrm{e}^{C\|f\|_{\mathbb{H}}^4} \quad \forall\, t \geqslant t_* + 2.$$

P r o o f. For $t \geqslant t_* + 1$, using (2.5), Lemmas 5.1 and 5.2, we obtain

$$\|\mathbb{B}(\Phi,\Phi)\|_{\mathbb{H}} \leqslant C\|\Phi\|_{\mathbb{H}}^{1/2}\|\Phi\|_{\mathbb{V}}\|\mathbb{A}\Phi\|_{\mathbb{H}}^{1/2} \leqslant C\|f\|_{\mathbb{H}}^{3/2}\mathrm{e}^{C\|f\|_{\mathbb{H}}^4}\|\mathbb{A}\Phi\|_{\mathbb{H}}^{1/2}.$$

Thus, Lemma 5.2 together with Hölder's inequality gives

$$\int_t^{t+1}\|\mathbb{B}(\Phi(s),\Phi(s))\|_{\mathbb{H}}^2\,\mathrm{d}s \leqslant C\|f\|_{\mathbb{H}}^3\mathrm{e}^{C\|f\|_{\mathbb{H}}^4}\biggl(\int_t^{t+1}\|\mathbb{A}\Phi(s)\|_{\mathbb{H}}^2\,\mathrm{d}s\biggr)^{1/2} \leqslant C\|f\|_{\mathbb{H}}^4\mathrm{e}^{C\|f\|_{\mathbb{H}}^4}.$$

Noticing that $\|f\|_{\mathbb{H}}^4 \leqslant C\|f\|_{\mathbb{H}}^2(1+\|f\|_{\mathbb{H}}^4) \leqslant C\|f\|_{\mathbb{H}}^2\mathrm{e}^{C\|f\|_{\mathbb{H}}^4}$,

$$\int_t^{t+1}\|\mathbb{B}(\Phi(s),\Phi(s))\|_{\mathbb{H}}^2\,\mathrm{d}s \leqslant C\|f\|_{\mathbb{H}}^2\mathrm{e}^{C\|f\|_{\mathbb{H}}^4}.$$

So Lemmas 5.1 and 5.2 together with (2.8) imply that

$$\int_t^{t+1}\|\partial_s\Phi(s)\|_{\mathbb{H}}^2\,\mathrm{d}s \leqslant C\|f\|_{\mathbb{H}}^2\mathrm{e}^{C\|f\|_{\mathbb{H}}^4}. \tag{5.2}$$

Recalling (3.15), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t\Phi\|_{\mathbb{H}}^2 + \|\partial_t\Phi\|_{\mathbb{V}}^2 \leqslant C\|\Phi\|_{\mathbb{H}}^2\|\Phi\|_{\mathbb{V}}^2\|\partial_t\Phi\|_{\mathbb{H}}^2.$$

By Lemma 3.4, Lemma 5.1, Lemma 5.2, and (5.2), we derive for all $t \geqslant t_* + 2$ that

$$\|\partial_t\Phi(t)\|_{\mathbb{H}}^2 \leqslant C\|f\|_{\mathbb{H}}^2\mathrm{e}^{C\|f\|_{\mathbb{H}}^4}. \tag{5.3}$$

Then

$$\begin{aligned}
\|\mathbb{A}\Phi\|_{\mathbb{H}}^2 &\leqslant C\|\partial_t\Phi\|_{\mathbb{H}}^2 + C\|B(\Phi,\Phi)\|_{\mathbb{H}}^2 + C\|\mathbb{C}\Phi\|_{\mathbb{H}}^2 + C\|f\|_{\mathbb{H}}^2\\
&\leqslant C\|f\|_{\mathbb{H}}^2\mathrm{e}^{C\|f\|_{\mathbb{H}}^4} + C\|f\|_{\mathbb{H}}^3\mathrm{e}^{C\|f\|_{\mathbb{H}}^4}\|\mathbb{A}\Phi(t)\|_{\mathbb{H}}\\
&\leqslant \frac{1}{2}\|\mathbb{A}\Phi\|_{\mathbb{H}}^2 + C\|f\|_{\mathbb{H}}^2\mathrm{e}^{C\|f\|_{\mathbb{H}}^4} + C\|f\|_{\mathbb{H}}^6\mathrm{e}^{C\|f\|_{\mathbb{H}}^4}\\
&\leqslant \frac{1}{2}\|\mathbb{A}\Phi\|_{\mathbb{H}}^2 + C\|f\|_{\mathbb{H}}^2\mathrm{e}^{C\|f\|_{\mathbb{H}}^4}.
\end{aligned}$$

Therefore,

$$\|\mathbb{A}\Phi(t)\|_{\mathbb{H}}^2 \leqslant C\|f\|_{\mathbb{H}}^2\mathrm{e}^{C\|f\|_{\mathbb{H}}^4} \quad \forall\, t \geqslant t_* + 2.$$

□

Lemma 5.1–Lemma 5.3 tell us that for any $\Phi_0 \in \mathbb{H}$ there exists a time t_* depending on Ω, $\|f\|_{\mathbb{H}}$, and $\|\Phi_0\|_{\mathbb{H}}$, such that $S(t)\Phi_0$ goes into a bounded ball in $D(\mathbb{A})$ after time t_*. Therefore, there exists a bounded absorbing set in $D(\mathbb{A})$. By Theorem 4.4, we can construct the global attractor.

Theorem 5.4. *Assume that* $f \in \mathbb{H}$. *Then there exists a connected global attractor* $\mathcal{A}$ *for* $\{S(t)\}_{t\geqslant 0}$ *in* $\mathbb{H}$. *Moreover,* $\mathcal{A}$ *is also the connected global attractor in* $\mathbb{V}$ *and is bounded in* $[H^2(\Omega)]^2 \times [H^2(\Omega)]^2 \times H^2(\Omega)$.

Proof. Let $\mathcal{B} = B_{D(\mathbb{A})}(0, C\|f\|_{\mathbb{H}}^2 \mathrm{e}^{C\|f\|_{\mathbb{H}}^4})$. By Lemma 5.3, for any bounded set $K \subset \mathbb{V}$, there exists $t_* > 0$ such that $S(t)K \subset \mathcal{B}$ for all $t \geqslant t_* + 2$. Thus, $\mathcal{B}$ is absorbing in $\mathbb{V}$. Because the imbedding $D(\mathbb{A}) \hookrightarrow \mathbb{V}$ is compact, $\bigcup_{t\geqslant t_*+2} S(t)K$ is relatively compact in $\mathbb{V}$ and then $\{S(t)\}_{t\geqslant 0}$ are uniformly compact in $\mathbb{V}$. Theorem 4.4 shows that $\mathcal{A} = \omega(\mathcal{B}; \mathbb{V})$ is a connected global attractor in $\mathbb{V}$.

By similar procedures, we can prove that $\omega(\mathcal{B}; \mathbb{H})$ is a connected global attractor in $\mathbb{H}$. To obtain $\omega(\mathcal{B}; \mathbb{H}) = \omega(\mathcal{B}; \mathbb{V})$, we only need to prove that if a sequence in $\mathcal{B}$ converges to some Φ_* in $\mathbb{H}$, then it converges to Φ_* in $\mathbb{V}$ too. This can be derived from the interpolation inequality

$$\|\varphi\|_{H^1(\Omega)} \leqslant C\|\varphi\|_{L^2(\Omega)}^{1/2}\|\varphi\|_{H^2(\Omega)}^{1/2} \quad \forall\, \varphi \in H^2(\Omega),$$

where C depends on Ω only. □

6. Dimension estimates

We shall prove that $S(t)$ is so-called uniformly differentiable in $\mathcal{A}$.

Lemma 6.1. *Let* $\mathcal{A}$ *be the global attractor obtained in Theorem* 5.4. *Then, for any fixed* $t > 0$, $S(t)$ *is uniformly differentiable in* $\mathcal{A} \subset \mathbb{H}$. *That is,*

$$\sup_{\substack{\psi,\varphi\in\mathcal{A}\\ 0<\|\psi-\varphi\|_{\mathbb{H}}\leqslant\varepsilon}} \frac{\|S(t)\psi - S(t)\varphi - L_{(\varphi,f)}(t)(\psi-\varphi)\|_{\mathbb{H}}}{\|\psi-\varphi\|_{\mathbb{H}}} \to 0 \quad \text{as } \varepsilon \to 0.$$

Proof. For $\varphi, \psi \in \mathbb{H}$, set $\Phi(t) = S(t)\varphi$, $\Psi(t) = S(t)\psi$, then $\Phi, \Psi \in C([0,T]; \mathbb{H}) \cap L^2(0,T;\mathbb{V}) \cap C((0,T]; D(\mathbb{A}))$ for all $T > 0$. Let $Z = \Psi - \Phi$, then Z satisfies

$$\text{(6.1)} \qquad \begin{cases} \partial_t Z + \mathbb{A}Z + \mathbb{B}(\Psi, Z) + \mathbb{C}Z = -\mathbb{B}(Z,\Phi), & 0 < t < T,\\ Z(0) = \psi - \varphi. \end{cases}$$

Estimate (3.19) gives

$$\text{(6.2)} \qquad \sup_{t\in[0,T]} \|Z(t)\|_{\mathbb{H}}^2 + \int_0^T \|Z(t)\|_{\mathbb{V}}^2 \,\mathrm{d}t \leqslant C\|\psi-\varphi\|_{\mathbb{H}}^2,$$

where $C = C(T, \|f\|_{\mathbb{V}'}, \|\varphi\|_{\mathbb{H}})$.

We rewrite (3.23) as

$$\begin{cases} \partial_t W + \mathbb{A}W + \mathbb{B}(\Psi, W) + \mathbb{C}W = -\mathbb{B}(W, \Phi) + \mathbb{B}(Z, W), \quad 0 < t < T, \\ W(0) = W_0, \end{cases} \tag{6.3}$$

and set

$$R = \Psi - \Phi - L_{(\varphi,f)}(t)(\psi - \varphi) = Z - W$$

with $W_0 = \psi - \varphi$ and $W(t) = L_{(\varphi,f)}(t)W_0$. Taking the difference of (6.1) and (6.3), we deduce

$$\begin{cases} \partial_t R + \mathbb{A}R + \mathbb{B}(\Psi, R) + \mathbb{C}R = -\mathbb{B}(R, \Phi) - \mathbb{B}(Z, W), \quad 0 < t < T, \\ R(0) = 0. \end{cases} \tag{6.4}$$

Multiplying $(6.4)_1$ by R, we deduce from (2.2) and the Poincaré inequality that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\|R\|_{\mathbb{H}}^2 + \|R\|_{\mathbb{V}}^2 \leqslant C(\|\Phi\|_{\mathbb{H}}^2\|\Phi\|_{\mathbb{V}}^2 + \|W\|_{\mathbb{V}}^2 + \|Z\|_{\mathbb{V}}^2)\|R\|_{\mathbb{H}}^2 \\ + C(\|Z\|_{\mathbb{H}} + \|W\|_{\mathbb{H}})(\|Z\|_{\mathbb{V}}^2 + \|W\|_{\mathbb{V}}^2). \end{aligned}$$

Combining (3.24) and (6.2) yields

$$\int_0^T (\|Z(t)\|_{\mathbb{H}} + \|W(t)\|_{\mathbb{H}})(\|Z(t)\|_{\mathbb{V}}^2 + \|W(t)\|_{\mathbb{V}}^2)\,\mathrm{d}t \leqslant C\|\psi - \varphi\|_{\mathbb{H}}^3.$$

Application of the Gronwall inequality gives

$$\sup_{0\leqslant t\leqslant T} \|R(t)\|_{\mathbb{H}}^2 + \int_0^T \|R(t)\|_{\mathbb{V}}^2\,\mathrm{d}t \leqslant C\|\psi - \varphi\|_{\mathbb{H}}^3$$

and

$$\frac{\|\Psi(t) - \Phi(t) - L_{(\varphi,f)}(t)(\psi - \varphi)\|_{\mathbb{H}}^2}{\|\psi - \varphi\|_{\mathbb{H}}^2} = \frac{\|R(t)\|_{\mathbb{H}}^2}{\|\psi - \varphi\|_{\mathbb{H}}^2} \leqslant C\|\psi - \varphi\|_{\mathbb{H}} \to 0 \tag{6.5}$$

as $\psi \to \varphi$ in $\mathbb{H}$, where $C = C(T, \|f\|_{\mathbb{H}}, \|\varphi\|_{\mathbb{H}}, \|\psi\|_{\mathbb{H}})$. This shows that $L_{(\varphi,f)}(t)$ is the Fréchet differential of $S(t)$ at $\varphi \in \mathbb{H}$. Moreover, if $\varphi, \psi \in \mathcal{A}$, then $\|\varphi\|_{\mathbb{H}}, \|\psi\|_{\mathbb{H}}$ are bounded according to Lemma 5.1, and the constant C in (6.5) is independent of φ. □

Except for the uniformly differentiable property of $S(t)$, we also need some bounds for the linear operator $L_{(\varphi,f)}(t)$. Estimate (3.24) says that

$$\sup_{t\in[0,1]}\sup_{\varphi\in\mathcal{A}}\|L_{(\varphi,f)}(t)\|_{\mathcal{L}(\mathbb{H})}\leqslant C_L=C_L(\|f\|_{\mathbb{H}}).$$

By Theorem 3.8,

$$L_{(\varphi,f)}(t)=L_{(S(\lfloor t\rfloor)\varphi,f)}(t-\lfloor t\rfloor)\circ L_{(S(\lfloor t\rfloor-1)\varphi,f)}(1)\circ\ldots\circ L_{(\varphi,f)}(1),$$

where $\lfloor t\rfloor$ is the integer part of t. Then

$$\sup_{\varphi\in\mathcal{A}}\|L_{(\varphi,f)}(t)\|_{\mathcal{L}(\mathbb{H})}\leqslant C_L^{\lfloor t\rfloor+1}. \tag{6.6}$$

For the global attractor $\mathcal{A}$ obtained in Theorem 5.4, we define

$$\tilde{q}_j=\limsup_{t\to\infty}\sup_{\varphi\in\mathcal{A}}\sup_{\substack{\psi_i\in\mathbb{H}\\ \|\psi_i\|_{\mathbb{H}}\leqslant 1\\ i=1,\ldots,j}}\frac{1}{t}\int_0^t\mathrm{Tr}[F'_{(\varphi,f)}(s)\circ\mathbb{Q}_j(s)]\,\mathrm{d}s,$$

where $\mathbb{Q}_j(s)=\mathbb{Q}_j(s,\varphi;\psi_1,\ldots,\psi_j)$ is the projection from $\mathbb{H}$ onto the space spanned by $L_{(\varphi,f)}(s)\psi_1$, …, $L_{(\varphi,f)}(s)\psi_j$. The number $\mathrm{Tr}[F'_{(\varphi,f)}(s)\circ\mathbb{Q}_j(s)]$ is the trace of the linear operator (of finite rank) $F'_{(\varphi,f)}(s)\circ\mathbb{Q}_j(s)$. The dimension of $\mathcal{A}$ relies on the negativeness of $\tilde{q}_j$. Lemma 6.1 together with (6.6) gives the following theorem.

Theorem 6.2. *If*

$$\tilde{q}_j\leqslant q_j,\quad j=1,2,\ldots$$

for a concave function q_j with respect to j, and

$$q_m\geqslant 0>q_{m+1}$$

for an integer m, then the Hausdorff dimension and the fractal dimension can be estimated by

$$\dim_H(\mathcal{A};\mathbb{H})\leqslant\dim_F(\mathcal{A};\mathbb{H})\leqslant m+\frac{q_m}{q_m-q_{m+1}}.$$

Moreover, if $m=0$, then $\dim_H(\mathcal{A};\mathbb{H})=\dim_F(\mathcal{A};\mathbb{H})=0$.

The proof of Theorem 6.2 (see [6]) relies on the (uniform) Lyapunov numbers and the (uniform) Lyapunov exponents which indicate the distortion of finite dimensional volumes produced by $S(t)$. If we can find some $q_{m+1}<0\leqslant q_m$, then we can estimate the dimension of $\mathcal{A}$ by m and q_m,q_{m+1}. To obtain our estimates for the dimension of $\mathcal{A}$, we need the following lemma (see [6]).

Lemma 6.3. *For a family of elements $e_1, \ldots, e_m$ of $\mathbb{V}$ which is orthonormal in $\mathbb{H}$,*

$$\sum_{i=1}^{m} \|e_i\|_{\mathbb{V}}^2 \geqslant \lambda_1 + \ldots + \lambda_m,$$

where $\{\lambda_i\}_{i\in\mathbb{N}}$ is the complete sequence of eigenvalues of $\mathbb{A}$, $0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots$.

According to Theorem 5.4, $\mathcal{A}$ is not only a global attractor in $\mathbb{H}$ but also in $\mathbb{V}$, so the dimension of $\mathcal{A}$ in $\mathbb{V}$, and the relation between dimensions of $\mathcal{A}$ in $\mathbb{V}$ and in $\mathbb{H}$ are also of interest. To this end, we introduce the following lemma which can be verified easily.

Lemma 6.4. *Let $(X, d_X), (Y, d_Y)$ be two complete metric spaces. Assume that $\Pi\colon X \to Y$ satisfies*

$$d_Y(\Pi(\varphi), \Pi(\psi)) \leqslant L d_X(\varphi, \psi) \quad \forall\, \varphi, \psi \in X,$$

for a nonnegative number $L \geqslant 0$. Then for any $K \subset X$ we have

$$\dim_H(\Pi(K); Y) \leqslant \dim_H(K; X),$$
$$\dim_F(\Pi(K); Y) \leqslant \dim_F(K; X).$$

Using Theorems 5.4, 6.2 and Lemmas 6.1, 6.3, 6.4, we can derive the following theorem.

Theorem 6.5. *Let $\mathcal{A}$ be the global attractor obtained in Theorem 5.4. Then*

$$\dim_H(\mathcal{A}; \mathbb{H}) = \dim_H(\mathcal{A}; \mathbb{V}) \leqslant \dim_F(\mathcal{A}; \mathbb{H}) = \dim_F(\mathcal{A}; \mathbb{V}) \leqslant C_0\|f\|_{\mathbb{H}}^4,$$

where C_0 depends on Ω only. Moreover, if $C_0\|f\|_{\mathbb{H}}^4 < 1$, then

$$\dim_H(\mathcal{A}; \mathbb{H}) = \dim_H(\mathcal{A}; \mathbb{V}) = \dim_F(\mathcal{A}; \mathbb{H}) = \dim_F(\mathcal{A}; \mathbb{V}) = 0.$$

P r o o f. Let $\Phi(t) = S(t)\varphi$. Then

$$\begin{aligned}\operatorname{Tr}[F'_{(\varphi,f)}(s) \circ \mathbb{Q}_j(s)] &= \sum_{i=1}^{j} \langle F'_{(\varphi,f)}(s) e_i(s), e_i(s)\rangle \\ &= -\sum_{i=1}^{j} [\|e_i(s)\|_{\mathbb{V}}^2 + b(e_i(s), \Phi(s), e_i(s))] \\ &= -\sum_{i=1}^{j} [\|e_i(s)\|_{\mathbb{V}}^2 - b(e_i(s), e_i(s), \Phi(s))].\end{aligned}$$

For the last term, we deduce from (2.2) that

$$\begin{aligned}\left|\sum_{i=1}^{j} b(e_i(s), e_i(s), \Phi(s))\right| &\leqslant C \sum_{i=1}^{j} \|\Phi(s)\|_{\mathbb{H}}^{1/2} \|\Phi(s)\|_{\mathbb{V}}^{1/2} \|e_i(s)\|_{\mathbb{H}}^{1/2} \|e_i(s)\|_{\mathbb{V}}^{3/2} \\ &\leqslant \frac{1}{2} \sum_{i=1}^{j} \|e_i(s)\|_{\mathbb{V}}^2 + Cj \|\Phi(s)\|_{\mathbb{H}}^2 \|\Phi(s)\|_{\mathbb{V}}^2.\end{aligned}$$

Therefore,

$$\operatorname{Tr}[F'_{(\varphi,f)}(s) \circ \mathbb{Q}_j(s)] \leqslant -\frac{1}{2} \sum_{i=1}^{j} \|e_i(s)\|_{\mathbb{V}}^2 + Cj \|\Phi(s)\|_{\mathbb{H}}^2 \|\Phi(s)\|_{\mathbb{V}}^2. \tag{6.7}$$

It is known that (in both the non-slip case and the periodic case) $\lambda_j \geqslant cj$, where c depends on Ω only. Therefore,

$$\operatorname{Tr}[F'_{(\varphi,f)}(s) \circ \mathbb{Q}_j(s)] \leqslant -\frac{c}{4} j^2 + Cj \|\Phi(s)\|_{\mathbb{H}}^2 \|\Phi(s)\|_{\mathbb{V}}^2,$$

which implies

$$\tilde{q}_j \leqslant -\frac{c}{4} j^2 + Cj \limsup_{t\to\infty} \frac{1}{t} \int_0^t \|\Phi(s)\|_{\mathbb{H}}^2 \|\Phi(s)\|_{\mathbb{V}}^2 \, \mathrm{d}s.$$

Taking advantage of (3.9) and Lemma 5.1, we derive

$$\tilde{q}_j \leqslant -\frac{c}{4} j^2 + Cj \|f\|_{\mathbb{H}}^4 = -\frac{c}{4} j (j - C_0 \|f\|_{\mathbb{H}}^4) \triangleq q_j.$$

Take $m = \lfloor C_0 \|f\|_{\mathbb{H}}^4 \rfloor$, then $q_m \geqslant 0 > q_{m+1}$. Thus

$$\dim_H(\mathcal{A}; \mathbb{H}) \leqslant \dim_F(\mathcal{A}; \mathbb{H}) \leqslant m + \frac{q_m}{q_m - q_{m+1}}.$$

Noticing that

$$\frac{q_m}{q_m - q_{m+1}} \leqslant \frac{m(C_0 \|f\|_{\mathbb{H}}^4 - m)}{2m - C_0 \|f\|_{\mathbb{H}}^4 + 1} \leqslant C_0 \|f\|_{\mathbb{H}}^4 - m,$$

we deduce

$$\dim_H(\mathcal{A}; \mathbb{H}) \leqslant \dim_F(\mathcal{A}; \mathbb{H}) \leqslant C_0 \|f\|_{\mathbb{H}}^4.$$

Now we are going to check that the dimensions in $\mathbb{H}$ are equal to those in $\mathbb{V}$, respectively. This results from two applications of Lemma 6.4.

On the one hand, let $X = \mathbb{V}$, $Y = \mathbb{H}$, $\Pi = \mathrm{Id}\colon \mathbb{V} \to \mathbb{H}$, then by Lemma 6.4,

$$\begin{aligned}\dim_H(\mathcal{A}; \mathbb{H}) &\leqslant \dim_H(\mathcal{A}; \mathbb{V}), \\ \dim_F(\mathcal{A}; \mathbb{H}) &\leqslant \dim_F(\mathcal{A}; \mathbb{V}).\end{aligned}$$

On the other hand, by Theorem 3.7, for $t > 0$, $S(t)$ is locally Lipschitz continuous from $\mathbb{H}$ into $\mathbb{V}$. Take $X = (\mathcal{B}, \|\cdot\|_{\mathbb{H}})$, $\mathcal{B} = B_{D(\mathbb{A})}(0, C\|f\|_{\mathbb{H}}^2 \mathrm{e}^{C\|f\|_{\mathbb{H}}^4})$, and $Y = \mathbb{V}$, $\Pi = S(t)$. Then by Lemma 6.4,

$$\dim_H(\mathcal{A}; \mathbb{H}) = \dim_H(S(t)\mathcal{A}; \mathbb{H}) \geqslant \dim_H(\mathcal{A}; \mathbb{V}),$$
$$\dim_F(\mathcal{A}; \mathbb{H}) = \dim_F(S(t)\mathcal{A}; \mathbb{H}) \geqslant \dim_F(\mathcal{A}; \mathbb{V}).$$

□

From Theorem 6.5, we know that if $\|f\|_{\mathbb{H}}$ is sufficiently small, then the Hausdorff dimension and the fractal dimension of $\mathcal{A}$ vanish. To be specific, the following proposition is valid.

Proposition 6.6. *There exists a small positive number $\varepsilon_0 > 0$ depending on Ω only, such that if $\|f\|_{\mathbb{H}} < \varepsilon_0$, then (2.8) has a unique stationary solution $\psi \in \mathbb{V}$ which is globally asymptotically stable, i.e., $\mathcal{A} = \{\psi\}$.*

Proof. Because ψ is a stationary solution to (2.8), we have

$$\mathbb{A}\psi + \mathbb{B}(\psi, \psi) + \mathbb{C}\psi = f.$$

Acting on ψ, we obtain that

$$\|\psi\|_{\mathbb{V}}^2 = \langle f, \psi\rangle \leqslant \lambda_1^{-1/2}\|f\|_{\mathbb{H}}\|\psi\|_{\mathbb{V}}.$$

Therefore,

$$\|\psi\|_{\mathbb{V}} \leqslant \lambda_1^{-1/2}\|f\|_{\mathbb{H}}. \tag{6.8}$$

Then we can construct a stationary weak solution ψ to (2.8) by the Galerkin method. We omit the details here. Each weak solution $\Phi(x,t)$ can be written as $\Phi(x,t) = \psi(x) + Z(x,t)$, where Z satisfies

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}Z + \mathbb{A}Z + \mathbb{B}(\Phi, Z) + \mathbb{B}(Z, \psi) + \mathbb{C}Z = 0, \\ Z(0) = \Phi(0) - \psi. \end{cases}$$

Similarly to (3.7), we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|Z\|_{\mathbb{H}}^2 + \|Z\|_{\mathbb{V}}^2 \leqslant C\|\psi\|_{\mathbb{H}}^2\|\psi\|_{\mathbb{V}}^2\|Z\|_{\mathbb{H}}^2.$$

Substituting (6.8) into the Poincaré inequality, we derive

$$\frac{\mathrm{d}}{\mathrm{d}t}\|Z\|_{\mathbb{H}}^2 + \lambda_1\|Z\|_{\mathbb{H}}^2 \leqslant C\|f\|_{\mathbb{H}}^4\|Z\|_{\mathbb{H}}^2.$$

Hence,
$$\frac{\mathrm{d}}{\mathrm{d}t}\|Z\|_{\mathbb{H}}^2 + (\lambda_1 - C\|f\|_{\mathbb{H}}^4)\|Z\|_{\mathbb{H}}^2 \leqslant 0.$$
If $\|f\|_{\mathbb{H}}$ is sufficiently small such that
$$\lambda_1 - C\|f\|_{\mathbb{H}}^4 \leqslant \frac{\lambda_1}{2},$$
then
$$\frac{\mathrm{d}}{\mathrm{d}t}\|Z\|_{\mathbb{H}}^2 + \frac{\lambda_1}{2}\|Z\|_{\mathbb{H}}^2 \leqslant 0.$$
This implies
$$\|Z(t)\|_{\mathbb{H}}^2 \leqslant \mathrm{e}^{-\lambda_1 t/2}\|\Phi(0) - \psi\|_{\mathbb{H}}^2 \to 0 \quad \text{as } t \to \infty.$$
Consequently, the stationary solution is unique and $\mathcal{A} = \{\psi\}$. □

Corollary 6.7. *Under the assumption of Proposition* 6.6, *for any* $\Phi_0 \in \mathbb{H}$, $\Phi(t) = S(t)\Phi_0$ *we have for all* $0 < \alpha < 1$,
$$\|\Phi(t) - \psi\|_{C^\alpha(\overline{\Omega})} \to 0 \quad \textit{as } t \to \infty.$$

P r o o f. Because $\psi \in \mathcal{A}$, we have $\psi \in D(\mathbb{A})$. Interpolation between L^2 and H^2 gives
$$\begin{aligned}\|\Phi(t) - \psi\|_{L^\infty(\Omega)} \leqslant C\|\Phi(t) - \psi\|_{L^2}^{1/2}\|\Phi(t) - \psi\|_{H^2}^{1/2} &\leqslant C\|\Phi(t) - \psi\|_{\mathbb{H}}^{1/2}\|\mathbb{A}\Phi(t) - \mathbb{A}\psi\|_{\mathbb{H}}^{1/2} \\ &\leqslant C\|\Phi(t) - \psi\|_{\mathbb{H}}^{1/2} \to 0.\end{aligned}$$
For any $0 < \alpha < 1$, we take $\alpha < \alpha' < 1$, the imbeddings $H^2(\Omega) \hookrightarrow C^{\alpha'}(\overline{\Omega}) \hookrightarrow C^\alpha(\overline{\Omega}) \hookrightarrow C(\overline{\Omega})$ are continuous. Then $\Phi(t) - \psi$ converges to 0 in $C(\overline{\Omega})$. Interpolation of Hölder's spaces yields
$$\|\Phi(t) - \psi\|_{C^\alpha(\overline{\Omega})} \leqslant C\|\Phi(t) - \psi\|_{C(\overline{\Omega})}^{1-\alpha/\alpha'}\|\Phi(t) - \psi\|_{C^{\alpha'}(\overline{\Omega})}^{\alpha/\alpha'} \leqslant C\|\Phi(t) - \psi\|_{C(\overline{\Omega})}^{1-\alpha/\alpha'} \to 0.$$
□

For lower bounds of the dimension of the global attractor, we should estimate the dimension of the local unstable set in a neighborhood of a stationary point. However, it is not easy to construct such a good stationary point in general case.

For a special case $f^2 = 0$, $f^3 = 0$, we let $\mathcal{A}_{NS}$ be the global attractor to the 2D Navier-Stokes equations, then $\mathcal{A}_{NS} \times \{0\} \times \{0\} \subset \mathcal{A}$. The lower bounds for the Hausdorff dimension and for the fractal dimension of the global attractor to the 2D Navier-Stokes equations can also be viewed as those for the tropical climate model,
$$\begin{aligned}\dim_H(\mathcal{A}; \mathbb{V}) = \dim_H(\mathcal{A}; \mathbb{H}) \geqslant \dim_H(\mathcal{A}_{NS}; \mathbb{H}),\\ \dim_F(\mathcal{A}; \mathbb{V}) = \dim_F(\mathcal{A}; \mathbb{H}) \geqslant \dim_F(\mathcal{A}_{NS}; \mathbb{H}).\end{aligned}$$
For lower bounds of the dimension of $\mathcal{A}_{NS}$, see [1].

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