ON THE MAXWELL-WAVE EQUATION COUPLING PROBLEM AND ITS EXPLICIT FINITE-ELEMENT SOLUTION

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Abstract. It is well known that in the case of constant dielectric permittivity and magnetic permeability, the electric field solving the Maxwell's equations is also a solution to the wave equation. The converse is also true under certain conditions. Here we study an intermediate situation in which the magnetic permeability is constant and a region with variable dielectric permittivity is surrounded by a region with a constant one, in which the unknown field satisfies the wave equation. In this case, such a field will be the solution of Maxwell's equation in the whole domain, as long as proper conditions are prescribed on its boundary. We show that an explicit finite-element scheme can be used to solve the resulting Maxwell-wave equation coupling problem in an inexpensive and reliable way. Optimal convergence in natural norms under reasonable assumptions holds for such a scheme, which is certified by numerical exemplification.

Keywords: constant magnetic permeability; dielectric permittivity; explicit scheme; finite element; mass lumping; Maxwell-wave equation

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1. Motivation

The aim of this work is to study a Maxwell-wave equation coupling problem for an unknown field defined in a bounded simply connected two- or three-dimensional Lipschitz domain Ω.

The solution of coupling problems for two or more types of partial differential equations that hold in complementary domains is a subject of growing interest. This is mostly due to the need of modeling interactive phenomena occurring in different neighboring regions, such as flow in a porous medium surrounded by shallow water flow. The coupling problem for the classical wave equation and Maxwell's equations

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has been shown to be a problem of paramount importance in the detection of objects through the solution of coefficient inverse problems for both equations, as reported in [8], [12], [34], [35]. However, to the best of the authors' knowledge, a rigorous study of the well-posedness of such a coupling problem and its numerical solution was lacking. This is precisely the main novelty of this article.

Actually, beyond the above aspect, our motivation to study such a coupling problem is multifold. First, noting that Maxwell's equations reduce to the wave equation for a solenoidal field, wherever ε is constant, the Maxwell-wave equation coupling problem does correspond to Maxwell's equations in Ω, as long as we can ensure that its solution is divergence free in the sub-domain where the wave equation hold. Actually, this is an issue directly connected to the conditions under which we can also recover Maxwell's equations from the wave equation, in which case both equations are equivalent. Many authors have studied such conditions. For instance, in [33] the author highlights a condition on the divergence and the curl of the initial data. Alternatively we can achieve equivalence by requiring a condition only on the divergence of the initial data, plus a divergence free boundary condition, in case a classical zero tangential-component boundary condition also holds for the electric field. However, this is not what we are going to do in this work, and hence in principle, Maxwell's equations will not necessarily hold in the sub-domain where the wave equation holds. Nevertheless, as a by-product, our study will trivially apply to the case where Maxwell's equations for the electric field do hold in the whole domain Ω. Moreover, since we will not be dealing with the latter case, we will be free to consider boundary conditions typical of the wave equation, and not as much of Maxwell's equations, such as Dirichlet or Neumann boundary conditions.

Whatever the case, the problem at hand is solved here by means of an explicit P_1 finite-element scheme. We can show that this is indeed a reliable numerical solution tool, provided the coupled equations are written in a suitable variational form. As far as the pure wave equation is concerned, we refer to [25] in this respect. On the other hand, for Maxwell's equations, this form corresponds to the AVF-*Augmented Variational Formulation* thoroughly studied by Ciarlet Jr. (cf. [18]) in the static and time-harmonic cases and in Jamelot [24] and Ciarlet Jr., Jamelot [19] in the time-dependent case. An adaptive method for the time-harmonic Maxwell equations, based on this approach, showed to be particularly efficient in [2], [3]. As we should emphasize, in the framework considered here the difficulties brought about by corner singularities are absent. However, nonnegligible additional complexities must be dealt with, stemming form the fact that the dielectric permittivity varies in space. This is one of the main reasons that compelled the authors to carry out in detail a rigorous analysis of a P_1 lumped-mass approximation of Maxwell's equations in [10]. As a matter of fact, to the best of their knowledge, such results were lacking in the literature. For instance, the case of a variable permittivity had been addressed in [20], [22], but not for conforming finite elements, while in [4], [18], [19] conforming finite elements are dealt with, though for constant coefficients.

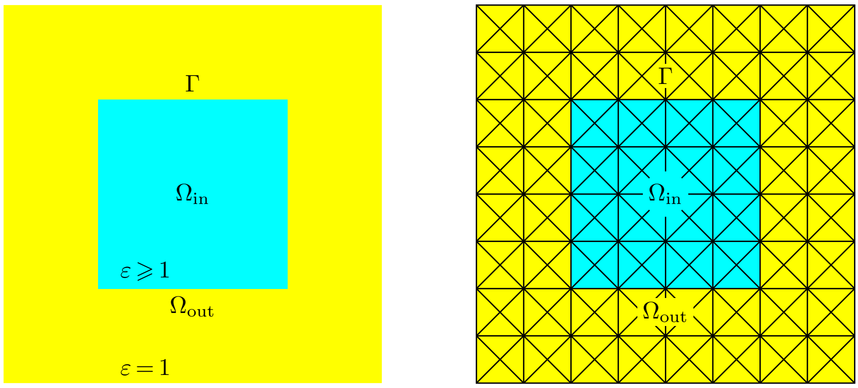
Incidentally, the standard conforming P_1 FEM is a particularly tempting possibility to solve Maxwell's equations as a simple and inexpensive method at a time. However, it is not always a reliable method for this purpose. Approximately four decades ago, this motivated Nédélec to propose ago a family of $\mathbf{H}(\mathrm{curl})$-conforming methods to solve these equations (cf. [32]), also known as edge elements. These methods are still widely in use, and comprehensive descriptions thereof can be found in [13] and [31]. Alternatively, the solution of the time-dependent Maxwell equations with conforming nodal finite elements was considered in the early nineties (cf. [4]) for convex domains. Later on, specialists studied formulations of the static or the time-harmonic Maxwell equations suitable for a numerical solution with nodal elements, even in nonconvex domains. In this respect, we refer to [21], [24], [19] and [5]. Such studies revealed the adequacy of nodal elements, at least in some relevant practical situations. Underlying this work lies precisely one such a case, characterized by the fact that Maxwell's equations in a given simply connected domain are coupled with the wave equation in a surrounding domain. Actually, by applying a mass-lumping explicit P_1 finite-element scheme to such a Maxwell-wave equation coupling problem, we establish here that optimal approximations of Maxwell's equations in terms of the electric field are also generated, provided a classical CFL condition is fulfilled.

An outline of this paper is as follows: In Section 2 the model problem being solved is described in detail together with its equivalent variational form. Well-posedness of this problem is established. In Section 3 a space discretization of the model problem by the finite element method combined with a standard finite-difference time discretization is presented; the stability condition to be fulfilled by this numerical model, together with the resulting optimal error estimates are also given. In Section 4 a numerical validation of the error estimates is provided, followed by conclusions and final remarks given in Section 5.

2. The model problem

We study a Maxwell-wave equation coupling problem in a bounded simply connected two- or three-dimensional Lipschitz domain Ω in the following particular framework:

Maxwell's equations hold in a simply connected sub-domain Ω_{in} whose closure is completely immersed in Ω, while the wave equation holds in the complementary sub-domain Ω_{out}. Let Γ be the interface between Ω_{in} and Ω_{out}. In Figure 1 (a) we sketch problem's geometry for a rectangular domain Ω, together with a typical finite-element mesh thereof in Figure 1 (b). We assume that the magnetic permeability of the medium in Ω_{in} is constant, so that Maxwell's equations can be expressed therein in terms of the sole electric field denoted by $\mathbf{e}$. We also denote by $\mathbf{e}$ the unknown field satisfying the wave equation in the sub-domain Ω_{out}.



(a) $\Omega = \Omega_{\text{in}} \cup \Omega_{\text{out}} \cup \Gamma$ (b) FE mesh in Ω

Figure 1. Maxwell-wave equation coupling in Ω.

For the sake of conciseness, in this work we only address the case where $\mathbf{e}$ satisfies homogeneous Dirichlet boundary conditions. In doing so we rule out the occurrence of singularities in the presence of re-entrant corners. As far as the authors can see, these conditions are only of academic interest in case Maxwell's equations also hold in the outer layer Ω_{out}. However, it can be established that the reliability results given in the sequel are derived from those that hold for other *uncoupling* boundary conditions carrying a physical meaning. In particular this assertion holds true for the case of absorbing boundary conditions studied in [10]. As pointed out in Section 1, the latter correspond to situations addressed in [8], [12], [30], [29], [28], [27]. Bearing in mind this introductory statement, let us switch to the description of the problem to solve.

2.1. Maxwell-wave equation coupling. Referring to Figure 1 (a), we wish to find a field $\mathbf{e}$ defined in a bounded simply connected domain Ω of $\Re^N$ with $N = 2$ or 3, satisfying Maxwell's equations in the sub-domain Ω_{in} occupied by a medium having dielectric permeability ε and the wave equation with constant wave speed v in the complementary sub-domain Ω_{out}. Although this is not essential, in order to ensure a suitable regularity of $\mathbf{e}$, we consider that the dielectric permittivity ε is continuous and takes a constant value ε_0 on the interface Γ between Ω_{out} and Ω_{in}, say, $\varepsilon_0 = 1$ and we take $v = 1$. In doing so, we extend the function ε by one in Ω_{out} for convenience (cf. Figure 1 (a)). In order to claim the validity of the analytical results given in [10], we assume in this work that ε belongs to $W^{2,\infty}(\Omega)$ and $\varepsilon \geqslant 1$ everywhere. Actually the assumption that ε attains a minimum in Ω_{out} is not essential for the methodology studied in this paper to work. However, as far as we can see, it guarantees optimal convergence results in the case of boundary conditions other than Dirichlet conditions (cf. [10]). In this respect we also refer to Remark (i) in Subsection 2.6 hereafter.

Let V be the space $H_0^1(\Omega)$ and $\mathbf{V}$ be $[V]^N$. Given $(\mathbf{e}_0;\mathbf{e}_1)$ in $\mathbf{V}\times\mathbf{V}$ satisfying $\nabla\cdot(\varepsilon\mathbf{e}_0) = \nabla\cdot(\varepsilon\mathbf{e}_1) = 0$, together with $\mathbf{f}\in\mathbf{H}(\text{div},\Omega)$, satisfying $\nabla\cdot\mathbf{f} = 0$ for all $t\in(0,T)$, we wish to

$$
(2.1)\qquad \begin{cases}
\text{Find } \mathbf{e}\in\mathcal{V} := H^2[(0,T);\{L^2(\Omega)\}^N]\cap L^2[(0,T);\mathbf{V}] \\
\text{such that} \\
\varepsilon\partial_{tt}\mathbf{e} + \nabla\times\nabla\times\mathbf{e} = \mathbf{f} & \text{in } \Omega_{\text{in}}\times(0,T), \\
\text{and} \\
\partial_{tt}\mathbf{e} - \Delta\mathbf{e} = \mathbf{f} & \text{in } \Omega_{\text{out}}\times(0,T), \\
\text{with the initial conditions} \\
\mathbf{e}(\cdot,0) = \mathbf{e}_0(\cdot),\ \text{and } \partial_t\mathbf{e}(\cdot,0) = \mathbf{e}_1(\cdot) & \text{in } \Omega.
\end{cases}
$$

At this point it is important to emphasize that, usually, in mathematical models coupling two equations in different domains, two sets of unknowns correspond to each one of the domains in which the coupling problem is defined. Then some compatibility condition at the interface Γ between both domains has to be enforced (see, e.g., [37]). Typically these conditions are expressed by the coincidence of traces and normal derivatives on Γ from both domains. However, (2.1) is a simpler problem, and therefore we allowed ourselves to disregard the statement of the problem with two different unknown fields in each domain, since such compatibility conditions are implicitly satisfied. Indeed, by requiring that $\mathbf{e}\in\{H^1(\Omega)\}^N$, we force the coincidence on Γ of the traces of the solution $\mathbf{e}$ restricted to Ω_{in} and Ω_{out}. For the same reason, normal derivatives on Γ will also coincide in a sense underlying the actual regularity of $\mathbf{e}$. Nevertheless, for the sake of clarity, we recast (2.1) in a usual equation coupling form posed in two nonoverlapping domains with a common interface. With this aim we first introduce the spaces

$$
\begin{aligned}
\mathcal{V}_{\text{in}} &:= H^2[(0,T);\{L^2(\Omega_{\text{in}})\}^N]\cap L^2[(0,T);\{H^1(\Omega_{\text{in}})\}^N], \\
\mathcal{V}_{\text{out}} &:= H^2[(0,T);\{L^2(\Omega_{\text{out}})\}^N]\cap L^2[(0,T);\{H^1(\Omega_{\text{out}})\}^N].
\end{aligned}
$$

Now denoting by $\gamma_0(\mathcal{A})$ the trace of a function or field $\mathcal{A}$ on Γ and by $\gamma_1(\mathcal{A})$ the first-order normal derivative of $\mathcal{A}$ on Γ in a given but fixed sense (outwards or inwards Ω_{out}, for instance), we have

$$
(2.2) \quad \begin{cases}
\text{Find } \mathbf{e}_{\text{in}} \in \mathcal{V}_{\text{in}} \text{ and } \mathbf{e}_{\text{out}} \in \mathcal{V}_{\text{out}} \text{ such that} & \\
\varepsilon \partial_{tt} \mathbf{e}_{\text{in}} + \nabla \times \nabla \times \mathbf{e}_{\text{in}} = \mathbf{f} & \text{in } \Omega_{\text{in}} \times (0,T), \\
\partial_{tt} \mathbf{e}_{\text{out}} - \Delta \mathbf{e}_{\text{out}} = \mathbf{f} & \text{in } \Omega_{\text{out}} \times (0,T), \\
\text{satisfying at } t = 0, & \\
\mathbf{e}_{\text{in}} = \mathbf{e}_0 \text{ and } \partial_t \mathbf{e}_{\text{in}} = \mathbf{e}_1 & \text{in } \Omega_{\text{in}}, \\
\mathbf{e}_{\text{out}} = \mathbf{e}_0 \text{ and } \partial_t \mathbf{e}_{\text{out}} = \mathbf{e}_1 & \text{in } \Omega_{\text{out}}, \\
\text{together with} & \\
\gamma_0(\mathbf{e}_{\text{out}}) = \gamma_0(\mathbf{e}_{\text{in}}) \text{ and } \gamma_1(\mathbf{e}_{\text{out}}) = -\gamma_1(\mathbf{e}_{\text{in}}) & \text{on } \Gamma \times (0,T].
\end{cases}
$$

Notice that, in general, the above compatibility condition on γ_1 is only satisfied in the sense of $L^2[(0,T);\{H^{-1/2}(\Gamma)\}^N]$, since it cannot hold in a stronger form for every function of the H^1-class. Of course if it happens that $\mathbf{e}_{\text{out}} \in \{H^2(\Omega_{\text{out}})\}^N$ and $\mathbf{e}_{\text{in}} \in \{H^2(\Omega_{\text{in}})\}^N$ for all $t \in (0,T]$, then it holds in $L^2[(0,T);\{H^{1/2}(\Gamma)\}^N]$.

Whatever the case, one can easily figure out that, setting $\mathbf{e}_{\text{out}} := \mathbf{e}_{|\Omega_{\text{out}}}$ and $\mathbf{e}_{\text{in}} := \mathbf{e}_{|\Omega_{\text{in}}}$, where $\mathbf{e}$ is a solution to (2.1), then the pair $(\mathbf{e}_{\text{out}};\mathbf{e}_{\text{in}})$ solves problem (2.2). Conversely, the field $\mathbf{e}$ defined by $\mathbf{e} = \mathbf{e}_{\text{out}}$ in Ω_{out} and $\mathbf{e} = \mathbf{e}_{\text{in}}$ in Ω_{in} lies in $\mathcal{V}$ owing to the compatibility condition for γ_0, and clearly solves (2.1).

Now we observe that (2.1) implies that Maxwell's equations hold in Ω_{in}. Indeed, let u be the function $\nabla \cdot (\varepsilon \mathbf{e})$. Taking the divergence of both sides of the second equation of (2.1) we have:

$$
u_{tt} = 0 \quad \text{in } \Omega_{\text{in}} \times (0,T).
$$

Taking into account that $u_{|t=0} = u_{t|t=0} = 0$ by our assumption on $\mathbf{e}_0$ and $\mathbf{e}_1$, it must hold that $\nabla \cdot (\varepsilon \mathbf{e}) = 0$ in $\Omega_{\text{in}} \times (0,T)$. This equation, together with the second equation, of (2.1) make up Maxwell's system of equations in Ω_{in} for the sole electric field.

Notice that the same conclusion cannot be drawn for $\mathbf{e}_{\text{out}}$. This is because in $\Omega_{\text{out}} \times (0,T)$, $u := \nabla \cdot \mathbf{e}$ solves a wave equation $u_{tt} - \Delta u = 0$ with zero initial conditions. But since zero boundary conditions do not necessarily hold for u, this is not sufficient to infer that $u \equiv 0$ in Ω_{out}.

We shall prove that a solution to (2.1) exists and is unique, which is a consequence of Lemma 2.1 given below. However, before addressing the well-posedness of (2.1), it is important to make some practical considerations about this problem.

2.2. Notations. Before pursuing, we present some nonstandard notation to be used in the sequel.

Whenever no confusion is possible we shall refer to $\mathbf{n}$ as the outer normal vector to the boundary of either Ω, Ω_{out} or Ω_{in}. Similarly, the outer normal derivative on such boundaries will be denoted by $\partial_n(\cdot)$.

For any integer $l \geqslant 1$, we denote the standard norm and semi-norm of $\{W^{m,\infty}(\Omega)\}^l$ or $\{C^m(\overline{\Omega})\}^l$ by $\|\cdot\|_{m,\infty}$ and $|\cdot|_{m,\infty}$ for $m > 0$ and the standard norm of $\{L^\infty(\Omega)\}^l$ or $C^0(\overline{\Omega})$ by $\|\cdot\|_\infty$. Let us further denote by $(\cdot,\cdot)_D$ the inner product of $\{L^2(D)\}^l$ if D is a proper subset of Ω and by $(\cdot,\cdot)$ the inner product of $\{L^2(\Omega)\}^l$, and also denote by $\|\cdot\|_D$ and $\|\cdot\|$ the respective norms. Finally $\|\cdot\|_{\infty,D}$ represents the standard norm of $\{C^0(\overline{D})\}^l$ or $\{L^\infty(D)\}^l$.

2.3. Well-posedness of (2.1)**.** The well-posedness of (2.1) strongly relies on the following result:

Lemma 2.1. *Let $\mathbf{V}_\varepsilon$ be the subspace of $\mathbf{V}$ of those fields $\mathbf{v}$ satisfying $\nabla\cdot(\varepsilon\mathbf{v}) = 0$ in Ω_{in}. Denoting the characteristic function of a subset D of Ω by χ_D, the problem*

$$\text{(2.3)}\qquad \begin{cases} \textit{Given } \mathbf{g} \in \{\mathbf{L}^2(\Omega)\}^N \textit{ find } \mathbf{u} \in \mathbf{V}_\varepsilon \textit{ such that} \\ \nabla\times\nabla\times\mathbf{u} - \nabla\nabla\cdot\mathbf{u}\chi_{\Omega_{\text{out}}} = \mathbf{g} \textit{ in } \{H^{-1}(\Omega)\}^N \end{cases}$$

has a unique solution.

P r o o f. First we note that (2.3) can be recast in the following variational form:

$$\text{(2.4)}\qquad \begin{cases} \text{Given } \mathbf{g} \in \{\mathbf{L}^2(\Omega)\}^N \text{ find } \mathbf{u} \in \mathbf{V}_\varepsilon \text{ such that} \\ (\nabla\times\mathbf{u}, \nabla\times\mathbf{v}) + (\nabla\cdot\mathbf{u}, \nabla\cdot\mathbf{v})_{\Omega_{\text{out}}} = (\mathbf{g},\mathbf{v}) \quad \forall\,\mathbf{v}\in\mathbf{V}_\varepsilon. \end{cases}$$

The other way around, it is easy to see that any solution of (2.4) is also a solution to (2.3), and hence both problems are perfectly equivalent. We may then examine the well-posedness of (2.4), which we do next.

First we define the bilinear form $a_\varepsilon\colon\ \mathbf{V}\times\mathbf{V} \longrightarrow \Re$ by

$$\text{(2.5)}\qquad a_\varepsilon(\mathbf{u},\mathbf{v}) := (\nabla\times\mathbf{u}, \nabla\times\mathbf{v}) + (\nabla\cdot\{\varepsilon\mathbf{u}\}, \nabla\cdot\{\varepsilon\mathbf{v}\}).$$

Equipping $\mathbf{V}$ with the standard norm of $H^1(\Omega)$, a_ε is seen to be continuous, according to the following argument:

$$\text{(2.6)}\qquad \begin{cases} a_\varepsilon(\mathbf{u},\mathbf{v}) \leqslant \|\nabla\times\mathbf{u}\|\|\nabla\times\mathbf{v}\| \\ +\{\|\varepsilon\nabla\cdot\mathbf{u}\| + \|\nabla\varepsilon\cdot\mathbf{u}\|\}\{\|\varepsilon\nabla\cdot\mathbf{v}\| + \|\nabla\varepsilon\cdot\mathbf{v}\|\} \leqslant \|\nabla\times\mathbf{u}\|\|\nabla\times\mathbf{v}\| \\ +\{\|\varepsilon\|_{\infty,\Omega}\|\nabla\cdot\mathbf{u}\| + \|\nabla\varepsilon\|_{\infty,\Omega}\|\mathbf{u}\|\}\{\|\varepsilon\|_{\infty,\Omega}\|\nabla\cdot\mathbf{v}\| + \|\nabla\varepsilon\|_{\infty,\Omega}\|\mathbf{v}\|\}. \end{cases}$$

Setting $c_\varepsilon = \sqrt{2}\max\{\|\varepsilon\|_{\infty,\Omega}, \|\nabla\varepsilon\|_{\infty,\Omega}\}$, by straightforward calculations we obtain

$$a_\varepsilon(\mathbf{u},\mathbf{v}) \leqslant c_\varepsilon\{\|\mathbf{u}\|^2 + \|\nabla\mathbf{u}\|^2\}^{1/2}\{\|\mathbf{v}\|^2 + \|\nabla\mathbf{v}\|^2\}^{1/2} \quad \forall\,(\mathbf{u};\mathbf{v}) \in \mathbf{V}\times\mathbf{V}, \tag{2.7}$$

since $\|\nabla\mathbf{v}\|^2 = \|\nabla\cdot\mathbf{v}\|^2 + \|\nabla\times\mathbf{v}\|^2$ for all $\mathbf{v}\in\mathbf{V}$. We can also establish that $a_\varepsilon(\mathbf{u},\mathbf{v})$ defines an inner product in the space $\mathbf{V}$. Indeed, if $a_\varepsilon(\mathbf{v},\mathbf{v}) = 0$ then $\nabla\times\mathbf{v} = \mathbf{0}$. Since Ω is simply connected, there exists a function $r \in H^2(\Omega)$ with zero normal derivative on $\partial\Omega$, such that $\mathbf{v} = \nabla r$. Noticing that $\nabla\cdot(\varepsilon\mathbf{v}) = 0$, r solves the second order elliptic equation $\nabla\cdot(\varepsilon\nabla r) = 0$ with Neumann boundary conditions. Therefore r must be constant and hence $\mathbf{v} = \mathbf{0}$.

Actually, the space $\mathbf{V}$ equipped with such an inner product is a Hilbert space. This assertion is the consequence of the following triple inequalities establishing the equivalence of three norms, with constants c_1, c_2 and c_3:

$$\begin{cases} c_1(\|\mathbf{v}\|^2 + \|\nabla\mathbf{v}\|^2)^{1/2} \leqslant (\|\mathbf{v}\|^2 + a_\varepsilon(\mathbf{v},\mathbf{v}))^{1/2} \leqslant c_2(\|\mathbf{v}\|^2 + \|\nabla\mathbf{v}\|^2)^{1/2} & \forall\,\mathbf{v}\in\mathbf{V}; \\ (\|\mathbf{v}\|^2 + \|\nabla\mathbf{v}\|^2)^{1/2} \leqslant c_3\{a_\varepsilon(\mathbf{v},\mathbf{v})\}^{1/2} & \forall\,\mathbf{v}\in\mathbf{V}. \end{cases} \tag{2.8}$$

Using again the identity $\|\nabla\mathbf{v}\|^2 = \|\nabla\times\mathbf{v}\|^2 + \|\nabla\cdot\mathbf{v}\|^2$ for all $\mathbf{v}\in\mathbf{V}$, and the assumed boundedness of both ε and its gradient, by straightforward calculations we can establish the double inequality in (2.8). On the other hand, the third inequality in (2.8) can be proved by using the same arguments as in classical proofs of Second Korn's inequality (see, e.g., [17], [36]).

We also note that, as a closed subspace of $\mathbf{V}$ equipped with the norm $\|\nabla(\cdot)\|$ (or with the norm $\{a_\varepsilon(\cdot,\cdot)\}^{1/2}$), $\mathbf{V}_\varepsilon$ equipped with the inner product $(\nabla\{\cdot\},\nabla\{\cdot\})$ (or yet with $a_\varepsilon(\{\cdot\},\{\cdot\})$ is also a Hilbert space.

Now we consider the problem

$$\begin{cases} \text{Find } (\tilde{\mathbf{u}};p)\in\mathbf{V}\times L^2(\Omega_{\text{in}}) \text{ such that} \\ a_\varepsilon(\tilde{\mathbf{u}},\mathbf{v}) - (p,\nabla\cdot\{\varepsilon\mathbf{v}\})_{\Omega_{\text{in}}} = (\mathbf{g},\mathbf{v}) & \forall\,\mathbf{v}\in\mathbf{V}, \\ (\nabla\cdot\{\varepsilon\tilde{\mathbf{u}}\},q)_{\Omega_{\text{in}}} = 0 & \forall\,q\in L^2(\Omega_{\text{in}}). \end{cases} \tag{2.9}$$

It is easy to check that $\tilde{\mathbf{u}}$ solves problem (2.4). Thus, in case a solution to (2.9) exists, (2.4) will have a unique solution. Indeed let $\bar{u}$ be a solution of (2.4) for $\mathbf{g}\equiv\mathbf{0}$. Taking $\mathbf{v} = \bar{\mathbf{u}}$ in (2.4), we infer that $\nabla\times\bar{\mathbf{u}} = \mathbf{0}$ and $\nabla\cdot(\varepsilon\bar{\mathbf{u}}) = 0$ in Ω. Thus, here again there exists a function $r\in H^2(\Omega)$, which is such that $\bar{\mathbf{u}} = \nabla r$, r is constant on $\partial\Omega$ and whose first-order normal derivative $\partial_n r$ vanishes on $\partial\Omega$. It follows that r solves the equation

$$\begin{cases} \nabla\cdot(\varepsilon\nabla r) = 0 & \text{in } \Omega, \\ \partial_n r = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.10}$$

Relations (2.10) only hold if r is equal to a constant in Ω and hence $\bar{\mathbf{u}}\equiv\mathbf{0}$ as required.

Now all that is left to do is establishing the existence and uniqueness of a solution to problem (2.9). With this aim we first note that this is a classical saddle-point problem, in which p plays the role of a Lagrange multiplier. Referring to [14], we know that it has a unique solution if both conditions below are satisfied:

▷ a_ε is coercive in $\mathbf{V}_\varepsilon$

▷ The following *inf-sup* condition holds:

$$\exists\, \beta > 0 \text{ such that } \quad \forall\, q \in L^2(\Omega_{\text{in}}) \inf_{\mathbf{v}\in\mathbf{V}\backslash\{\mathbf{0}\}} \frac{(q, \nabla\cdot\{\varepsilon\mathbf{v}\})_{\Omega_{\text{in}}}}{\|\nabla\mathbf{v}\|} \geqslant \beta\|q\|_{\Omega_{\text{in}}}. \tag{2.11}$$

The coercivity of a_ε in $\mathbf{V}_\varepsilon$ trivially follows from (2.8).

Let us then prove that (2.11) holds. First of all, we associate with a given $q \in L^2(\Omega_{\text{in}})$ the function $\tilde{q} \in L^2(\Omega)$, which equals a constant c in Ω_{out}, coincides with q in Ω_{in} and fulfills $\int_\Omega \tilde{q} = 0$. It is clear that $\text{meas}(\Omega_{\text{out}})c + \int_{\Omega_{\text{in}}} q = 0$. Therefore $c = -\{\text{meas}(\Omega_{\text{out}})\}^{-1}\int_{\Omega_{\text{in}}} q$ and we have

$$\|\tilde{q}\|^2 = \frac{\{\int_{\Omega_{\text{in}}} q\}^2}{\text{meas}(\Omega_{\text{out}})} + \int_{\Omega_{\text{in}}} q^2. \tag{2.12}$$

By the Cauchy-Schwarz inequality it trivially follows that

$$\|\tilde{q}\| \leqslant \left\{\frac{\text{meas}(\Omega_{\text{in}})}{\text{meas}(\Omega_{\text{out}})} + 1\right\}^{1/2} \|q\|_{\Omega_{\text{in}}}. \tag{2.13}$$

Now we recall (see, e.g., [14]) that there exists a constant $\widetilde{C}$ independent of q, and a field $\mathbf{w} \in \mathbf{V}$ such that

$$\begin{cases} \nabla\cdot\mathbf{w} = \tilde{q} \in \Omega, \\ \|\nabla\mathbf{w}\| \leqslant \widetilde{C}\|\tilde{q}\|. \end{cases} \tag{2.14}$$

Combining (2.14) and (2.13) we find that for $\overline{C} = \widetilde{C}\{\text{meas}(\Omega_{\text{in}})/\text{meas}(\Omega_{\text{out}}) + 1\}^{1/2}$ it holds that

$$\begin{cases} \nabla\cdot\mathbf{w} = q \in \Omega_{\text{in}}, \\ \|\nabla\mathbf{w}\| \leqslant \overline{C}\|q\|_{\Omega_{\text{in}}}. \end{cases} \tag{2.15}$$

Setting $\mathbf{v} = \varepsilon^{-1}\mathbf{w}$, after straightforward calculations, $\mathbf{v}$ is seen to satisfy

$$\begin{cases} \|\nabla\mathbf{w}\| \geqslant (\widehat{C})^{-1}\|\nabla\mathbf{v}\| \text{ with} \\ \widehat{C} := \|\nabla\varepsilon\|_\infty C_{FP} + \|\varepsilon\|_\infty, \end{cases} \tag{2.16}$$

where C_{FP} is the constant of the Friedrichs-Poincaré inequality $\|v\| \leqslant C_{FP}\|\nabla v\|$ for all $v \in H_0^1(\Omega)$.

Finally, combining (2.15) and (2.16) we come up with

$$
\begin{cases} \nabla \cdot \{\varepsilon \mathbf{v}\} = q \in \Omega_{\text{in}}, \\ \|\nabla \mathbf{v}\| \leqslant \widehat{C}\overline{C}\|q\|_{\Omega_{\text{in}}}, \end{cases} \tag{2.17}
$$

which readily yields (2.11) with $\beta = (\widehat{C}\overline{C})^{-1}$.

Now, in order to complete the proof we still have to establish that (2.4), or yet (2.9), implies (2.3). With this aim, we integrate by parts on the left-hand side of the equality in (2.9). Using a self-explanatory notation we obtain

$$
\langle \nabla \times \nabla \mathbf{u} - \varepsilon \nabla \nabla \cdot (\varepsilon \mathbf{u}) + \varepsilon \nabla p \chi_{\Omega_{\text{in}}}, \mathbf{v} \rangle_{\{H^{-1}(\Omega)\}^N - \mathbf{V}} - \oint_\Gamma \varepsilon p \mathbf{v} \cdot \mathbf{n} \, \mathrm{d}\Gamma = (\mathbf{g}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}. \tag{2.18}
$$

Next we take successively $\mathbf{v} = \mathbf{w}\chi_{\Omega_{\text{out}}}$ and $\mathbf{v} = \mathbf{w}\chi_{\Omega_{\text{in}}}$, for $\mathbf{w}$ arbitrarily chosen in $\{H_0^1(\Omega_{\text{out}})\}^N$ and $\{H_0^1(\Omega_{\text{in}})\}^N$, respectively, and use the well-known identity $-\Delta \mathbf{u} = \nabla \times \nabla \times \mathbf{u} - \nabla \nabla \cdot \mathbf{u}$, to immediately obtain

$$
\begin{cases} -\Delta \mathbf{u} = \mathbf{g} & \text{in } \Omega_{\text{out}}, \\ \nabla \times \nabla \times \mathbf{u} + \varepsilon \nabla p = \mathbf{g} & \text{in } \Omega_{\text{in}}, \\ \nabla \cdot (\varepsilon \mathbf{u}) = 0 & \text{in } \Omega_{\text{in}}, \\ p = 0 & \text{on } \Gamma, \\ \mathbf{u} = \mathbf{0} & \text{on } \partial\Omega. \end{cases} \tag{2.19}
$$

The condition $p = 0$ on Γ results from (2.18) and the first two equations of (2.19).

Finally, applying the divergence operator to the second equation of (2.19), we further obtain

$$
\begin{cases} \nabla \cdot (\varepsilon \nabla p) = 0 & \text{in } \Omega_{\text{in}}, \\ p = 0 & \text{on } \Gamma, \end{cases} \tag{2.20}
$$

which implies that $p \equiv 0$. Therefore, (2.19) leads to (2.3). □

Theorem 2.1. *Problem* (2.1) *has a unique solution.*

P r o o f. Consider the same problem as (2.3), rewritten by dividing both sides of the equation by ε. Such a problem is a vector elliptic equation, which has a unique solution. Therefore, from well-known results (see, e.g., [16]), its linear second-order hyperbolic counterpart assorted with proper initial conditions also has a unique solution. This is precisely the case of (2.1), whose well-posedness is thus guaranteed. □

2.4. Variational formulation. In the previous section we avoided the explicit enforcement of the constraint $\nabla \cdot (\varepsilon \mathbf{e}) = 0$ in $\Omega_{\rm in}$ in the solution of the auxiliary stationary problem (2.3) by using the symmetric bilinear form a_ε in its equivalent variational formulation. In order to solve the time-dependent problem (2.1), we resort to a similar equivalent variational form, in which the term $(\nabla \cdot \{\varepsilon \mathbf{u}\}, \nabla \cdot \{\varepsilon \mathbf{v}\})$ is replaced by $(\nabla \cdot \{\varepsilon \mathbf{u}\}, \nabla \cdot \mathbf{v})$. The resulting formulation is (2.21), given below, which will be dealt with throughout the remainder of this article. Actually, it can be viewed as the augmented formulation of Maxwell's equations exploited in [19].

R e m a r k 2.1. We could have used the latter term in the variational formulation of the auxiliary problem (2.3). The reason why we preferred the former (symmetric) term is the fact that a_ε is coercive for any ε, while in the latter case coercivity would only hold under the assumption that both $\|\varepsilon\|_{\infty,\Omega}$ and $\|\nabla\varepsilon\|_{\infty,\Omega}$ are sufficiently small. The other way around, in the case of problem (2.1), the above augmented formulation has some nice properties from the analytical and numerical points of view, which its symmetric counterpart doesn't (cf. [10]).

Now, requiring that $\tilde{\mathbf{e}}_{|t=0} = \mathbf{e}_0$ and $\{\partial_t \tilde{\mathbf{e}}\}_{|t=0} = \mathbf{e}_1$, we wish to find $\tilde{\mathbf{e}}$ in the space $\mathcal{V}$ defined by (2.1), such that for all $\mathbf{v} \in \mathbf{V}$ it holds

$$(\varepsilon \partial_{tt} \tilde{\mathbf{e}}, \mathbf{v}) + (\nabla \tilde{\mathbf{e}}, \nabla \mathbf{v}) + (\nabla \cdot \{\varepsilon \tilde{\mathbf{e}}\}, \nabla \cdot \mathbf{v}) - (\nabla \cdot \tilde{\mathbf{e}}, \nabla \cdot \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \quad \forall\, t \in (0,T). \tag{2.21}$$

Proposition 2.1. *The following assertions hold:*

(1) *The solution to* (2.1) *is a solution to* (2.21).
(2) *Any solution to* (2.21) *is unique, and thus it is the solution to equation* (2.1).

P r o o f. (1) First we use the well known operator identity $\nabla \times \nabla \times \equiv -\Delta + \nabla\nabla\cdot$ to rewrite the first equation of (2.1) as

$$\varepsilon \partial_{tt} \mathbf{e} - \Delta \mathbf{e} + \nabla\nabla \cdot \mathbf{e} = \mathbf{f} \quad \text{in } \Omega_{\rm in} \times (0,T). \tag{2.22}$$

We know that the solution of (2.1) satisfies $\nabla\nabla \cdot (\varepsilon \mathbf{u}) = \mathbf{0}$ in $\Omega_{\rm in}$. If we subtract the above equation from (2.22), we note that the resulting equation also holds in $\Omega_{\rm out}$, since $\varepsilon = 1$ therein. Thus, taking an arbitrary $\mathbf{v} \in \mathbf{V}$ and using integration by parts, we readily obtain

$$(\varepsilon \partial_{tt} \mathbf{e}, \mathbf{v}) + (\nabla \mathbf{e}, \nabla \mathbf{v}) - (\nabla \cdot \mathbf{e}, \nabla \cdot \mathbf{v}) + (\nabla \cdot \{\varepsilon \mathbf{e}\}, \nabla \cdot \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \quad \forall\, t \in (0,T), \tag{2.23}$$

which establishes that $\mathbf{e}$ solves (2.21).

(2) In order to prove the uniqueness of a solution to (2.21) we resort to the following energy estimate, with a constant $\mathcal{E}$ independent of $\mathbf{f}$, $\mathbf{e}_0$ and $\mathbf{e}_1$:

$$\begin{aligned}(2.24)\qquad &\|\partial_t\tilde{\mathbf{e}}\|_\varepsilon^2 + \|\nabla\tilde{\mathbf{e}}\|^2 + \|\nabla\cdot\tilde{\mathbf{e}}\|_{\varepsilon-1}^2 \\ &\leqslant \mathcal{E}\Big\{\int_0^T \|\mathbf{f}(\cdot,t)\|^2\,\mathrm{d}t + \|\mathbf{e}_0\|^2 + \|\nabla\mathbf{e}_0\|^2 + \|\nabla\cdot\mathbf{e}_0\|_{\varepsilon-1}^2 + \|\mathbf{e}_1\|_\varepsilon^2\Big\}.\end{aligned}$$

The inequality (2.24) trivially derives from the energy estimate given in [6] for the variational problem analogous to (2.21) in the case of homogeneous first-order absorbing boundary conditions. The uniqueness follows from (2.24), owing to the linearity of problem (2.21). □

2.5. Complementary remarks. Before going into the numerical scheme employed here to solve (2.21) (or yet (2.1)), some important remarks are in order, namely,

(i) Throughout this paper we make the assumption that $\varepsilon \geqslant 1$ everywhere. However, strictly speaking, this was required only to derive the energy estimate (2.24) in [6]. Otherwise stated, it is a sufficient condition for the uniqueness of a solution to (2.1), while it is not necessary for its existence.

(ii) Besides the coincidence of traces γ_0, a compatibility condition on the curl of the solution $\mathbf{e}$ of (2.1) (i.e., (2.21)) can be proven to hold on Γ. Indeed, the first equation of (2.1) implies that $\nabla\times\mathbf{e} \in \mathbf{H}(\mathrm{curl},\Omega)$ for all t. Thus by a well-known result (cf. [23]) $\mathbf{n}\times\nabla\times\mathbf{e}$ is the same distribution in $\{H^{-1/2}(\Gamma)\}^N$ from both sides of Γ. Of course, if $\mathbf{e}\in\{H^2(\Omega)\}^N$ then the tangential traces of $\nabla\times\mathbf{e}$ from both sides of Γ coincide in the sense of $\{H^{1/2}(\Gamma)\}^N$ for all t.

(iii) A more interesting property states that the divergence of $\mathbf{e}$ is a function in $L^2(\Omega)$ with no jumps across Γ. Indeed, first we note that $\nabla\cdot(\varepsilon\mathbf{e}_{\mathrm{in}}) = 0$ in Ω_{in} together with $\varepsilon\equiv 1$ on Γ imply that $\gamma_0(\nabla\cdot\mathbf{e}_{\mathrm{in}})$ is well defined and equals precisely $-\gamma_0(\nabla\varepsilon\cdot\mathbf{e}_{\mathrm{in}})\in H^{1/2}(\Gamma)$. Since $\nabla\varepsilon\in C^1(\overline{\Omega})$ by assumption and $\nabla\varepsilon\equiv\mathbf{0}$ in Ω_{out}, it follows that $\gamma_0(\nabla\cdot\mathbf{e}_{\mathrm{in}})\equiv 0$.

Now let us take in (2.21) $\mathbf{v}=\nabla w$ where w is any function in $H_0^2(\Omega)$ satisfying $\gamma_0(w)\equiv 0$. For all such w we trivially obtain, $(\varepsilon\partial_{tt}\mathbf{e},\nabla w)+(\nabla\cdot\mathbf{e},\Delta w)_{\Omega_{\mathrm{out}}} = (\mathbf{f},\nabla w)$, which yields for all such functions w,

$$(\nabla\cdot\varepsilon\partial_{tt}\mathbf{e} - \nabla\cdot\Delta\mathbf{e}\chi_{\Omega_{\mathrm{out}}}, w) - \oint_\Gamma \gamma_0(\nabla\cdot\mathbf{e}_{\mathrm{out}})\partial_n w\,\mathrm{d}\Gamma = (\nabla\cdot\mathbf{f}, w)$$

after integrations by parts and straightforward calculations[1].

[1] The term related to Γ is to be understood as a real integral, under the reasonable assumption that for all t, $\mathbf{e}$ lies in $\{H^{1+\mu}(\Omega)\}^N$ for some $\mu > 1/2$. Otherwise this integral has to be replaced by the duality product between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$.

Now, by the Trace Theorem, $(\gamma_0;\gamma_1)$ is a surjection from $H_0^2(\Omega)$ onto $H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$. Hence, taking the divergence of both sides of the wave equation in Ω_{out} and using the fact that both $\varepsilon\mathbf{e}$ and $\mathbf{f}$ are solenoidal in Ω_{in}, it follows that $\gamma_0(\nabla\cdot\mathbf{e}_{\text{out}})$ also vanishes on Γ.

3. The numerical scheme

Henceforth, without loss of essential results, we restrict the presentation to the case where Ω, Ω_{in} and Ω_{out} are polygonal domains.

3.1. Semi-discretization in space. Let V_h be the usual P_1 FE-space of continuous functions related to a mesh $\mathcal{T}_h$ consisting of triangles with maximum edge length h, fitting both Ω and Ω_{in} in the sense that the closure of both domains are the union of the closure of elements in the mesh. We assume that $\mathcal{T}_h$ belongs to a quasi-uniform family of meshes (cf. [17], [26]).

Setting $\mathbf{V}_h := [V_h \cap H_0^1(\Omega)]^2$ we define $\mathbf{e}_{0h}$ (resp. $\mathbf{e}_{1h}$) to be the usual $\mathbf{V}_h$-interpolate of $\mathbf{e}_0$ (resp. $\mathbf{e}_1$). Then the semi-discretized problem in space that we wish to solve reads:

Find $\mathbf{e}_h \in \mathbf{V}_h$ such that for all $\mathbf{v} \in \mathbf{V}_h$

$$
\begin{aligned}
&(\varepsilon\partial_{tt}\mathbf{e}_h,\mathbf{v}) + (\nabla\mathbf{e}_h,\nabla\mathbf{v}) + (\nabla\cdot[\varepsilon\mathbf{e}_h],\nabla\cdot\mathbf{v}) - (\nabla\cdot\mathbf{e}_h,\nabla\cdot\mathbf{v}) = 0,\\
&\qquad \mathbf{e}_h(\cdot,0) = \mathbf{e}_{0h}(\cdot) \quad \text{and} \quad \partial_t\mathbf{e}_h(\cdot,0) = \mathbf{e}_{1h}(\cdot) \quad \text{in } \Omega.
\end{aligned}
\tag{3.1}
$$

3.2. Full discretization. To begin with, we consider a natural centered time-discretization scheme to solve (3.1), namely: Given a number M of time steps we define the time increment $\tau := T/M$. Then we approximate $\mathbf{e}_h(k\tau)$ by $\mathbf{e}_h^k \in \mathbf{V}_h$ for $k = 1,2,\ldots,M$ according to the following scheme for $k = 1,2,\ldots,M-1$:

$$
\begin{aligned}
&\Big(\varepsilon\frac{\mathbf{e}_h^{k+1} - 2\mathbf{e}_h^k + \mathbf{e}_h^{k-1}}{\tau^2},\mathbf{v}\Big) + (\nabla\mathbf{e}_h^k,\nabla\mathbf{v})\\
&\qquad\qquad + (\nabla\cdot\varepsilon\mathbf{e}_h^k,\nabla\cdot\mathbf{v}) - (\nabla\cdot\mathbf{e}_h^k,\nabla\cdot\mathbf{v}) = 0 \quad \forall\,\mathbf{v}\in\mathbf{V}_h,\\
&\mathbf{e}_h^0 = \mathbf{e}_{0h} \quad \text{and} \quad \mathbf{e}_h^1 = \mathbf{e}_h^0 + \tau\mathbf{e}_{1h} \quad \text{in } \Omega.
\end{aligned}
\tag{3.2}
$$

Owing to its coupling with $\mathbf{e}_h^k$ and $\mathbf{e}_h^{k-1}$ on the left-hand side of (3.2), $\mathbf{e}_h^{k+1}$ cannot be determined explicitly by (3.2) at every time step. In order to enable an explicit solution, we resort to the classical mass-lumping technique. We recall that for a constant ε this consists of replacing on the left-hand side the inner product $(\varepsilon\mathbf{u},\mathbf{v})$ by a discrete inner product $(\varepsilon\mathbf{u},\mathbf{v})_h$, using the trapezoidal rule to compute the integral of $\int_K \varepsilon\mathbf{u}_{|K}\cdot\mathbf{v}_{|K}\,d\mathbf{x}$ for every element K in $\mathcal{T}_h$, where $\mathbf{u}$ stands for $\mathbf{e}_h^{k+1} - 2\mathbf{e}_h^k + \mathbf{e}_h^{k-1}$.

It is well known that in this case the matrix associated with $(\varepsilon \mathbf{e}_h^{k+1}, \mathbf{v})_h$ for $\mathbf{v} \in \mathbf{V}_h$, is a diagonal matrix. In our case ε is not constant, but the same property will hold if we replace in each element K the integral of $\varepsilon \mathbf{u}_{|K} \cdot \mathbf{v}_{|K}$ in a triangle $K \in \mathcal{T}_h$ as follows:

$$\int_K \varepsilon \mathbf{u}_{|K} \cdot \mathbf{v}_{|K} \, \mathrm{d}\mathbf{x} \approx \varepsilon(G_K) \mathrm{area}(K) \sum_{i=1}^{3} \frac{\mathbf{u}(S_{K,i}) \cdot \mathbf{v}(S_{K,i})}{3},$$

where $S_{K,i}$ are the vertexes of K, $i = 1, 2, 3$, G_K is the centroid of K.

Before proceeding, we define the auxiliary function ε_h whose value in each $K \in \mathcal{T}_h$ is constant equal to $\varepsilon(G_K)$. Then, still denoting the approximation of $\mathbf{e}_h(k\tau)$ by $\mathbf{e}_h^k$, for $k = 1, 2, \ldots, M$ we determine $\mathbf{e}_h^{k+1}$ by

$$\begin{aligned} &\left(\varepsilon_h \frac{\mathbf{e}_h^{k+1} - 2\mathbf{e}_h^k + \mathbf{e}_h^{k-1}}{\tau^2}, \mathbf{v}\right)_h + (\nabla \mathbf{e}_h^k, \nabla \mathbf{v}) \\ &\qquad + (\nabla \cdot \varepsilon \mathbf{e}_h^k, \nabla \cdot \mathbf{v}) - (\nabla \cdot \mathbf{e}_h^k, \nabla \cdot \mathbf{v}) = 0 \quad \forall \, \mathbf{v} \in \mathbf{V}_h, \\ &\mathbf{e}_h^0 = \mathbf{e}_{0h} \quad \text{and} \quad \mathbf{e}_h^1 = \mathbf{e}_h^0 + \tau \mathbf{e}_{1h} \quad \text{in } \Omega. \end{aligned} \tag{3.3}$$

Remark 3.1. As long as the finite element subspaces are suitably adapted, our methodology works as well, as applied to other boundary conditions for which the wave equation in Ω_{out} is equivalent to Maxwell's equations with a constant dielectric permittivity. In the case of zero tangential-component boundary conditions, existence of a solution to the resulting Maxwell's equation in the whole Ω is guaranteed (cf. [1]). On the other hand, uniqueness and equivalence can be established under the supplementary (Neumann) condition, that the divergence of the electric field vanishes on $\partial\Omega$.

3.3. Stability and error estimates.

Let us first set

$$\eta := 2 + |\varepsilon|_{1,\infty} + 2|\varepsilon|_{2,\infty}. \tag{3.4}$$

Next we recall the classical inverse inequality (cf. [17]) together with a result in [15] according to which

$$\|\nabla v\| \leqslant C_\varepsilon h^{-1} \|(\varepsilon_h)^{1/2} v\|_h \quad \forall \, v \in V_h, \tag{3.5}$$

where C_ε is a mesh-independent constant.

The explicit scheme (3.3) is stable in the same sense specified in [10], under the assumption that τ satisfies the following CFL-condition:

$$\tau \leqslant \min\{h/\nu, 1/(2\eta)\} \text{ with } \nu = C_\varepsilon (1 + 3\|\varepsilon - 1\|_\infty)^{1/2}. \tag{3.6}$$

Next we further assume that the solution $\mathbf{e}$ to equation (2.1) belongs to $[H^4\{\Omega \times (0,T)\}]^2$.

Let us define a function $\mathbf{e}_h$ in $\overline{\Omega} \times [0,T]$ whose value at $t = k\tau$ equals $\mathbf{e}_h^k$ for $k = 1,2,\ldots,M$ and that varies linearly with t in each time interval $([k-1]\tau, k\tau)$, in such a way that $\partial_t \mathbf{e}_h(\mathbf{x},t) = (\mathbf{e}_h^k(\mathbf{x}) - \mathbf{e}_h^{k-1}(\mathbf{x}))/\tau$ for every $\mathbf{x} \in \overline{\Omega}$ and $t \in ([k-1]\tau, k\tau)$. We also define $\mathbf{a}^{m+1/2}(\cdot)$ for any field $\mathbf{a}(\cdot,t)$ to be $\mathbf{a}(\cdot,[m+1/2]\tau)$.

Theorem 3.1. *Assume that Ω is convex. Provided the CFL condition* (3.6) *is fulfilled, under the above regularity assumption on* $\mathbf{e}$, *there exists a constant $\mathcal{C}$ depending only on Ω, ε and T such that*

$$\tag{3.7} \begin{aligned} \max_{1\leqslant m\leqslant M-1} \|[\partial_t(\mathbf{e}_h - \mathbf{e})]^{m+1/2}\| + \max_{2\leqslant m\leqslant M} \|\nabla(\mathbf{e}_h^m - \mathbf{e}^m)\| \\ \leqslant \mathcal{C}(\tau + h + h^2/\tau)\{\|\mathbf{e}\|_{H^4[\Omega\times(0,T)]} + |\mathbf{e}_0|_2 + |\mathbf{e}_1|_2\}. \end{aligned}$$

The proof of Theorem 3.1 is nothing but a simplification of the proof of an identical error estimate given in [10] for the same problem as (2.21), except for the fact that absorbing boundary conditions are prescribed instead of Dirichlet boundary conditions.

In short, (3.7) means that, as long as τ varies linearly with h, the error of the numerical solution generated by scheme (3.3) measured in the norms on the left-hand side of (3.7) goes to zero roughly proportionally to either τ or h.

Remark 3.2. The regularity assumption on $\mathbf{e}$ of Theorem 3.1 is certainly too strict. It is easy to see from the analysis given in [10] that it can be weakened by replacing it with

$$\mathbf{e}_{|\Omega_{\text{in}}} \in [H^4\{\Omega_{\text{in}} \times (0,T)\}]^2$$

and

$$\mathbf{e}_{|\Omega_{\text{out}}} \in [H^4\{\Omega_{\text{out}} \times (0,T)\}]^2.$$

Actually, presumably this is still stringent, as suggested by the numerical experiments reported in the next section.

Remark 3.3. The case of a nonconvex domain is not a real problem, because the corner paradox does not come into play here. This is because we are dealing with boundary conditions which do not couple the components of the electric field.

However, if the domain has re-entrant corners, the customary convergence-rate downgrade from one to μ with $0 < \mu < 1$ is to be expected.

Remark 3.4. Another issue that is worth a comment is the practical calculation of the term $(\nabla\cdot\varepsilon\mathbf{e}_h^k, \nabla\cdot\mathbf{v})$ in (3.3). Unless ε is a simple function such as a polynomial, it is not possible to compute this term exactly. That is why we advocate the use of the trapezoidal rule do carry out these computations. At the price of small adjustments in some terms involving norms of ε, the thus modified scheme remains stable in the same sense as before. Moreover the qualitative convergence result (3.7) also holds, provided we require a little more regularity from ε. We skip the details for the sake of brevity.

4. Numerical validation

In this section, we proceed to the validation of scheme (3.3). More precisely, scheme (3.3) is used to solve (2.1) for $T = 0.5$, $\Omega = (0,1) \times (0,1)$, with $\mathbf{e} = (e_1, e_2)$.

Similarly to [11], the source data $\mathbf{f}$ is chosen in such a way that the components of the exact solution are given by

$$\begin{aligned} e_1 &= \frac{1}{\varepsilon} 2\pi \sin^2 \pi x \cos \pi y \sin \pi y \frac{t^2}{2}, \\ e_2 &= -\frac{1}{\varepsilon} 2\pi \sin^2 \pi y \cos \pi x \sin \pi x \frac{t^2}{2}. \end{aligned} \tag{4.1}$$

In (4.1) the function ε is defined to be

$$\varepsilon(x,y) = \begin{cases} 1 + \sin^m \pi(2x - 0.5) \cdot \sin^m \pi(2y - 0.5) & \text{in } [0.25, 0.75] \times [0.25, 0.75], \\ 1 & \text{otherwise,} \end{cases} \tag{4.2}$$

where m is an integer, $m > 1$. In Figure 2, ε is depicted for different values of m.

The solution given by (4.1) satisfies homogeneous initial conditions together with homogeneous Dirichlet conditions on the boundary $\partial\Omega$ of the square Ω for every time t. In our computations, we used the software package WavES [7] only for the finite element method applied to the solution of (2.1). We note that this package was also used in [6] to solve the same model problem by a domain decomposition FEM/FDM method.

We discretized the computational domain $\Omega \times (0, T]$ using a partition of the spatial domain Ω into triangles of sizes $h_l = 2^{-l}$, $l = 1, \ldots, 6$. and a partition of the time domain $(0, T]$ into intervals $(t_{k-1}, t_k]$ of uniform length τ_l for a given number of time intervals N, $l = 1, \ldots, 6$. We choose the time step $\tau_l = 0.025 \times 2^{-l}$, $l = 1, \ldots, 6$, which provides numerical stability for all meshes.

In [11] we performed numerical tests taking $m = 2$ and $m = 7$. Here we push further these experiments by supplying results taking $m = 3, 4, 5, 6, 8, 9$ in (4.2). We computed the maximum value over the time steps of the relative errors measured in the L_2-norm and the H^1-semi-norm and in the L_2 norm for the time-derivative,

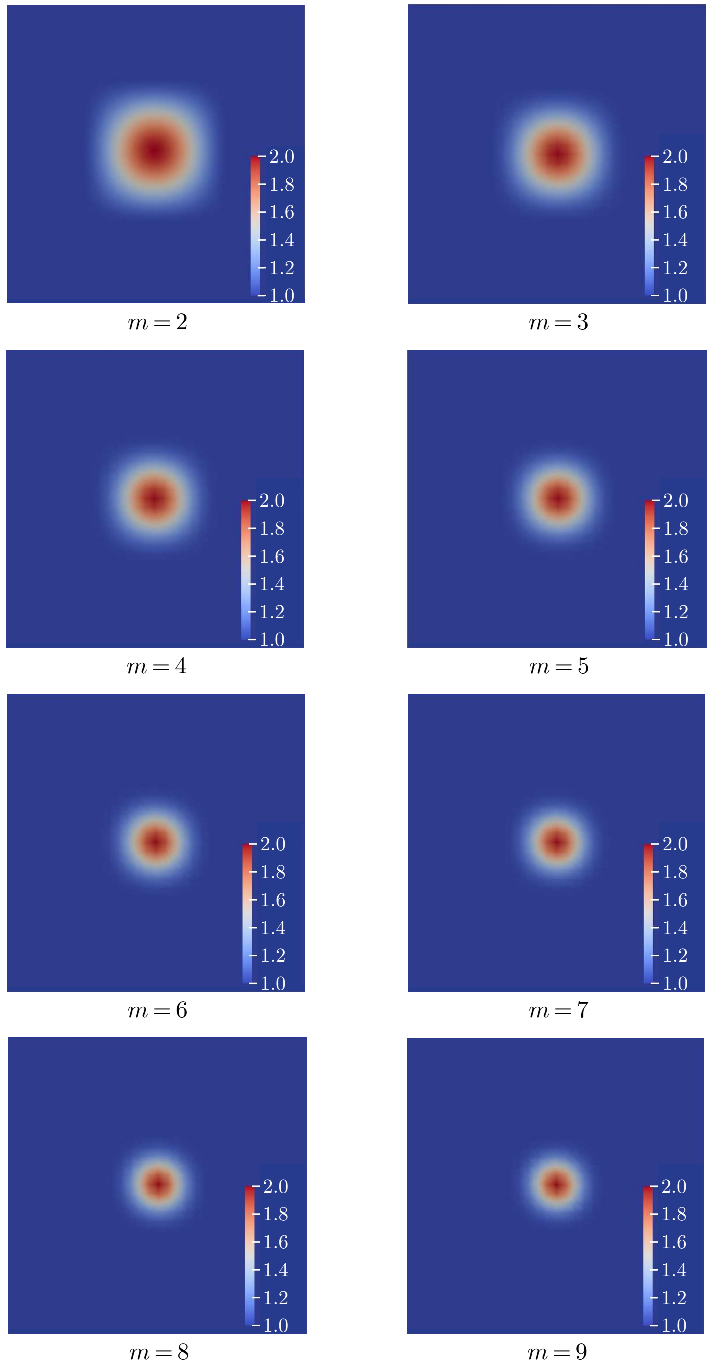


Figure 2. Function $\varepsilon(x, y)$ in the domain $\Omega = (0, 1) \times (0, 1)$ for different values of m in (4.2).

respectively represented by

$$e_l^1 = \frac{\max\limits_{1\leqslant k\leqslant N} \|\mathbf{e}^k - \mathbf{e}_h^k\|}{\max\limits_{1\leqslant k\leqslant N} \|\mathbf{e}^k\|}, \quad e_l^2 = \frac{\max\limits_{1\leqslant k\leqslant N} \|\nabla(\mathbf{e}^k - \mathbf{e}_h^k)\|}{\max\limits_{1\leqslant k\leqslant N} \|\nabla\mathbf{e}^k\|},$$
$$e_l^3 = \frac{\max\limits_{1\leqslant k\leqslant N-1} \|\{\partial_t(\mathbf{e} - \mathbf{e}_h)\}^{k+1/2}\|}{\max\limits_{1\leqslant k\leqslant N-1} \|\{\partial_t\mathbf{e}\}^{k+1/2}\|}. \tag{4.3}$$

Here, $\mathbf{e}_h$ is the computed solution, while $N = T/\tau_l$.

In Tables 1–2 convergence in these three senses can be observed, taking $m = 3, 6$ in (4.2). Similar results were obtained for $m = 2, 4, 5, 7, 8, 9$. The acronyms nel and nno stand for *number of elements* and *number of nodes* therein. Figure 3 shows convergence rates of our numerical scheme based on a P_1 space discretization, taking the function ε defined by (4.2) with $m = 2$ (on the left) and $m = 3$ (on the right) for $\varepsilon(x)$. Roughly the same convergence rates are observed in Figures 4 and 5 taking $m = 6, 7, 8, 9$ in (4.2).

l	nel	nno	e_l^1	e_{l-1}^1/e_l^1	e_l^2	e_{l-1}^2/e_l^2	e_l^3	e_{l-1}^3/e_l^3
1	8	9	0.043394		0.2784		1.0869	
2	32	25	0.011451	3.789538	0.1098	2.5355	0.5305	2.0488
3	128	81	0.003343	3.425366	0.06	1.83	0.2586	2.0514
4	512	289	0.000781	4.385873	0.0248	2.4194	0.1306	1.9801
5	2048	1089	0.000202	3.866337	0.0119	2.0840	0.0654	1.9969
6	8192	4225	0.000052	3.884615	0.0059	2.0169	0.0327	2

Table 1. Maximum over the time steps of relative errors in the L_2-norm, in the H^1-seminorm and in the L^2-norm of the time-derivative for mesh sizes $h_l = 2^{-l}, l = 1, \ldots, 6$ taking $m = 3$ in (4.2).

l	nel	nno	e_l^1	e_{l-1}^1/e_l^1	e_l^2	e_{l-1}^2/e_l^2	e_l^3	e_{l-1}^3/e_l^3
1	8	9	0.054228		0.2837		1.1120	
2	32	25	0.012241	4.430030	0.0906	3.1313	0.4937	2.2524
3	128	81	0.002973	4.117389	0.0408	2.2206	0.2665	1.8525
4	512	289	0.000590	5.038983	0.0150	2.7200	0.1335	1.9963
5	2048	1089	0.000163	3.619631	0.0079	1.8987	0.0667	2.0015
6	8192	4225	0.000043	3.790698	0.0040	1.9750	0.0334	1.9970

Table 2. Maximum over the time steps of relative errors in the L_2-norm, in the H^1-seminorm and in the L^2-norm of the time-derivative for mesh sizes $h_l = 2^{-l}, l = 1, \ldots, 6$ taking $m = 6$ in (4.2).

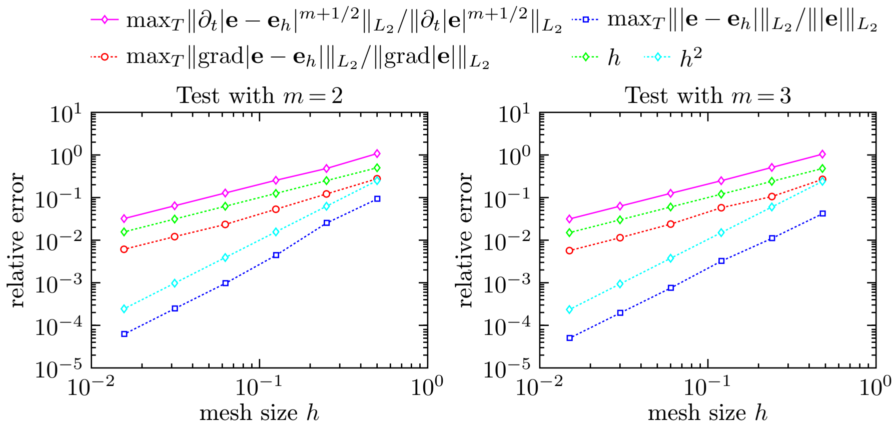


Figure 3. Maximum in time of relative errors for $m = 2$ (left) and $m = 3$ (right).

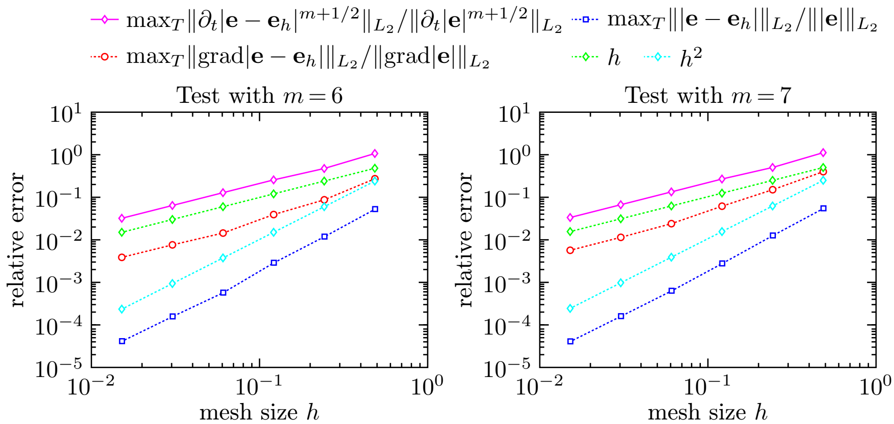


Figure 4. Maximum in time of relative errors for $m = 6$ (left) and $m = 7$ (right).

These tables and figures clearly indicate that our scheme behaves like a first order method in the (semi-)norm of $L^\infty[(0,T);H^1(\Omega)]$ for $\mathbf{e}$ and in the norm of $L^\infty[(0,T);L^2(\Omega)]$ for $\partial_t\mathbf{e}$ for all the chosen values of m. As far as the values of m greater or equal to 6 are concerned this perfectly conforms to the a priori error estimates (3.7). However, those tables and figures also show that such theoretical predictions extend to estimates not included in our analysis, such as $m = 2$ and $m = 3$, in which the regularity of the exact solution is lower than assumed. In view of this, it turns out that some of our assumptions seem to be of academic interest only and a lower regularity of the solution such as $H^2[\Omega \times (0,T)]$ should be sufficient to attain optimal first-order convergence in both senses. On the other hand, second-order convergence can be expected from our scheme in $L^\infty[(0,T);L^2(\Omega)]$, according to Tables 1–2 and Figures 2–4, even though we did not establish error estimates in the norm of this space.

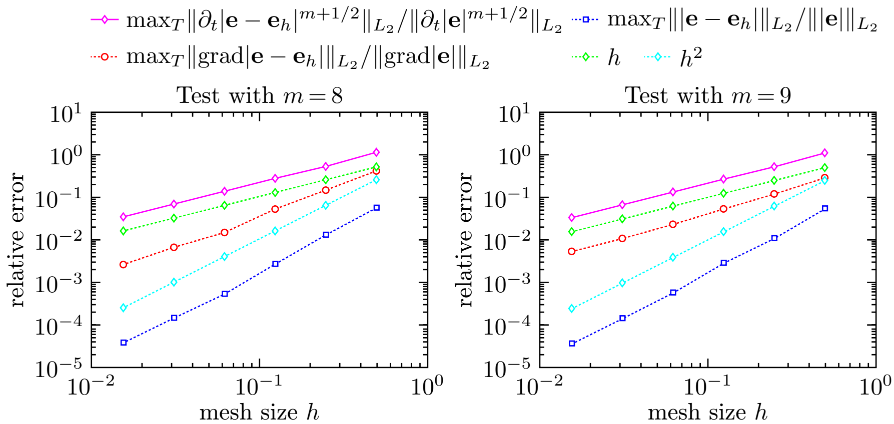


Figure 5. Maximum in time of relative errors for $m = 8$ (left) and $m = 9$ (right).

5. Conclusions and final remarks

In this section we draw the main conclusions from the results presented in the Sections 2, 3 and 4, followed by a few important remarks.

5.1. Conclusions. In this work we validated the extension of the reliability analysis conducted in [10] for a numerical scheme to solve Maxwell's equations in terms of the sole electric field to a Maxwell-wave equation coupling problem (2.1), for a combination of an explicit finite-difference time discretization with a lumped-mass P_1 finite-element space discretization. The scheme is effective in the case where the dielectric permittivity is not constant in a sub-domain completely surrounded by an outer layer, in which the wave equation holds. The problem's well-posedness and equivalence with a suitable variational form are addressed, in the particular case where Dirichlet boundary conditions are prescribed for the unknown field. We conjecture that our qualitative results apply as well to the case of Neumann boundary conditions, and eventually to the case of zero boundary conditions for both the divergence and the tangential components of the unknown field. Notice that in the latter case the coupling problem reduces to Maxwell's equation in the whole domain.

A detailed description of the numerical scheme in [10] adapted to (2.1) was given, together with underlying a priori error estimates, under suitable regularity assumptions. Then we showed that such convergence results are confirmed in practice by means of numerical experiments performed for a test-problem in two-dimension space with known exact solution. Furthermore we presented convincing evidence that our theoretical predictions extend to solutions with much lower regularity than the one assumed in our analysis. Similarly, optimal second-order convergence is observed in a norm other than those in which convergence is formally established. In short, we undoubtedly indicated that the Maxwell-wave equation coupling problem can be efficiently solved with classical conforming P_1 finite elements in relevant particular cases, among which lies the model problem (2.1). As a by-product, this conclusion extends to the case where Maxwell's equations hold in the whole problem definition domain, as long as the the dielectric permittivity is constant in an outer domain.

5.2. Final remarks. (1) The methodology and the results addressed in Sections 3 and 4 can be easily adapted to the case of polyhedral domains of the same kind. In this respect the authors refer to their paper [10], which focuses on the three-dimensional case.

(2) It is also noteworthy that the qualitative results given in Section 4 extend to the case of curved domains, at the price of customary minor modifications. Roughly speaking, if Ω is convex, it suffices to replace the norm $\|\cdot\|$ by the norm of $L^2(\Omega_h)$, where Ω_h is the union of the elements in $\mathcal{T}_h$. If Ω is not convex, the same conclusion holds, as long as the exact solution is suitably extended to $\Omega_h \setminus \Omega$.

(3) An interesting feature of problem (2.1) is the fact that it suggests the use of a simple solution method in Ω_{out} and a more sophisticated one in Ω_{in}. This is particularly handy in the case where ε varies abruptly in Ω_{in}, which is often the case of coefficient inverse problems governed by Maxwell's equations (see, e.g., [9], [8], [29], [34], [35] and [28], [27]). Referring to [2], [3], assume for instance that Ω is a rectangular domain, and that Ω_{in} is also rectangular (cf. Figure 1 (b)). In this case, a straightforward solution of the wave equation in Ω_{out} by the finite difference method on a structured grid can be achieved, as long as conveniently approximated values of $\mathbf{e}$ on Γ are available. On the other hand, if such values are known, Maxwell's equations in Ω_{in} can be solved by a more elaborate procedure, in order to capture sharp gradients of the solution, owing to a "wild" behavior of the function ε therein. It is well known that one of the best ways to handle such a situation is to use the adaptive finite element method. This actually suggests that we first compute an approximate solution in the whole Ω, by using the finite difference method on a structured grid in Ω_{in} as well. To simplify things, the latter grid could have the same grid points on the interface frame Γ as the grid constructed for Ω_{out}. Such an initial solution can be interpolated at the nodes of an underlying triangular or tetrahedral mesh, whose nodes coincide with the grid points in Ω_{in}. From this point a classical adaptive finite element procedure can start in Ω_{in} for solving Maxwell's equations, by prescribing the solution values on Γ to be those computed at the initial step. We refer to [9], [8], [29] and [28], [27] for more details.

(4) An issue strongly connected to the previous remark is the time discretization. Indeed, we have chosen a fully explicit scheme for time integration, precisely because otherwise the adaptive solution would become too costly. Furthermore, in the framework of the iterative solution of the underlying coefficient inverse problem, the fact that ε changes at every iteration advises against the use of implicit time integrations schemes, since in this case a linear system of algebraic equations with a new matrix has to be solved at every iteration.

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