GLOBAL EXPONENTIAL STABILITY OF PSEUDO ALMOST AUTOMORPHIC SOLUTIONS FOR DELAYED COHEN-GROSBERG NEURAL NETWORKS WITH MEASURE

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Abstract. We investigate the Cohen-Grosberg differential equations with mixed delays and time-varying coefficient: Several useful results on the functional space of such functions like completeness and composition theorems are established. By using the fixed-point theorem and some properties of the doubly measure pseudo almost automorphic functions, a set of sufficient criteria are established to ensure the existence, uniqueness and global exponential stability of a (μ,ν)-pseudo almost automorphic solution. The theory of this work generalizes the classical results on weighted pseudo almost automorphic functions. Finally, a numerical example is provided to illustrate the validity of the proposed theoretical results.

Keywords: pseudo almost automorphic solution; double measure; mixed delays

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1. Introduction

Bochner was the first to introduce the concept of almost automorphy in [5]. As a generalization of both the classical almost automorphy and the pseudo almost periodicity introduced by Zhang [14], [15], the pseudo almost automorphy was developed by Liang et al. [9], [13]. Blot et al. have introduced a new approach dealing with the weighted pseudo almost automorphic functions using measure theory noted (μ-pseudo almost automorphic functions) as a more general concept than classical concept of weighted pseudo almost automorphy [1].

Lately, the authors of [4] have proposed the weighted pseudo almost automorphic functions as an extension of pseudo almost automorphic functions. The authors of [3] focused on the existence, uniqueness and global exponential stability of pseudo

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almost automorphic solutions of recurrent neural networks with time varying coefficients and mixed delays. The authors of [8] investigated the problem of almost automorphic solutions of Cohen-Grosberg with time varying coefficients. Regarding [2], authors investigated the problem of global exponential stability of asymptotic almost automorphic solutions.

More recently, in [1], [6], the notion of (μ, ν)-pseudo almost automorphic is introduced as a generalization of μ-pseudo almost automorphic and fundamental properties of measure pseudo almost automorphic functions established. Therefore, the notions of pseudo almost automorphic functions and weighted pseudo almost automorphic functions become particular cases of this theory. In [10], Miraoui and Yaakoubi studied the existence, uniqueness, and the exponential stability of the measure pseudo almost periodic solution of shunting neural networks with mixed delays. The existence of (μ_1, μ_2)-pseudo almost automorphic of fractional differential equations was established in [7].

Motivated by the above discussion, in our article, we introduce the concept of (μ, ν)-pseudo almost automorphic functions. Then we establish new approach dealing with the existence, uniqueness and global exponential stability of (μ, ν)-pseudo almost automorphic solutions of the following differential equations:

$$\begin{aligned}(1)\qquad x_i'(t) = & -a_i(x_i(t))\Big[b_i(x_i(t)) - \sum_{j=1}^{n} o_{ij}(t) f_j(t, x_j(t)) \\ & - \sum_{j=1}^{n} w_{ij}(t) g_j(t, x_j(t - \tau_{ij}(t))) \\ & - \sum_{j=1}^{n} d_{ij}(t) \int_{-\infty}^{t} k_{ij}(t-s) h_j(s, x_j(s))\,\mathrm{d}s - L_i(t)\Big], \quad 1 \leqslant i \leqslant n;\end{aligned}$$

- ▷ n is the number of neurons,
- ▷ $x(\cdot) = (x_1(\cdot), \dots, x_n(\cdot))^\top \in \mathbb{R}^n$ denotes the state of the neuron,
- ▷ $a_i(\cdot)$ is the amplification function,
- ▷ $b_i(\cdot) > 0$ is the appropriately behaved function,
- ▷ the functions $o_{ij}(\cdot)$, $w_{ij}(\cdot)$ and $d_{ij}(\cdot)$ denote respectively the connection weights, the discretely delayed connection weights and the distributively delayed connection weights,
- ▷ f_j, g_j and h_j are the activation functions of the jth neuron,
- ▷ $L_i(t)$ is the external bias on the ith neuron,
- ▷ for all $1 \leqslant i$, $j \leqslant n$, $\tau_{ij}(\cdot) \in [0, \tau^+]$ corresponds to the transmission time varying delay.

The main object of this paper is to establish sufficient new conditions to ensure the existence, uniqueness and global exponential stability of (μ,ν)-pseudo almost automorphic solution. To our best knowledge, there are no works about (μ,ν)-pseudo almost automorphic Cohen-Grosberg differential equations with time varying-coefficients using the measure theory.

The rest of this paper is organized as follows: In Section 2, we present some preliminary results. These results play an important role in Section 3, where we give new conditions for the existence, uniqueness and global exponential stability of doubly measure pseudo almost automorphic solutions of (1) in the suitable convex set of (μ,ν)-pseudo almost automorphic solutions. In Section 4, a numerical example is given to illustrate the feasibility of the results. In Section 5, we end up drawing meaningful conclusions.

2. Preliminaries

In this section, we introduce some basic notations, definitions, preliminary results and assumptions.

▷ For a bounded and continuous function h defined on $\mathbb{R}$, we notice

$$h^* = \sup_{t\in\mathbb{R}} |h(t)|, \quad h_* = \inf_{t\in\mathbb{R}} |h(t)|.$$

▷ The set of bounded continuous functions from $\mathbb{R}$ to $\mathbb{R}^n$ is denoted by $BC(\mathbb{R},\mathbb{R}^n)$.
▷ $\mathcal{B}$ is the Lebesgue σ-field of $\mathbb{R}$.
▷ $\mathcal{M}$ is the set of positive measures ν on $\mathcal{B}$ such that $\nu(\mathbb{R}) = \infty$ and $\nu([e,c]) < \infty$ for all $e, c \in \mathbb{R}$, $e \leqslant c$.
▷ For $\mu,\nu \in \mathcal{M}$, the ergodic space is defined as

$$\mathcal{E}(\mathbb{R},\mathbb{R}^n,\mu,\nu) = \left\{ g \in BC(\mathbb{R},\mathbb{R}^n);\ \lim_{\varrho\to\infty} \frac{1}{\nu([-\varrho,\varrho])} \int_{-\varrho}^{\varrho} |g(t)|\,\mathrm{d}\mu(t) = 0 \right\},$$

where $\nu([-\varrho,\varrho]) := \int_{-\varrho}^{\varrho} \mathrm{d}\nu(t)$.
▷ $\Lambda = \{1,\dots,n\}$.

Definition 1 ([6])**.** Let $g \in BC(\mathbb{R},\mathbb{R}^n)$. We say that g is almost automorphic if for any sequence of real numbers $\{s_n\}_{n=1}^{\infty}$ there exists a subsequence $\{\sigma_n\}_{n=1}^{\infty} \subset \{s_n\}_{n=1}^{\infty}$ such that $f(t) := \lim_{n\to\infty} g(t+\sigma_n)$ is well defined for each real t and

$$\lim_{n\to\infty} f(t-\sigma_n) = g(t) \quad \forall\, t \in \mathbb{R}.$$

Denote by $\mathcal{AA}(\mathbb{R},\mathbb{R}^n)$ the set of all almost automorphic functions from $\mathbb{R}$ to $\mathbb{R}^n$.

R e m a r k 1 ([11]). Definition 1 can be given in the following form:

Let $g \in BC(\mathbb{R}, \mathbb{R}^n)$. We say that g is almost automorphic if for any sequence of real numbers $\{s_n\}_{n=1}^{\infty}$ there exists a subsequence $\{\sigma_n\}_{n=1}^{\infty} \subset \{s_n\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} \lim_{m\to\infty} g(t + \sigma_n - \sigma_m) = g(t) \quad \forall\, t \in \mathbb{R}.$$

R e m a r k 2. We observe that the function f in Definition 1 is from $\mathbb{R}$ into $\mathbb{R}^n$ measurable but not necessarily continuous. Moreover, if f is continuous, then g is uniformly continuous. If the convergence in both limits in Definition 1 is uniform on $\mathbb{R}$, the function g is said to be almost periodic (in Bochner's sense) [12], [16].

Definition 2 ([6]). Let $\mu, \nu \in \mathcal{M}$, a function $g \in \mathcal{C}(\mathbb{R}, \mathbb{R}^n)$ is said to be doubly measure pseudo almost automorphic or (μ, ν)-pseudo almost automorphic if it can be expressed as $g = g_1 + g_0$, $g_1 \in \mathcal{AA}(\mathbb{R}, \mathbb{R}^n)$ and $g_0 \in \mathcal{E}(\mathbb{R}, \mathbb{R}^n, \mu, \nu)$. Denote by $\mathcal{PAA}(\mathbb{R}, \mathbb{R}^n, \mu, \nu)$ the set of all such functions.

Definition 3 ([1]). A continuous function $\phi\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is said to be almost automorphic in t uniformly with respect to the second variable x if the following conditions are true:

(1) For all $x \in \mathbb{R}$, $\phi(\cdot, x) \in \mathcal{AA}(\mathbb{R}, \mathbb{R})$.

(2) ϕ is uniformly continuous on each compact set K in $\mathbb{R}$ with respect to the second variable x.

Denote by $AAU(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ the collection of such functions.

Definition 4 ([1]). Let $\mu, \nu \in \mathcal{M}$. The continuous function $\phi\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is said to be (μ, ν)-ergodic in t uniformly with respect to the second variable $x \in \mathbb{R}$ if the following conditions hold:

(1) For all $y \in \mathbb{R}$, $\phi(\cdot, y) \in \mathcal{E}(\mathbb{R}, \mathbb{R}, \mu, \nu)$.

(2) ϕ is uniformly continuous on each compact set K in $\mathbb{R}$ with respect to the second variable y.

Denote by $\mathcal{E}U(\mathbb{R} \times \mathbb{R}, \mathbb{R}, \mu, \nu)$ the space of all such functions.

Definition 5 ([1]). Let $\mu, \nu \in \mathcal{M}$. The continuous function $\phi\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is said to be (μ, ν)-pseudo almost automorphic if it can be expressed as $\phi = \varphi + \psi$, where $\varphi \in \mathcal{AAU}(\mathbb{R}\times\mathbb{R}, \mathbb{R})$ and $\psi \in \mathcal{E}U(\mathbb{R}\times\mathbb{R}, \mathbb{R}, \mu, \nu)$. We denote by $\mathcal{PAAU}(\mathbb{R}\times\mathbb{R}, \mathbb{R}, \mu, \nu)$ the space of all such functions.

E x a m p l e 1. Let $\mu, \nu \in \mathcal{M}$, the following function is a classical example of (μ, ν)-pseudo almost automorphic function

$$h(s) = \cos \frac{1}{1 + \cos(\sqrt{3}s) + \sin(\sqrt{5}s)} + \frac{1}{1 + s^2}, \quad s \in \mathbb{R}.$$

Let us formulate the following conditions:

(H_1) For all $\varsigma \in \mathbb{R}$ there exist $\gamma > 0$ and a bounded interval U such that

$$\mu(\{c+\varsigma\colon\ c \in C\}) \leqslant \gamma\mu(C);\ C \in \mathcal{B} \text{ satisfies } C \cap U = \emptyset.$$

(H_2) For all $\mu, \nu \in \mathcal{M}$, $\limsup_{\varrho\to\infty} \mu[-\varrho,\varrho]/\nu[-\varrho,\varrho] < \infty$.

Lemma 1 ([6]). *If $g, h \in \mathcal{AA}(\mathbb{R},\mathbb{R})$ and if $\varpi \in \mathbb{R}$, then we have $g+h$, ϖg, $gh \in \mathcal{AA}(\mathbb{R},\mathbb{R})$.*

Lemma 2 ([6]). *Let $\mu, \nu \in \mathcal{M}$, under conditions* (H_1) *and* (H_2) *we have:*

(a) *$\mathcal{PAA}(\mathbb{R},\mathbb{R}^N,\mu,\nu)$ is translation invariant (i.e., if $f \in \mathcal{PAA}(\mathbb{R},\mathbb{R}^N,\mu,\nu)$, then $t \mapsto f(t+\sigma) \in \mathcal{PAA}(\mathbb{R},\mathbb{R}^N,\mu,\nu)$ for all $\sigma \in \mathbb{R}$).*

(b) *The decomposition of a (μ,ν)-pseudo almost automorphic function is unique.*

(c) *The space $(\mathcal{PAA}(\mathbb{R},\mathbb{R}^N,\mu,\nu), \|\cdot\|_\infty)$ is a Banach space.*

Lemma 3. *If $g, h \in \mathcal{PAA}(\mathbb{R},\mathbb{R},\mu,\nu)$, then $gh \in \mathcal{PAA}(\mathbb{R},\mathbb{R},\mu,\nu)$.*

P r o o f. From Definition 2 we have $g = g_1 + g_0$ and $h = h_1 + h_0$, where

$$\begin{cases} g_1, h_1 \in \mathcal{AA}(\mathbb{R},\mathbb{R}),\\ g_0, h_0 \in \mathcal{E}(\mathbb{R},\mathbb{R},\mu,\nu).\end{cases}$$

Then we have $gh = g_1h_1 + g_1h_0 + g_0h_1 + g_0h_0$.

By Lemma 1, $g_1h_1 \in \mathcal{AA}(\mathbb{R},\mathbb{R})$. Since

$$\begin{aligned}\lim_{\varrho\to\infty} \frac{1}{\nu([-\varrho,\varrho])} \int_{-\varrho}^{\varrho} &|g_1(t)h_0(t) + g_0(t)h_1(t) + g_0(t)h_0(t)| \,\mathrm{d}\mu(t)\\ &\leqslant \lim_{\varrho\to\infty} \frac{\|g_1\|_\infty}{\nu([-\varrho,\varrho])} \int_{-\varrho}^{\varrho} |h_0(t)| \,\mathrm{d}\mu(t)\\ &\quad + \lim_{\varrho\to\infty} \frac{\|h_1\|_\infty}{\nu([-\varrho,\varrho])} \int_{-\varrho}^{\varrho} |g_0(t)| \,\mathrm{d}\mu(t)\\ &\quad + \lim_{\varrho\to\infty} \frac{\|g_0\|_\infty}{\nu([-\varrho,\varrho])} \int_{-\varrho}^{\varrho} |h_0(t)| \,\mathrm{d}\mu(t) = 0,\end{aligned}$$

$g_1h_0 + g_0h_1 + g_0h_0$ is (μ,ν)-ergodic function. Therefore, $hg \in \mathcal{PAA}(\mathbb{R},\mathbb{R},\mu,\nu)$. □

Lemma 4. *Let $\mu, \nu \in \mathcal{M}$ satisfy* (H_2) *and $\widetilde{F} \in \mathcal{PAAU}(\mathbb{R}\times\mathbb{R},\mathbb{R},\mu,\nu)$ such that*

$$|\widetilde{F}(t,x) - \widetilde{F}(t,y)| \leqslant l(t)|x-y|,\ l \in \mathcal{L}^p(\mathbb{R},\mathrm{d}\mu),\quad 1 \leqslant p < \infty.$$

If $x \in \mathcal{PAA}(\mathbb{R},\mathbb{R},\mu,\nu)$, then $[t \mapsto \widetilde{F}(t,x(t))] \in \mathcal{PAA}(\mathbb{R},\mathbb{R},\mu,\nu)$.

P r o o f. The function $t \mapsto \widetilde{F}(t, x(t))$ is continuous. Since $x \in \mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu, \nu)$, we can write $x = x_1 + x_0$, where $x_1 \in \mathcal{AA}(\mathbb{R}, \mathbb{R})$ and $x_0 \in \mathcal{E}(\mathbb{R}, \mathbb{R}, \mu, \nu)$. Since $F \in \mathcal{PAAU}(\mathbb{R} \times \mathbb{R}, \mathbb{R}, \mu, \nu)$, we have $\widetilde{F} = \widetilde{F}_1 + \widetilde{F}_0$, where $\widetilde{F}_1 \in \mathcal{AAU}(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $\widetilde{F}_0 \in \mathcal{E}U(\mathbb{R} \times \mathbb{R}, \mathbb{R}, \mu, \nu)$. The function F can be written as

$$\begin{aligned}\widetilde{F}(t, x(t)) &= \widetilde{F}_1(t, x_1(t)) + [\widetilde{F}(t, x(t)) - \widetilde{F}(t, x_1(t))] + [\widetilde{F}(t, x_1(t)) - \widetilde{F}_1(t, x_1(t))] \\ &= \widetilde{F}_1(t, x_1(t)) + [\widetilde{F}(t, x(t)) - \widetilde{F}(t, x_1(t))] + \widetilde{F}_0(t, x_1(t)).\end{aligned}$$

From Theorem 8 in [1] we have $[t \mapsto \widetilde{F}_1(t, x_1(t))] \in \mathcal{AA}(\mathbb{R}, \mathbb{R})$. On the other hand, we have two cases:

Case 1: $p > 1$,

$$\begin{aligned}&\frac{1}{\nu([-\varrho, \varrho])} \int_{-\varrho}^{\varrho} |\widetilde{F}(t, x(t)) - \widetilde{F}(t, x_1(t))| \, \mathrm{d}\mu(t) \\ &\quad\leqslant \frac{1}{\nu([-\varrho, \varrho])} \int_{-\varrho}^{\varrho} |l(t)||x_0(t)| \, \mathrm{d}\mu(t) \\ &\quad\leqslant \frac{1}{\nu([-\varrho, \varrho])} \Big(\int_{-\varrho}^{\varrho} |l(t)|^p \, \mathrm{d}\mu(t)\Big)^{1/p} \Big(\int_{-\varrho}^{\varrho} |x_0(t)|\|x_0(t)\|_{\mathbb{R}}^{q-1}\Big)^{1/q} \\ &\quad\leqslant \frac{\|l\|_p \|x_0\|_\infty^{1/p}}{\nu([-\varrho, \varrho])^{1/p}} \Big[\frac{1}{\nu([-\varrho, \varrho])} \int_{-\varrho}^{\varrho} |x_0(t)| \, \mathrm{d}\mu(t)\Big]^{1/q} \to 0 \text{ as } r \to \infty.\end{aligned}$$

Case 2: $p = 1$,

$$\begin{aligned}&\frac{1}{\nu([-\varrho, \varrho])} \int_{-\varrho}^{\varrho} |\widetilde{F}(t, x(t)) - \widetilde{F}(t, x_1(t))| \, \mathrm{d}\mu(t) \\ &\qquad\leqslant \frac{1}{\nu([-\varrho, \varrho])} \int_{-\varrho}^{\varrho} |l(t)||x_0(t)| \, \mathrm{d}\mu(t) \\ &\qquad\leqslant \frac{\|l\|_1 \|x_0\|_\infty}{\nu([-\varrho, \varrho])} \to 0 \text{ as } r \to \infty.\end{aligned}$$

The remains to show that the function $[t \mapsto \widetilde{F}_0(t, x_1(t))] \in \mathcal{E}U(\mathbb{R} \times \mathbb{R}, \mathbb{R}, \mu, \nu)$. By using the fact that $\widetilde{F}_0$ is uniformly continuous on the compact to $\mathcal{K} = \overline{\{x_1(t), t \in \mathbb{R}\}}$ with respect, the second variable, we deduce that for given $\varepsilon > 0$ there exists $\delta > 0$ such that for all $t \in \mathbb{R}$ and $z_1, z_2 \in \mathcal{K}$ one has

$$|z_1 - z_2| \leqslant \delta \text{ implies } |\widetilde{F}_0(t, z_1(t)) - \widetilde{F}_0(t, z_2(t)| \leqslant \varepsilon.$$

Using the compactness of $\mathcal{K}$, there exists $\eta(\varepsilon) \in \mathbb{N}^*$ and a family $\{\chi_i\}_{i=1}^{\eta(\varepsilon)} \subset \mathcal{K}$ such that $\mathcal{K} \subset \bigcup_{i=1}^{\eta(\varepsilon)} \mathcal{B}(\chi_i, \delta)$. Then there exists $i_0 \in \{1, \ldots, \eta(\varepsilon)\}$ such that for all $t \in \mathbb{R}$

$$|\widetilde{F}_0(t, x_1(t))| \leqslant \varepsilon + |\widetilde{F}_0(t, \chi_{i_0})|.$$

Since we have

$$\lim_{\varrho\to\infty}\frac{1}{\nu([-\varrho,\varrho])}\int_{-\varrho}^{\varrho}|\widetilde{F}_0(t,\chi_{i_0})|\,\mathrm{d}\mu(t)=0,$$

we deduce that for all $\varepsilon>0$:

$$\limsup_{\varrho\to\infty}\frac{1}{\nu([-\varrho,\varrho])}\int_{-\varrho}^{\varrho}|\widetilde{F}_0(t,x_1(t))|\,\mathrm{d}\mu(t)\leqslant\varepsilon\limsup_{\varrho\to\infty}\frac{\mu([-\varrho,\varrho])}{\nu([-\varrho,\varrho])}.$$

Therefore,

$$\lim_{\varrho\to\infty}\frac{1}{\nu([-\varrho,\varrho])}\int_{-\varrho}^{\varrho}|\widetilde{F}_0(t,x_1(t))|\,\mathrm{d}\mu(t)=0$$

which implies that $[t\mapsto\widetilde{F}_0(t,x_1(t))]$ is (μ,ν)-ergodic. □

Throughout this manuscript, the following hypotheses are given:

($\mathbb{A}_1$) For $i\in\Lambda$, $a_i(\cdot)$ are almost automorphic and there exist positive constants a_{i*} and a_i^* such that $0<a_{i*}\leqslant a_i(\cdot)\leqslant a_i^*$.

($\mathbb{A}_2$) For $i\in\Lambda$, $b_i(\cdot)$ are almost automorphic and there exist positive constants b_{i*} and b_i^* such that $b_{i*}\leqslant(b_i(x_1)-b_i(x_2))/(x_1-x_2)\leqslant b_i^*$ for $x_1,x_2\in\mathbb{R}$ and $b_i(0)=0$.

($\mathbb{A}_3$) The functions $t\mapsto o_{ij}(t)$, $t\mapsto w_{ij}(t)$, $t\mapsto d_{ij}(t)$ and $t\mapsto L_i(t)$ are (μ,ν)-pseudo almost automorphic and $t\mapsto\tau_{ij}(t)$ is almost automorphic.

($\mathbb{A}_4$) For $p>1$ and $j\in\Lambda$, the activation functions f_j, g_j and h_j are (μ,ν)-pseudo almost automorphic and there exist positive continuous functions M_j^f, M_j^g, $M_j^h\in\mathcal{L}^p(\mathbb{R},\mathrm{d}\mu)\cap\mathcal{L}^p(\mathbb{R},\mathrm{d}x)$ such that for all $y,z\in\mathbb{R}$

$$|f_j(t,y)-f_j(t,z)|\leqslant M_j^f(t)|y-z|,$$
$$|g_j(t,y)-g_j(t,z)|\leqslant M_j^g(t)|y-z|,$$
$$|h_j(t,y)-h_j(t,z)|\leqslant M_j^h(t)|y-z|,$$

and for $t\in\mathbb{R}$, $j\in\Lambda$ we suppose that $f_j(t,0)=g_j(t,0)=h_j(t,0)=0$.

($\mathbb{A}_5$) For $i,j\in\Lambda$, the delay kernels $k_{ij}\colon\mathbb{R}^+\to\mathbb{R}^+$ satisfy

$$\int_0^\infty k_{ij}(s)=\overline{k}_{ij}.$$

($\mathbb{A}_6$) Assume that there exists $\theta>0$ such that

$$\theta=\max_{i\in\Lambda}\left\{\frac{1}{(a_{i*}b_{i*})^{1-1/p}}\sum_{j=1}^{n}a_j^*(o_{ij}^*\|M_j^f\|_p+w_{ij}^*\|M_j^g\|_p+d_{ij}^*\overline{k}_{ij}\|M_j^h\|_p)\right\}<\frac{1}{2}.$$

3. Main result

In this section, we establish some results for the existence, uniqueness and the global exponential stability of the (μ,ν)-pseudo almost automorphic solution of system (1).

Now, using assumption $(\mathbb{A}_1)$, the antiderivative of $1/a_i(x_i)$ exists. Then we choose an antiderivative $C_i(x_i)$ of $1/a_i(x_i)$ such that $C_i(0)=0$. Evidently, $C_i'(x_i)=1/a_i(x_i)$. Since $a_i(x_i)>0$, we can see that $C_i(x_i)$ is strictly monotone increasing on x_i. Using the inverse function theorem, there exists an inverse function $C_i^{-1}(x_i)$ of $C_i(x_i)$ which is continuous and differential. Then we have $(C_i^{-1}(x_i))'=a_i(x_i)$. Denote $y_i(t)=C_i(x_i(t))$, it is easy to see that $C_i'(x_i)x_i'(t)=x_i'(t)/a_i(x_i(t))=y_i'(t)$ and $x_i(t)=C_i^{-1}(y_i(t))$. As a result, system (1) can be expressed as

$$
\begin{aligned}
(2)\qquad y_i'(t) = & -b_i(C_i^{-1}(y_i(t)))+\sum_{j=1}^{n} o_{ij}(t)f_j(t,C_j^{-1}(y_j(t)))\\
&+\sum_{j=1}^{n} w_{ij}(t)g_j(t,C_j^{-1}(y_j(t-\tau_{ij}(t))))\\
&+\sum_{j=1}^{n} d_{ij}(t)\int_{-\infty}^{t} k_{ij}(t-s)h_j(s,C_j^{-1}(y_j(s)))\,\mathrm{d}s+L_i(t).
\end{aligned}
$$

From assumption $(\mathbb{A}_2)$ and the mean value theorem, there exist a constant $l_i\in[0,1]$ such that

$$b_i(C_i^{-1}(y_i(t)))=(b_i(C_i^{-1}(l_iy_i(t))))'y_i(t):=\bar{b}_i(y_i(t))y_i(t).$$

Then system (2) can be rewritten as

$$
\begin{aligned}
(3)\qquad y_i'(t) = & -\bar{b}_i(y_i(t))y_i(t)+\sum_{j=1}^{n} o_{ij}(t)f_j(t,C_j^{-1}(y_j(t)))\\
&+\sum_{j=1}^{n} w_{ij}(t)g_j(t,C_j^{-1}(y_j(t-\tau_{ij}(t))))\\
&+\sum_{j=1}^{n} d_{ij}(t)\int_{-\infty}^{t} k_{ij}(t-s)h_j(s,C_j^{-1}(y_j(s)))\,\mathrm{d}s+L_i(t).
\end{aligned}
$$

R e m a r k 3. Equation (1) has a unique (μ,ν)-pseudo almost automorphic solution if and only if equation (3) has a unique (μ,ν)-pseudo almost automorphic solution. Then we only need to consider the almost automorphic solution of system (3), see [2], [8], [16].

By Lagrange theorem, we have

$$|C_i^{-1}(u)-C_i^{-1}(v)|=|(C_i^{-1}(v+l_i(u-v)))'(u-v)|=|(a_i(v+l_i(u-v))||u-v|.$$

From assumption $(\mathbb{A}_1)$ we get

$$a_{i*}|u-v| \leqslant |C_i^{-1}(u) - C_i^{-1}(v)| \leqslant a_i^*|u-v|.$$

Combined with $(\mathbb{A}_2)$, we obtain

$$b_{i*}a_{i*} \leqslant (b_i(C_i^{-1}(\cdot)))' \leqslant b_i^* a_i^*.$$

Lemma 5. *Under conditions* (H_1), (H_2), $(\mathbb{A}_1)$–$(\mathbb{A}_3)$, *if for all* $j \in \Lambda$, $y_j \in \mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu, \nu)$, *then for all* $i, j \in \Lambda$ *the function*

$$\psi_{ij} \mapsto \int_{-\infty}^{t} k_{ij}(t-s) h_j(s, C_j^{-1}(y_j(s)))\,\mathrm{d}s$$

belongs to $\mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu, \nu)$.

Proof. From Lemma 3 and Assumption $(\mathbb{A}_1)$, $[t \mapsto h_j(t, C_j^{-1}(y_j(s)))] \in \mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu, \nu)$ for all $j \in \Lambda$. Then it is bounded and there exists $L_j^h > 0$ such that for all $t \in \mathbb{R}$ we have

$$|h_j(t, C_j^{-1}(y_j(s)))| \leqslant L_j^h.$$

It follows that for all $i, j \in \Lambda$, the functions ψ_{ij} are bounded and satisfy

$$|\psi_{ij}(t)| \leqslant \int_{-\infty}^{t} |k_{ij}(t-s) h_j(s, C_j^{-1}(y_j(s)))|\,\mathrm{d}s \leqslant L_j^h \int_0^{\infty} k_{ij}(s)\,\mathrm{d}s \leqslant L_j^h \overline{k}_{ij}.$$

By the same arguments given in [3], we prove that ψ_{ij} is continuous and bounded.

Since $[t \mapsto h_j(t, C_j^{-1}(y_j(t)))] \in \mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu, \nu)$ for all $j \in \Lambda$, using the decomposition theorem of (μ, ν)-pseudo almost automorphic functions, one has that

$$h_j(t, C_j^{-1}(y_j(t))) = h_j^1(t) + h_j^0(t),$$

where $h_j^1 \in \mathcal{AA}(\mathbb{R}, \mathbb{R})$ and $h_j^0 \in \mathcal{E}(\mathbb{R}, \mathbb{R}, \mu, \nu)$. Consequently,

$$\psi_{ij}(t) = \int_{-\infty}^{t} k_{ij}(t-s) h_j^1(s)\,\mathrm{d}s + \int_{-\infty}^{t} k_{ij}(t-s) h_j^0(s)\,\mathrm{d}s := \psi_{ij}^1(t) + \psi_{ij}^0(t).$$

Step 1: We will prove that $\psi_{ij}^1 \in \mathcal{AA}(\mathbb{R}, \mathbb{R})$.

Let (σ_n') be a sequence of real numbers, we can extract a subsequence (σ_n) of (σ_n') such that

$$\lim_{n\to\infty} h_j^1(t+\sigma_n) = \overline{h}_j^1(t) \quad \text{and} \quad \lim_{n\to\infty} \overline{h}_j^1(t-\sigma_n) = h_j^1(t)$$

for all $t \in \mathbb{R}$. Denote

$$\psi_{ij}^{-1}(t) = \int_{-\infty}^{t} k_{ij}(t-s) \overline{h}_j^1(s)\,\mathrm{d}s.$$

Then we have

$$|\psi_{ij}^1(t+\sigma_n)-\psi_{ij}^{-1}(t)| = \left|\int_{-\infty}^{t+\sigma_n} k_{ij}(t+\sigma_n-s)h_j^1(s)\,\mathrm{d}s - \int_{-\infty}^{t} k_{ij}(t-s)\overline{h}_j^1(s)\,\mathrm{d}s\right|$$
$$\leqslant \int_{-\infty}^{t} k_{ij}(t-s)|h_j^1(s+\sigma_n)-\overline{h}_j^1(s)|\,\mathrm{d}s.$$

Applying the *Lebesgue* dominated convergence theorem, we get

$$\lim_{n\to\infty}\psi_{ij}^1(t+\sigma_n)=\psi_{ij}^{-1}(t)\quad\text{for each } t\in\mathbb{R}.$$

Similarly, we can obtain

$$\lim_{n\to\infty}\psi_{ij}^{-1}(t-\sigma_n)=\psi_{ij}^{1}(t)\quad\text{for each } t\in\mathbb{R},$$

which implies that $\psi_{ij}^1\in\mathcal{AA}(\mathbb{R},\mathbb{R})$.

Step 2: We will show that $\psi_{ij}^0\in\mathcal{E}(\mathbb{R},\mathbb{R},\mu,\nu)$. Since

$$\begin{aligned}\frac{1}{\nu([-\varrho,\varrho])}\int_{-\varrho}^{\varrho}|\psi_{ij}^0(t)|\,\mathrm{d}\mu(t) &= \frac{1}{\nu([-\varrho,\varrho])}\int_{-\varrho}^{\varrho}\left(\left|\int_{-\infty}^{t}k_{ij}(t-s)h_j^0(s)\,\mathrm{d}s\right|\right)\mathrm{d}\mu(t)\\ &= \frac{1}{\nu([-\varrho,\varrho])}\int_{-\varrho}^{\varrho}\left(\left|\int_{\infty}^{0}k_{ij}(s)h_j^0(t-s)\,\mathrm{d}s\right|\right)\mathrm{d}\mu(t)\\ &\leqslant \frac{1}{\nu([-\varrho,\varrho])}\int_{-\varrho}^{\varrho}\left(\int_{-\infty}^{t}k_{ij}(s)|h_j^0(t-s)\,\mathrm{d}s|\right)\mathrm{d}\mu(t)\\ &\leqslant \int_{-\infty}^{t}k_{ij}(s)\left(\frac{1}{\nu([-\varrho,\varrho])}\int_{-\varrho}^{\varrho}|h_j^0(t-s)\,\mathrm{d}s|\,\mathrm{d}\mu(t)\right)\mathrm{d}s\end{aligned}$$

and since μ satisfies (H_1), we have $[t\mapsto h_j^0(t-s)]\in\mathcal{E}(\mathbb{R},\mathbb{R},\mu,\nu)$ for all $s\in\mathbb{R}$. By the Lebesgue's dominated convergence theorem, we have

$$\lim_{\varrho\to\infty}\frac{1}{\nu([-\varrho,\varrho])}\int_{-\varrho}^{\varrho}|\psi_{ij}^0(t)|\,\mathrm{d}\mu(t)=0,$$

which implies that $\psi_{ij}^0\in\mathcal{E}(\mathbb{R},\mathbb{R},\mu,\nu)$. Consequently, $\psi_{ij}\in\mathcal{PAA}(\mathbb{R},\mathbb{R},\mu,\nu)$. □

Lemma 6. *For $p>1$ and $h\in\mathcal{C}(\mathbb{R}\times\mathbb{R},\mathbb{R})$ such that*

$$|h(t,x)-h(t,y)|\leqslant M^h(t)|x-y|,$$

where $M^h\in\mathcal{L}^p(\mathbb{R},\,\mathrm{d}\mu)\cap\mathcal{L}^p(\mathbb{R},\,\mathrm{d}x)$, if $C^{-1}(\varphi)\in\mathcal{PAA}(\mathbb{R},\mathbb{R},\mu,\nu)$ and $\alpha\in\mathcal{AA}(\mathbb{R},\mathbb{R})$, then $[t\mapsto h(t,C^{-1}(\varphi(t-\alpha(t))))]\in\mathcal{PAA}(\mathbb{R},\mathbb{R},\mu,\nu)$.

P r o o f. Let $C^{-1}(\varphi)=\varphi_1+\varphi_0$, where $\varphi_1\in\mathcal{AA}(\mathbb{R},\mathbb{R})$ and $\varphi_0\in\mathcal{E}(\mathbb{R},\mathbb{R},\mu,\nu)$. Let us consider the following function:

$$\begin{aligned}\Theta(t)&=h(t,\varphi_1(t-\alpha(t)))+h(t,\varphi_1(t-\alpha(t))+\varphi_0(t-\alpha(t)))-h(t,\varphi_1(t-\alpha(t)))\\&:=\Theta_1(t)+\Theta_0(t),\end{aligned}$$

where $\Theta_1(t)=h(t,\varphi_1(t-\alpha(t)))$ and $\Theta_0(t)=h(t,\varphi_1(t-\alpha(t))+\varphi_0(t-\alpha(t)))-h(t,\varphi_1(t-\alpha(t)))$. From Theorem 8 in [1] we have $\Theta_1\in\mathcal{AA}(\mathbb{R},\mathbb{R})$. Since

$$\begin{aligned}&\lim_{\varrho\to\infty}\frac{1}{\nu([-\varrho,\varrho])}\int_{-\varrho}^{\varrho}|\Theta_0(t)|\,\mathrm{d}\mu(t)\\&=\lim_{\varrho\to\infty}\frac{1}{\nu([-\varrho,\varrho])}\int_{-\varrho}^{\varrho}|h(t,\varphi_1(t-\alpha(t))+\varphi_0(t-\alpha(t)))-h(t,\varphi_1(t-\alpha(t)))|\,\mathrm{d}\mu(t)\\&\leqslant\lim_{\varrho\to\infty}\frac{1}{\nu([-\varrho,\varrho])}\int_{-\varrho}^{\varrho}M^h(t)|\varphi_0(t-\alpha(t))|\,\mathrm{d}\mu(t)\end{aligned}$$

and since $\varphi_0\in\mathcal{E}(\mathbb{R},\mathbb{R},\mu,\nu)$, we get

$$\begin{aligned}&\lim_{\varrho\to\infty}\frac{1}{\nu([-\varrho,\varrho])}\int_{-\varrho}^{\varrho}|\Theta_0(t)|\,\mathrm{d}\mu(t)\\&\quad\leqslant\lim_{\varrho\to\infty}\frac{1}{\nu([-\varrho,\varrho])}\int_{-\varrho}^{\varrho}M^h(t)|\varphi_0(t-\alpha(t))|\,\mathrm{d}\mu(t)\\&\quad\leqslant\lim_{\varrho\to\infty}\frac{\|\varphi_0\|_\infty}{\nu([-\varrho,\varrho])}\int_{-\varrho}^{\varrho}M^h(t)\,\mathrm{d}\mu(t)\\&\quad\leqslant\lim_{\varrho\to\infty}\frac{\|\varphi_0\|_\infty}{\nu([-\varrho,\varrho])}\Big\{\Big(\int_{-\varrho}^{\varrho}(M^h(t))^p\,\mathrm{d}\mu(t)\Big)^{1/p}\Big(\int_{-\varrho}^{\varrho}\mathrm{d}\mu(t)\Big)^{1/q}\Big\}\text{ with }\frac{1}{p}+\frac{1}{q}=1\\&\quad\leqslant\lim_{\varrho\to\infty}\frac{\mu([-\varrho,\varrho])^{1/q}\|\varphi_0\|_\infty}{\nu([-\varrho,\varrho])}\Big(\int_{-\varrho}^{\varrho}(M^h(t))^p\,\mathrm{d}\mu(t)\Big)^{1/p}\\&\quad=\lim_{\varrho\to\infty}\frac{\mu([-\varrho,\varrho])\|\varphi_0\|_\infty\|M^h\|_p}{\nu([-\varrho,\varrho])\mu([-\varrho,\varrho])^{1/p}}=0.\end{aligned}$$

Then $[t\mapsto\Theta_0(t)]\in\mathcal{E}(\mathbb{R},\mathbb{R},\mu,\nu)$. Consequently, $[t\mapsto h(t,C^{-1}(\varphi(t-\alpha(t))))]\in\mathcal{PAA}(\mathbb{R},\mathbb{R},\mu,\nu)$. □

3.1. Existence and uniqueness of the (μ,ν)-pseudo almost automorphic solution. In this subsection, we present some useful lemmas for the existence and the uniqueness of the (μ,ν)-pseudo almost automorphic solution of system (3).

Lemma 7. *For each $\varphi = (\varphi_1, \ldots, \varphi_n) \in \mathcal{PAA}(\mathbb{R}, \mathbb{R}^n, \mu, \nu)$, define the nonlinear operator Γ as*

$$\Gamma_\varphi(t) = \begin{pmatrix} \displaystyle\int_{-\infty}^t \exp\Big(-\int_s^t \bar{b}_1(v_1(u))\,\mathrm{d}u\Big) F_1(s)\,\mathrm{d}s \\ \vdots \\ \displaystyle\int_{-\infty}^t \exp\Big(-\int_s^t \bar{b}_n(v_n(u))\,\mathrm{d}u\Big) F_n(s)\,\mathrm{d}s \end{pmatrix}$$

and for all $i \in \Lambda$

$$\begin{aligned} F_i(s) = {} & \sum_{j=1}^n o_{ij}(s) f_j(s, C_j^{-1}(\varphi_j(s))) + \sum_{j=1}^n w_{ij}(s) g_j(s, C_j^{-1}(\varphi_j(s - \tau_i(s)))) \\ & + \sum_{j=1}^n d_{ij}(s) \int_0^\infty k_{ij}(u) h_j(s, C_j^{-1}(\varphi_j(s-u)))\,\mathrm{d}u + L_i(s). \end{aligned}$$

Suppose that the assumptions (H$_1$)–(H$_2$) *and* ($\mathbb{A}_1$)–($\mathbb{A}_4$) *hold. Then Γ maps $\mathcal{PAA}(\mathbb{R}, \mathbb{R}^n, \mu, \nu)$ into itself.*

Proof. From Lemmas 2–6, for all $i \in \Lambda$, the function $s \mapsto F_i(s)$ is (μ, ν)-pseudo almost automophic. Consequently, F_i can be expressed as $F_i = F_i^0 + F_i^1$, where $F_i^0(\cdot) \in \mathcal{E}(\mathbb{R}, \mathbb{R}, \mu, \nu)$ and $F_i^1(\cdot) \in \mathcal{AA}(\mathbb{R}, \mathbb{R})$. It follows that

$$\begin{aligned} \Gamma_i\varphi(t) &= \int_{-\infty}^t \exp\Big(-\int_s^t \bar{b}_i(y_i(u))\,\mathrm{d}u\Big) F_i(s)\,\mathrm{d}s \\ &= \int_{-\infty}^t \exp\Big(-\int_s^t \bar{b}_i(v_1(u))\,\mathrm{d}u\Big) F_i^1(s)\,\mathrm{d}s \\ &\quad + \int_{-\infty}^t \exp\Big(-\int_s^t \bar{b}_i(v_1(u))\,\mathrm{d}u\Big) F_i^0(s)\,\mathrm{d}s \\ &:= (\Gamma_i F_i^1)(t) + (\Gamma_i F_i^0)(t). \end{aligned}$$

On the one hand, by the same arguments as are given in [1], $\Gamma_i F_i^1 \in \mathcal{AA}(\mathbb{R}, \mathbb{R})$. On the other hand, from Fubini's theorem we have

$$\begin{aligned} & \lim_{\varrho\to\infty} \frac{1}{\nu([-\varrho, \varrho])} \int_{-\varrho}^{\varrho} |(\Gamma_i F_i^0)(t)|\,\mathrm{d}\mu(t) \\ & \quad \leqslant \lim_{\varrho\to\infty} \frac{1}{\nu([-\varrho, \varrho])} \int_{-\varrho}^{\varrho} \int_{-\infty}^t \exp(-(t-s)a_{i*}b_{i*}) |F_i^0(t)|\,\mathrm{d}s\,\mathrm{d}\mu(t) \\ & \quad \leqslant \lim_{\varrho\to\infty} \frac{1}{\nu([-\varrho, \varrho])} \int_{-\varrho}^{\varrho} \Big(\int_0^\infty \exp(-y a_{i*} b_{i*}) |F_i^0(t-y)|\,\mathrm{d}y\Big)\,\mathrm{d}\mu(t) \\ & \quad \leqslant \int_0^\infty \exp(-y\bar{b}_{i*}) \Big(\lim_{\varrho\to\infty} \frac{1}{\nu([-\varrho, \varrho])} \int_{-\varrho}^{\varrho} |F_i^0(t-y)|\,\mathrm{d}\mu(t)\Big)\,\mathrm{d}y. \end{aligned}$$

From Lemma 1, $t \mapsto F_i^0(t-y) \in \mathcal{E}(\mathbb{R}, \mathbb{R}, \mu, \nu)$. By the Lebesgue dominated convergence theorem, we get

$$\lim_{\varrho \to \infty} \frac{1}{\nu([-\varrho, \varrho])} \int_{-\varrho}^{\varrho} |\Gamma_i F_i^0(t)| \, \mathrm{d}\mu(t) = 0,$$

consequently, $\Gamma_i F_i^0(t) \in \mathcal{E}(\mathbb{R}, \mathbb{R}, \mu, \nu)$. As a conclusion, for all $i \in \Lambda$, $\Gamma_i \varphi \in \mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu, \nu)$ and $\Gamma_\varphi \in \mathcal{PAA}(\mathbb{R}, \mathbb{R}^n, \mu, \nu)$. $\square$

Theorem 1. *Suppose that* (H_1), (H_2) *and* $(\mathbb{A}_1)$–$(\mathbb{A}_4)$ *are fulfilled. Then there exists only one* (μ, ν)*-pseudo almost automorphic solution of the delayed differential equations* (1) *in the region*

$$\Im = \Big\{ \vartheta \in \mathcal{PAA}(\mathbb{R}, \mathbb{R}^n, \mu, \nu), \ \|\vartheta - \vartheta_0\|_\infty \leqslant \frac{\theta \xi}{1-\theta} \Big\},$$

where

$$\xi = \max_{i \in \Lambda} \Big\{ \frac{L_i^*}{b_{i*} a_{i*}} \Big\} \quad \text{and} \quad \vartheta_0(t) = \begin{pmatrix} \displaystyle\int_{-\infty}^{t} \exp\Big(-\int_s^t \bar{b}_1(v_1(u)) \, \mathrm{d}u\Big) L_1(s) \, \mathrm{d}s \\ \vdots \\ \displaystyle\int_{-\infty}^{t} \exp\Big(-\int_s^t \bar{b}_n(v_n(u)) \, \mathrm{d}u\Big) L_n(s) \, \mathrm{d}s \end{pmatrix}.$$

Proof. Firstly, it is easy to see that the function $\vartheta_0 \in \mathcal{L}^\infty(\mathbb{R}, \mathbb{R}^n)$ and $\|\vartheta_0\|_\infty \leqslant \xi$. Consider the set

$$\Im = \Big\{ \vartheta \in \mathcal{PAA}(\mathbb{R}, \mathbb{R}^n, \mu, \nu), \ \|\vartheta - \vartheta_0\|_\infty \leqslant \frac{\theta \xi}{1-\theta} \Big\}.$$

It is clear that $\Im$ is a closed convex subset of $\mathcal{PAA}(\mathbb{R}, \mathbb{R}^n, \mu, \nu)$. Moreover, for any $\vartheta \in \Im$ we have

$$\begin{aligned}
&\|\Gamma_\vartheta - \vartheta_0\|_\infty \\
&\quad = \max_{i \in \Lambda} \sup_{t \in \mathbb{R}} \Big\{ \Big| \int_{-\infty}^{t} \exp\Big(-\int_s^t \bar{b}_i(y_i(u)) \, \mathrm{d}u\Big) \Big(F_i(s) - L_i(s)\Big) \, \mathrm{d}s \Big| \Big\} \\
&\quad \leqslant \max_{i \in \Lambda} \sup_{t \in \mathbb{R}} \Big\{ \int_{-\infty}^{t} \exp\Big(-\int_s^t \bar{b}_i(y_i(u)) \, \mathrm{d}u\Big) \Big| \sum_{j=1}^{n} o_{ij}(s) f_j(s, C_j^{-1}(\varphi_j(s))) \Big| \, \mathrm{d}s \\
&\qquad + \int_{-\infty}^{t} \exp\Big(-\int_s^t \bar{b}_i(y_i(u)) \, \mathrm{d}u\Big) \Big| \sum_{j=1}^{n} w_{ij}(s) g_j(s, C_j^{-1}(\varphi_j(s - \tau_i(s)))) \Big| \, \mathrm{d}s \\
&\qquad + \int_{-\infty}^{t} \exp\Big(-\int_s^t \bar{b}_i(y_i(u)) \, \mathrm{d}u\Big) \\
&\qquad \times \Big| \sum_{j=1}^{n} d_{ij}(s) \int_0^\infty k_{ij}(u) h_j(s, C_j^{-1}(\varphi_j(s-u))) \, \mathrm{d}u \, \mathrm{d}s \Big| \Big\}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \max_{i\in\Lambda}\sup_{t\in\mathbb{R}}\bigg\{\int_{-\infty}^{t}\exp\bigg(-\int_{s}^{t}\bar{b}_i(y_i(u))\,\mathrm{d}u\bigg)\sum_{j=1}^{n}|o_{ij}(s)||f_j(s,C_j^{-1}(\varphi_j(s)))|\,\mathrm{d}s\\
&\quad+\int_{-\infty}^{t}\exp\bigg(-\int_{s}^{t}\bar{b}_i(y_i(u))\,\mathrm{d}u\bigg)\sum_{j=1}^{n}|w_{ij}(s)||g_j(s,C_j^{-1}(\varphi_j(s-\tau_i(s))))|\,\mathrm{d}s\\
&\quad+\int_{-\infty}^{t}\exp\bigg(-\int_{s}^{t}\bar{b}_i(y_i(u))\,\mathrm{d}u\bigg)\\
&\quad\times\sum_{j=1}^{n}|d_{ij}(s)|\int_{0}^{\infty}k_{ij}(u)|h_j(s,C_j^{-1}(\varphi_j(s-u)))|\,\mathrm{d}u\,\mathrm{d}s\bigg\}\\
&\leqslant \max_{i\in\Lambda}\sup_{t\in\mathbb{R}}\sum_{j=1}^{n}\bigg\{\int_{-\infty}^{t}\exp\left(-(t-s)a_{i*}b_{i*}\right)o_{ij}^{*}a_j^{*}L_j^{f}(s)|\varphi_j(s)|\,\mathrm{d}s\\
&\quad+\int_{-\infty}^{t}\exp\left(-(t-s)a_{i*}b_{i*}\right)w_{ij}^{*}a_j^{*}L_j^{g}(s)|\varphi_j(s-\tau_i(s))|\,\mathrm{d}s\\
&\quad+\int_{-\infty}^{t}\exp\left(-(t-s)a_{i*}b_{i*}\right)d_{ij}^{*}a_j^{*}\int_{0}^{\infty}k_{ij}(u)L_j^{h}(s)|\varphi_j(s-u)|\,\mathrm{d}u\,\mathrm{d}s\bigg\}\\
&\leqslant \|\vartheta\|_\infty\max_{i\in\Lambda}\sup_{t\in\mathbb{R}}\sum_{j=1}^{n}\bigg\{\int_{-\infty}^{t}\exp\left(-(t-s)a_{i*}b_{i*}\right)o_{ij}^{*}a_j^{*}L_j^{f}(s)\,\mathrm{d}s\\
&\quad+\int_{-\infty}^{t}\exp\left(-(t-s)a_{i*}b_{i*}\right)w_{ij}^{*}a_j^{*}L_j^{g}(s)\,\mathrm{d}s\\
&\quad+\int_{-\infty}^{t}\exp\left(-(t-s)a_{i*}b_{i*}\right)d_{ij}^{*}a_j^{*}\int_{0}^{\infty}k_{ij}(u)L_j^{h}(s)\,\mathrm{d}u\,\mathrm{d}s\bigg\}\\
&\leqslant \|\vartheta\|_\infty\Big(\frac{p-1}{p}\Big)^{1-1/p}\max_{i\in\Lambda}\sum_{j=1}^{n}\frac{(o_{ij}^{*}\|M_j^{f}\|_p+w_{ij}^{*}\|M_j^{g}\|_p+d_{ij}^{*}a_j^{*}\overline{k}_{ij}\|M_j^{h}\|_p)a_j^{*}}{(a_{i*}b_{i*})^{1-1/p}}\\
&\leqslant \|\vartheta\|_\infty\max_{i\in\Lambda}\sum_{j=1}^{n}\frac{(o_{ij}^{*}\|M_j^{f}\|_p+w_{ij}^{*}\|M_j^{g}\|_p+d_{ij}^{*}\overline{k}_{ij}\|M_j^{h}\|_p)a_j^{*}}{(a_{i*}b_{i*})^{1-1/p}}=:\theta\|\vartheta\|_\infty\leqslant\frac{\theta\xi}{1-\theta}.
\end{aligned}
$$

From $(\mathbb{A}_2)$, for any $\vartheta,\bar{\vartheta}\in\Im$ we get

$$
\begin{aligned}
&\|\Gamma_\vartheta-\Gamma_{\bar{\vartheta}}\|_\infty\\
&\leqslant \|\vartheta-\bar{\vartheta}\|_\infty\max_{i\in\Lambda}\sup_{t\in\mathbb{R}}\sum_{j=1}^{n}\bigg\{\int_{-\infty}^{t}\exp\left(-(t-s)a_{i*}b_{i*}\right)[|o_{ij}(s)|a_j^{*}M_j^{f}(s)\\
&\quad+|w_{ij}(s)|a_j^{*}M_j^{g}(s)+|d_{ij}(s)|a_j^{*}\overline{k}_{ij}M_j^{h}(s)]\,\mathrm{d}s\bigg\}\\
&\leqslant \|\vartheta-\bar{\vartheta}\|_\infty\Big(\frac{p-1}{p}\Big)^{1-1/p}\max_{i\in\Lambda}\sum_{j=1}^{n}\frac{(o_{ij}^{*}\|M_j^{f}\|_p+w_{ij}^{*}\|M_j^{g}\|_p+d_{ij}^{*}\overline{k}_{ij}\|M_j^{h}\|_p)a_j^{*}}{(a_{i*}b_{i*})^{1-1/p}}
\end{aligned}
$$

$$\leqslant \|\vartheta - \bar{\vartheta}\|_\infty \max_{i\in\Lambda} \sum_{j=1}^{n} \frac{(o_{ij}^* \|M_j^f\|_p + w_{ij}^* \|M_j^g\|_p + d_{ij}^* \overline{k}_{ij} \|M_j^h\|_p) a_j^*}{(a_{i*} b_{i*})^{1-1/p}} := \theta \|\vartheta - \bar{\vartheta}\|_\infty.$$

That is, Γ is contraction mapping from the region $\Im$ into itself. Then using the Banach fixed-point theorem, the mapping Γ has a unique fixed point in $\Im$, which is the (μ,ν)-pseudo almost automorphic solution of model (3). □

3.2. Global exponential stability of weighted pseudo almost automorphic solution. In this subsection, we will look into the global exponential stability of (μ,ν)-pseudo almost automorphic solutions of system (3) by using reduction to absurdity.

Let us define the phase space $\mathcal{C}((-\infty,0],\mathbb{R}^n)$ as a Banach space of continuous mappings from $(-\infty,0]$ to $\mathbb{R}^n$ equipped with the following norm:

$$\|\vartheta\|_\infty = \max_{i\in\Lambda} \sup_{-\infty\leqslant t\leqslant 0} |\vartheta_i(t)|.$$

For $\vartheta = (\vartheta_1,\ldots,\vartheta_n)^\top \in \mathcal{C}((-\infty,0],\mathbb{R}^n)$, the initial conditions associated with (3) are given by

$$y_i(s) = \vartheta_i(s), \quad s \in (-\infty,0], \quad i \in \Lambda,$$

with $\vartheta = (\vartheta_1,\ldots,\vartheta_n)^\top \in \mathcal{C}((-\infty,0],\mathbb{R}^n)$.

Definition 6. Let $y = (y_1,\ldots,y_n)^\top$ be a (μ,ν)-pseudo almost automorphic solution of system (3) with initial value $\vartheta \in \mathcal{C}((-\infty,0],\mathbb{R}^n)$ and $\overline{y}$ an arbitrary solution of equation (3) with initial value $\bar{\vartheta} \in \mathcal{C}((-\infty,0],\mathbb{R}^n)$. If there exist positive constants ϖ and $M(\vartheta)$ such that

$$|y(t) - \overline{y}(t)| \leqslant M(\vartheta)\|\vartheta - \bar{\vartheta}\|_\infty \exp(-\varpi t), \quad t \geqslant 0,$$

then the (μ,ν)-pseudo almost automorphic solution x of system (3) is said to be globally exponentially stable.

Theorem 2. *Suppose that* (H_1)–(H_2) *and* $(\mathbb{A}_1)$–$(\mathbb{A}_4)$ *are fulfilled and suppose further that there exists* $\varpi_0 > 0$ *such that*

$$\int_0^\infty k_{ij}(s)\exp(\varpi_0 s)\,\mathrm{d}s < \infty.$$

Then equation (3) *has a unique* (μ,ν)*-pseudo almost automorphic solution* $\overline{y}$ *which is globally exponentially stable.*

P r o o f. By Theorem 1, equation (3) has a unique (μ,ν)-pseudo almost automorphic solution $\overline{y}$. Let y be an arbitrary solution of equation (3) with initial value $\bar{\vartheta}$ and $\overline{y}$ be the (μ,ν)-pseudo almost automorphic solution of equation (3) with initial value ϑ. Let $z_i(t)=y_i(t)-\overline{y}_i(t)$. Then for $i\in\Lambda$ we have

$$
\begin{aligned}
(4)\quad \dot{z}(t)= &-[b_i(C^{-1}(z_j(t)+\overline{y}_j(t)))-b_i(C^{-1}(\overline{y}_j(t)))]\\
&+\sum_{j=1}^{n} o_{ij}(t)[f_j(t,C^{-1}(z_j(t)+\overline{y}_j(t)))-f_j(t,C^{-1}(\overline{y}_j(t)))]\\
&+\sum_{j=1}^{n} w_{ij}(t)\\
&\times[g_j(t,C^{-1}(z_j(t-\tau_{ij}(t))+\overline{y}_j(t-\tau_{ij}(t))))-g_j(t,C^{-1}(\overline{y}_j(t-\tau_{ij}(t))))]\\
&+\sum_{j=1}^{n} d_{ij}(t)\int_{-\infty}^{t} k_{ij}(t-s)\\
&\times[h_j(t,C^{-1}(z_j(s)+\overline{y}_j(s)))-h_j(t,C^{-1}(\overline{y}_j(s)))]\,\mathrm{d}s.
\end{aligned}
$$

Let Π be defined by

$$
\begin{aligned}
\Pi_i(\zeta)=a_{i*}b_{i*}-\zeta-2(a_{i*}b_{i*})^{1/p}\sum_{j=1}^{n} a_j^*\Big\{&o_{ij}^*\|M_j^f\|_p+w_{ij}^*\exp(\zeta\tau^+)\|M_j^g\|_p\\
&+d_{ij}^*\|M_j^h\|_p\int_0^\infty k_{ij}(s)\exp(\zeta s)\,\mathrm{d}s\Big\},
\end{aligned}
$$

where $\zeta\in[0,\infty)$.

From $(\mathbb{A}_3)$ and $(\mathbb{A}_4)$ we have that for any $i\in\Lambda$,

$$
\Pi_i(0)=a_{i*}b_{i*}-2(a_{i*}b_{i*})^{1/p}\sum_{j=1}^{n} a_j^*\{o_{ij}^*\|M_j^f\|_p+w_{ij}^*\|M_j^g\|_p+d_{ij}^*\|M_j^h\|_p\overline{k}_{ij}\}>0,
$$

and Π_i is a continuous function on $[0,\infty)$ satisfying $\lim\limits_{\zeta\to\infty}\Pi_i(\zeta)=-\infty$. Therefore, there exists $\eta_i^\star>0$ such that $\Pi_i(\eta_i^\star)=0$ and $\Pi_i(\eta_i)>0$ for $\eta_i\in]0,\eta_i^\star[$. Let $\eta=\min\{\eta_1^\star,\dots,\eta_n^\star\}$. We obtain $\Pi_i(\eta)\geqslant 0$ for all $i\in\Lambda$. Then we can take $\varpi>0$ such that

$$
0<\varpi<\min\{\eta,a_{1*}b_{1*},\dots,a_{n*}b_{n*},\varpi_0\}\quad\text{and}\quad\Pi_i(\varpi)>0
$$

with ϖ_0 satisfying $\int_0^\infty k_{ij}(s)\exp(\varpi_0 s)<\infty$. So, we have

$$
\begin{aligned}
(5)\qquad \frac{2(a_{i*}b_{i*})^{1/p}}{a}{}_{i*}\,b_{i*}-\varpi\sum_{j=1}^{n}\Big\{&o_{ij}^*a_j^*\|M_j^f\|_p+w_{ij}^*a_j^*\exp(\varpi\tau^+)\|M_j^g\|_p\\
&+d_{ij}^*a_j^*\|M_j^h\|_p\int_0^\infty k_{ij}(s)\exp(\varpi s)\,\mathrm{d}s\Big\}<1.
\end{aligned}
$$

Multiplying (4) by $\exp\left(-\int_0^s \bar{b}_i(z_i(u))\,\mathrm{d}u\right)$ and integrating $[0,t]$, we obtain

$$
\begin{aligned}
(6)\quad z_i(t) &= \vartheta_i(0)\exp\Big(-\int_0^t \bar{b}_i(z_i(u))\,\mathrm{d}u\Big) + \int_0^t \exp\Big(-\int_s^t \bar{b}_i(z_i(u))\,\mathrm{d}u\Big) \\
&\quad \times \sum_{j=1}^n o_{ij}(s)[f_j(s, C^{-1}(z_j(s)+\overline{y}_j(s))) - f_j(s, C^{-1}(\overline{y}_j(s)))]\,\mathrm{d}s \\
&\quad + \int_0^t \exp\Big(-\int_0^t \bar{b}_i(z_i(u))\,\mathrm{d}u\Big)\sum_{j=1}^n w_{ij}(s)[g_j(t, C^{-1}(z_j(s-\tau_{ij}(s)) \\
&\quad + \overline{y}_j(s-\tau_{ij}(s)))) - g_j(s, C^{-1}(\overline{y}_j(s-\tau_{ij}(s))))]\,\mathrm{d}s \\
&\quad + \int_0^t \exp\Big(-\int_0^t \bar{b}_i(z_i(u))\,\mathrm{d}u\Big)\sum_{j=1}^n d_{ij}(s) \\
&\quad \times \int_{-\infty}^s k_{ij}(s-\sigma)[h_j(s, C^{-1}(z_j(s)+\overline{y}_j(s))) - h_j(s, C^{-1}(\overline{y}_j(\sigma)))]\,\mathrm{d}\sigma\,\mathrm{d}s \\
&\leqslant |\vartheta_i(0)|\exp\Big(-\int_0^t \bar{b}_i(z_i(u))\,\mathrm{d}u\Big) + \sum_{j=1}^n \int_0^t \exp(-(t-s)a_{i*}b_{i*})o_{ij}^* \\
&\quad \times |f_j(s, C^{-1}(z_j(s)+\overline{y}_j(s))) - f_j(s, C^{-1}(\overline{y}_j(s)))|\,\mathrm{d}s \\
&\quad + \sum_{j=1}^n \int_0^t \exp(-(t-s)a_{i*}b_{i*})w_{ij}^*|g_j(t, C^{-1}(z_j(s-\tau_{ij}(s)) \\
&\quad + \overline{y}_j(s-\tau_{ij}(s)))) - g_j(s, C^{-1}(\overline{y}_j(s-\tau_{ij}(s))))|\,\mathrm{d}s \\
&\quad + \sum_{j=1}^n \int_0^t \exp(-(t-s)a_{i*}b_{i*})d_{ij}^* \int_{-\infty}^s k_{ij}(s-\sigma) \\
&\quad \times |h_j(s, C^{-1}(z_j(s)+\overline{y}_j(s))) - h_j(s, C^{-1}(\overline{y}_j(\sigma)))|\,\mathrm{d}\sigma\,\mathrm{d}s.
\end{aligned}
$$

Then for all $i \in \Lambda$ we have

$$
\begin{aligned}
(7)\quad |z_i(t)| &\leqslant |\vartheta_i(0)|\exp\Big(-\int_0^t a_i(u)\,\mathrm{d}u\Big) \\
&\quad + \sum_{j=1}^n \int_0^t \exp(-(t-s)a_{i*})o_{ij}^* a_j^* M_j^f(s)|z_j(s)|\,\mathrm{d}s \\
&\quad + \sum_{j=1}^n \int_0^t \exp(-(t-s)a_{i*})w_{ij}^* a_j^* M_j^g(s)|z_j(s-\tau_{ij}(s)|\,\mathrm{d}s \\
&\quad + \sum_{j=1}^n \int_0^t \exp(-(t-s)a_{i*})d_{ij}^* a_j^* \\
&\quad \times \int_{-\infty}^s k_{ij}(s-\sigma)M_j^h(s)|z_j(\sigma)|\,\mathrm{d}\sigma\,\mathrm{d}s.
\end{aligned}
$$

Let

$$N = \max_{i\in\Lambda} \frac{(a_{i*}b_{i*})^{1-1/p}}{\sum\limits_{j=1}^{n}(o_{ij}^*\|M_j^f\|_p + w_{ij}^*\|M_j^g\|_p + d_{ij}^*\|M_j^h\|_p \int_0^\infty k_{ij}(s)\exp(\varpi s)\,\mathrm{d}s)a_j^*}.$$

In view of $(\mathbb{A}_4)$, we can deduce that $M > 2$ and

$$\begin{aligned}(8)\qquad \frac{1}{N} - \frac{(a_{i*}b_{i*})^{1/p}}{a_{i*}b_{i*}-\varpi}\sum_{j=1}^{n} a_j^*\Big\{ & o_{ij}^*\|M_j^f\|_p + w_{ij}^*\exp(\xi\tau^+)\|M_j^g\|_p \\ & + d_{ij}^*\|M_j^h\|_p\int_0^\infty k_{ij}(s)\exp(\varpi s)\,\mathrm{d}s\Big\} \leqslant 0.\end{aligned}$$

We can see that for all $t \in (-\infty, 0]$, $\|z(t)\|_\infty = \|\vartheta\|_\infty \leqslant N\|\vartheta\|_\infty \exp(-\varpi t)$. In the following, we suppose that

$$(9)\qquad \|z(t)\|_\infty \leqslant N\|\vartheta\|_\infty \exp(-\varpi t), \quad t > 0.$$

To prove (9), we first show that for any $\mathfrak{J} > 1$ we have

$$\|z_i(t)\|_\infty \leqslant \mathfrak{J}N\|\vartheta\|_\infty \exp(-\varpi t), \quad t > 0.$$

Suppose that (8) is false, so there must be some $t_1 > 0$ and some $i \in \Lambda$ such that

$$(10)\qquad \|z(t_1)\|_\infty = \|z_i(t_1)\|_\infty = \mathfrak{J}N\|\vartheta\|_\infty \exp(-\varpi t_1)$$

and

$$(11)\qquad \|z(t)\|_\infty \leqslant \mathfrak{J}N\|\vartheta\|_\infty \exp(-\varpi t_1) \quad \forall\, t \in (-\infty, t_1).$$

From (5), (7), (8), (11) and assumption $(\mathbb{A}_2)$ we have

$$\begin{aligned}|z_i(t_1)| \leqslant{} & |\vartheta_i(0)|\exp(-t_1(a_{i*}b_{i*})) \\ & + \sum_{j=1}^{n}\int_0^{t_1}\exp(-(t_1-s)(a_{i*}b_{i*}))o_{ij}^*a_j^*M_j^f(s)|z_j(s)|\,\mathrm{d}s \\ & + \sum_{j=1}^{n}\int_0^{t_1}\exp(-(t_1-s)(a_{i*}b_{i*}))w_{ij}^*a_j^*M_j^g(s)|z_j(s-\tau_{ij}(s))|\,\mathrm{d}s \\ & + \sum_{j=1}^{n}\int_0^{t_1}\exp(-(t_1-s)(a_{i*}b_{i*}))d_{ij}^*a_j^*\int_{-\infty}^{s}k_{ij}(s-\sigma)M_j^h(s)|z_j(\sigma)|\,\mathrm{d}\sigma\,\mathrm{d}s \\ \leqslant{} & \|\vartheta\|_\infty\exp(-t_1(a_{i*}b_{i*})) \\ & + \sum_{j=1}^{n}\int_0^{t_1}\exp(-(t_1-s)(a_{i*}b_{i*}))o_{ij}^*a_j^*M_j^f(s)\|z_j(s)\|_\infty\,\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{n}\int_0^{t_1}\exp\left(-(t_1-s)a_{i*}b_{i*}\right)w_{ij}^*a_j^*M_j^g(s)\|z_j(s-\tau_{ij}(s))\|_\infty\,\mathrm{d}s\\
&+\sum_{j=1}^{n}\int_0^{t_1}\exp\left(-(t_1-s)(a_{i*}b_{i*})\right)d_{ij}^*a_j^*\int_{-\infty}^{s}k_{ij}(s-\sigma)M_j^h(s)\|z_j(\sigma)\|_\infty\,\mathrm{d}\sigma\,\mathrm{d}s\\
\leqslant{}&\|\vartheta\|_\infty\exp\left(-t_1(a_{i*}b_{i*})\right)\\
&+\sum_{j=1}^{n}\int_0^{t_1}\exp\left(-(t_1-s)(a_{i*}b_{i*})\right)o_{ij}^*a_j^*M_j^f(s)\mathfrak{J}N\|\vartheta\|_\infty\exp\left(-\varpi s\right)\mathrm{d}s\\
&+\sum_{j=1}^{n}\int_0^{t_1}\exp\left(-(t_1-s)(a_{i*}b_{i*})\right)\\
&\times w_{ij}^*a_j^*M_j^g(s)\mathfrak{J}N\|\vartheta\|_\infty\exp\left(-\varpi(s-\tau_{ij}(s))\right)\mathrm{d}s\\
&+\sum_{j=1}^{n}\int_0^{t_1}\exp\left(-(t_1-s)(a_{i*}b_{i*})\right)d_{ij}^*a_j^*\\
&\times\int_{-\infty}^{s}k_{ij}(s-\sigma)M_j^h(s)\mathfrak{J}N\|\vartheta\|_\infty\exp\left(-\varpi\sigma\right)\mathrm{d}\sigma\,\mathrm{d}s\\
\leqslant{}&\|\vartheta\|_\infty\exp\left(-t_1(a_{i*}b_{i*})\right)+\exp\left(-t_1(a_{i*}b_{i*})\right)\\
&\times\sum_{j=1}^{n}\int_0^{t_1}\exp\left(((a_{i*}b_{i*})-\varpi)s\right)o_{ij}^*a_j^*M_j^f(s)\mathfrak{J}N\|\vartheta\|_\infty\,\mathrm{d}s\\
&+\exp\left(-t_1(a_{i*}b_{i*})\right)\sum_{j=1}^{n}w_{ij}^*a_j^*\exp\left(\varpi\tau^+\right)\\
&\times\int_0^{t_1}\exp\left(((a_{i*}b_{i*})-\varpi)s\right)M_j^g(s)\mathfrak{J}N\|\vartheta\|_\infty\,\mathrm{d}s\\
&+\exp\left(-t_1(a_{i*}b_{i*})\right)\sum_{j=1}^{n}d_{ij}^*a_j^*\int_0^{t_1}M_j^h(s)\exp\left((a_{i*}b_{i*}-\varpi)s\right)\mathrm{d}s\\
&\times\int_0^{\infty}k_{ij}(m)\mathfrak{J}N\|\vartheta\|_\infty\exp\left(\varpi m\right)\mathrm{d}m\\
\leqslant{}&\mathfrak{J}N\|\vartheta\|_\infty\exp\left(-\varpi t_1\right)\biggl[\frac{\exp\left((\varpi-a_{i*}b_{i*})t_1\right)}{\mathfrak{J}N}+\exp\left((\varpi-(a_{i*}b_{i*}))t_1\right)\sum_{j=1}^{n}o_{ij}^*a_j^*\\
&\times\int_0^{t_1}\exp\left((a_{i*}b_{i*}-\varpi)s\right)M_j^f(s)\,\mathrm{d}s+\exp\left((\varpi-a_{i*}b_{i*})t_1\right)\sum_{j=1}^{n}w_{ij}^*a_j^*\\
&\times\exp\left(\varpi\tau^+\right)\int_0^{t_1}\exp\left((a_{i*}b_{i*}-\varpi)s\right)M_j^g(s)\,\mathrm{d}s+\exp\left((\varpi-a_{i*}b_{i*})t_1\right)\\
&\times\sum_{j=1}^{n}d_{ij}^*a_j^*\int_0^{t_1}M_j^h(s)\exp\left((a_{i*}b_{i*}-\varpi)s\right)\mathrm{d}s\int_0^{\infty}k_{ij}(m)\exp\left(\varpi m\right)\mathrm{d}m\biggr].
\end{aligned}
$$

Let $q>1$ such that $1/p+1/q=1$. It follows that for all $i \in \Lambda$

$$\begin{aligned}
|z_i(t_1)| &= \mathfrak{J}N\|\vartheta\|_\infty \exp(-\varpi t_1)\bigg[\frac{\exp((\varpi - a_{i*}b_{i*})t_1)}{\mathfrak{J}N} \\
&\quad + \exp((\varpi - a_{i*}b_{i*})t_1)\sum_{j=1}^n o_{ij}^* a_j^* \|M_j^f\|_p \bigg(\int_0^{t_1} \exp(q(a_{i*}b_{i*} - \varpi)s)\,\mathrm{d}s\bigg)^{1/q} \\
&\quad + \exp((\varpi - a_{i*}b_{i*})t_1)\sum_{j=1}^n w_{ij}^* a_j^* \exp(\varpi\tau^+)\|M_j^g\|_p \\
&\quad \times \bigg(\int_0^{t_1} \exp(q(a_{i*}b_{i*} - \varpi)s)\,\mathrm{d}s\bigg)^{1/q} \\
&\quad + \exp((\varpi - a_{i*}b_{i*})t_1)\sum_{j=1}^n d_{ij}^* a_j^* \|M_j^h\|_p \\
&\quad \times \bigg(\int_0^{t_1} \exp(q(a_{i*}b_{i*} - \varpi)s)\,\mathrm{d}s\bigg)^{1/q} \int_0^\infty k_{ij}(m)\exp(\varpi m)\,\mathrm{d}m\bigg] \\
&\leqslant \mathfrak{J}N\|\vartheta\|_\infty \exp(-\varpi t_1)\bigg[\frac{\exp((\varpi - a_{i*}b_{i*})t_1)}{\mathfrak{J}N} + \frac{1}{[q(a_{i*}b_{i*} - \varpi)]^{1/q}} \\
&\quad \times \sum_{j=1}^n o_{ij}^* a_j^* \|M_j^f\|_p \exp((\varpi - a_{i*}b_{i*})t_1)(\exp(q(a_{i*}b_{i*} - \varpi)t_1) - 1)^{1/q} \\
&\quad + \frac{1}{[q(a_{i*}b_{i*} - \varpi)]^{1/q}}\sum_{j=1}^n w_{ij}^* a_j^* \exp(\varpi\tau^+)\|M_j^g\|_p \\
&\quad \times \exp((\varpi - a_{i*}b_{i*})t_1)(\exp(q(a_{i*}b_{i*} - \varpi)t_1) - 1)^{1/q} \\
&\quad + \frac{1}{[q(a_{i*}b_{i*} - \varpi)]^{1/q}}\sum_{j=1}^n d_{ij}^* a_j^* \|M_j^h\|_p \exp((\varpi - a_{i*}b_{i*})t_1) \\
&\quad \times (\exp(q(a_{i*}b_{i*} - \varpi)t_1) - 1)^{1/q} \int_0^\infty k_{ij}(m)\exp(\varpi m)\,\mathrm{d}m\bigg] \\
&\leqslant \mathfrak{J}N\|\vartheta\|_\infty \exp(-\varpi t_1)\bigg[\frac{\exp((\varpi - a_{i*}b_{i*})t_1)}{\mathfrak{J}N} + \frac{1}{[q(a_{i*}b_{i*} - \varpi)]^{1/q}} \\
&\quad \times \sum_{j=1}^n o_{ij}^* a_j^* \|M_j^f\|_p \exp((\varpi - a_{i*}b_{i*})t_1)(1 - \exp(q(a_{i*}b_{i*} - \varpi)t_1))^{1/q} \\
&\quad + \frac{1}{[q(a_{i*}b_{i*} - \varpi)]^{1/q}}\sum_{j=1}^n w_{ij}^* a_j^* \exp(\varpi\tau^+)\|M_j^g\|_p \\
&\quad \times \exp((\varpi - a_{i*}b_{i*})t_1)(1 - \exp(q(a_{i*}b_{i*} - \varpi)t_1))^{1/q} \\
&\quad + \frac{1}{[q(a_{i*}b_{i*} - \varpi)]^{1/q}}\sum_{j=1}^n d_{ij}^* a_j^* \|M_j^h\|_p \exp((\varpi - a_{i*}b_{i*})t_1) \\
&\quad \times (1 - \exp(q(a_{i*}b_{i*} - \varpi)t_1))^{1/q} \int_0^\infty k_{ij}(m)\exp(\varpi m)\,\mathrm{d}m\bigg]
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \mathfrak{J}N\|\vartheta\|_\infty \exp(-\varpi t_1)\bigg[\frac{\exp((\varpi - a_{i*}b_{i*})t_1)}{\mathfrak{J}N} + \frac{(a_{i*}b_{i*}-\varpi)^{1/p}}{a_{i*}b_{i*}-\varpi} \\
&\quad \times \bigg(\sum_{j=1}^{n} o_{ij}^* a_j^* \|M_j^f\|_p + \sum_{j=1}^{n} w_{ij}^* a_j^* \exp(\varpi\tau^+)\|M_j^g\|_p \\
&\quad + \sum_{j=1}^{n} d_{ij}^* a_j^* \|M_j^h\|_p \int_0^\infty k_{ij}(m)\exp(\varpi m)\,\mathrm{d}m\bigg)(2 - \exp((\varpi - a_{i*}b_{i*})t_1))\bigg] \\
&\leqslant \mathfrak{J}N\|\vartheta\|_\infty \exp(-\varpi t_1)\bigg[\frac{\exp((\varpi - a_{i*}b_{i*})t_1)}{\mathfrak{J}N} \\
&\quad + \frac{(a_{i*}b_{i*})^{1/p}}{a_{i*}b_{i*}-\varpi}\bigg(\sum_{j=1}^{n} o_{ij}^* a_j^* \|M_j^f\|_p + \sum_{j=1}^{n} w_{ij}^* a_j^* \exp(\varpi\tau^+)\|M_j^g\|_p \\
&\quad + \sum_{j=1}^{n} d_{ij}^* a_j^* \|M_j^h\|_p \int_0^\infty k_{ij}(m)\exp(\varpi m)\,\mathrm{d}m\bigg)(2 - \exp((\varpi - a_{i*}b_{i*})t_1))\bigg] \\
&\leqslant \mathfrak{J}N\|\vartheta\|_\infty \exp(-\varpi t_1)\bigg[\exp((\varpi - a_{i*}b_{i*})t_1)\frac{1}{N} \\
&\quad + \frac{(a_{i*}b_{i*})^{1/p}}{a_{i*}b_{i*}-\varpi}\bigg(\sum_{j=1}^{n} o_{ij}^* a_j^* \|M_j^f\|_p + \sum_{j=1}^{n} w_{ij}^* a_j^* \exp(\varpi\tau^+)\|M_j^g\|_p \\
&\quad + \sum_{j=1}^{n} d_{ij}^* a_j^* \|M_j^h\|_p \int_0^\infty k_{ij}(m)\exp(\varpi m)\,\mathrm{d}m\bigg) \\
&\quad + \frac{2(a_{i*}b_{i*})^{1/p}}{(a_{i*}b_{i*}-\varpi)}\bigg(\sum_{j=1}^{n} o_{ij}^* a_j^* \|M_j^f\|_p + \sum_{j=1}^{n} w_{ij}^* a_j^* \exp(\varpi\tau^+)\|M_j^g\|_p \\
&\quad + \sum_{j=1}^{n} d_{ij}^* a_j^* \|M_j^h\|_p \int_0^\infty k_{ij}(m)\exp(\varpi m)\,\mathrm{d}m\bigg)\bigg],
\end{aligned}
$$

which means that for all $i \in \Lambda$ we get

$$
\begin{aligned}
\|z_i(t_1)\|_\infty \leqslant \mathfrak{J}N\|\vartheta\|_\infty \exp(-\varpi t_1)\bigg[&\frac{2(a_{i*}b_{i*})^{1/p}}{a_{i*}b_{i*}-\varpi}\bigg(\sum_{j=1}^{n} o_{ij}^* a_j^* \|M_j^f\|_p \\
&+ \sum_{j=1}^{n} w_{ij}^* a_j^* \exp(\varpi\tau^+)\|M_j^g\|_p \\
&+ \sum_{j=1}^{n} d_{ij}^* a_j^* \|M_j^h\|_p \int_0^\infty k_{ij}(m)\exp(\varpi m)\,\mathrm{d}m\bigg)\bigg] \\
&< \mathfrak{J}N\|\vartheta\|_\infty \exp(-\varpi t_1),
\end{aligned}
$$

which is contrary to (10). Then for any $\mathfrak{J} > 1$ we get

$$
\|z_i(t)\|_\infty < \mathfrak{J}N\|\vartheta| \exp(-\varpi t), \quad t > 0.
$$

As $\mathfrak{J} \to 1$, we get $\|z(t)\|_\infty \leqslant N\|\vartheta\|_\infty \exp(-\varpi t)$, $t > 0$. Thus, from Definition 2, the (μ,ν)-pseudo almost automorphic solution of system (3) is globally exponentially stable. □

R e m a r k 4. In light of Theorems 1–2, the existence, the uniqueness and the global exponential stability of (μ,ν)-pseudo almost automorphic solution of system in equation (1) are obtained.

4. Numerical example

Consider the differential equations

$$
\begin{aligned}
(12)\qquad x_i'(t) = &-a_i(x_i(t))\Big[b_i(x_i(t)) - \sum_{j=1}^{2} o_{ij}(t) f_j(t, x_j(t)) \\
&- \sum_{j=1}^{2} w_{ij}(t) g_j(t, x_j(t-\tau_{ij}(t))) \\
&- \sum_{j=1}^{2} d_{ij}(t) \int_{-\infty}^{t} k_{ij}(t-s) h_j(s, x_j(s))\,\mathrm{d}s - L_i(t)\Big], \quad 1 \leqslant i \leqslant 2
\end{aligned}
$$

with $a_1(\cdot) = \cos(\cdot) + 3$, $a_2(\cdot) = -\sin(\cdot) + 3$, $b_1(x_1(t)) = 3x_1(t)$, $b_2(x_2(t)) = 2x_2(t)$,

$$x_i(s) = \vartheta_i(s), \quad s \in (-\infty, 0]; \quad i \in \Lambda,$$

$$o(t) = (o_{ij}(t)) = \begin{bmatrix} \frac{1}{20}\cos(\sqrt{3}t) + \frac{1}{100}\exp(-|t|) & \frac{1}{100}\exp(-|t|) \\ \frac{1}{50}\sin(\sqrt{3}t) & \frac{1}{50}\cos(\sqrt{3}t) + \frac{1}{100}\exp(-|t|) \end{bmatrix},$$

$$w(t) = (w_{ij}(t)) = \begin{bmatrix} \frac{1}{20}\sin(t) & \frac{1}{50}\cos(t) + \frac{1}{100}\exp(-|t|) \\ \frac{1}{20}\sin(t) & \frac{1}{50}\cos(t) + \frac{1}{100}\exp(-|t|) \end{bmatrix},$$

$$d(t) = (d_{ij}(t)) = \begin{bmatrix} 0.03\cos(\sqrt{3}t) + \frac{1}{100}\exp(-|t|) & \frac{1}{100}\sin(\sqrt{3}t) + \frac{1}{100}\exp(-|t|) \\ 0.1\cos(t) + \frac{1}{100}\exp(-|t|) & 0.03\cos(t) + \frac{1}{100}\exp(-|t|) \end{bmatrix},$$

$$k_{ij}(t) = \exp(-t), \quad \tau_{ij}(t) = |\cos(\sqrt{3}t)|;$$

$$L_1(t) = 0.6\sin(\sqrt{5}t), \quad L_2(t) = 0.3\cos(\sqrt{11}t) + \frac{1}{100}\exp(-|t|),$$

$$f_j(t,x) = g_j(t,x) = h_j(t,x) = \cos\Big(\frac{1}{2+\sin(t)+\sin(\sqrt{2}t)}\Big)\frac{\tanh(x(t))}{3\sqrt{1+t^2}}.$$

By computing, we have

$$o^* = (o^*_{ij}) = \begin{bmatrix} 0.06 & 0.01 \\ 0.02 & 0.03 \end{bmatrix}, \quad w^* = (w^*_{ij}) = \begin{bmatrix} 0.05 & 0.03 \\ 0.05 & 0.03 \end{bmatrix}, \quad d* = (d^*_{ij}) = \begin{bmatrix} 0.04 & 0.02 \\ 0.02 & 0.04 \end{bmatrix},$$

$$L^*_1 = 0.6, \quad L^*_2 = 0.31,$$

$$|f_j(t,x) - f_j(t,z)| \leqslant M^f_j(t)|x-z|,$$

$$|g_j(t,x) - g_j(t,z)| \leqslant M^g_j(t)|x-z|,$$

$$|h_j(t,x) - h_j(t,z)| \leqslant M^h_j(t)|x-z|$$

with

$$M^f_j(t) = M^g_j(t) = M^h_j(t) = \frac{1}{3\sqrt{1+t^2}}.$$

Since $\|M^f_j\|_{\mathcal{L}^2(\mathbb{R},\,\mathrm{d}x)} = \|M^g_j\|_{\mathcal{L}^2(\mathbb{R},\,\mathrm{d}x)} = \|M^h_j\|_{\mathcal{L}^2(\mathbb{R},\,\mathrm{d}x)} = \sqrt{\pi}/3$ for all $j = 1,2$, then M^f_j, M^g_j, $M^h_j \in \mathcal{L}^2(\mathbb{R},\,\mathrm{d}x)$. Now, we consider the measure μ, where its *Radon-Nikodym* derivative is $\overline{\varrho}_1(t) = \exp(t)$ for all $t \in \mathbb{R}$. Then $\mu \in \mathcal{M}$. Since if $\overline{\varrho}_1(t) > 0$, then from [1], condition (H_1) is equivalent to

$$\limsup_{|t|\to\infty} \frac{\overline{\varrho}_1(t+\tau)}{\overline{\varrho}_1(t)} < \infty \quad \forall\, \tau \in \mathbb{R}.$$

Then μ satisfies hypothesis (H_1) and $M^f_j, M^g_j, M^h_j \in \mathcal{L}^2(\mathbb{R},\,\mathrm{d}\mu)$.

On the other hand, we take the measure ν, where its *Radon-Nikodym* derivative is $\overline{\varrho}_2(t) = \exp(\sin(t))$. Since

$$\limsup_{\varrho\to\infty} \frac{\mu[-\varrho,\varrho]}{\nu[-\varrho,\varrho]} = \limsup_{\varrho\to\infty} \frac{\int_{-\varrho}^{\varrho} \overline{\varrho}_1(t)\,\mathrm{d}t}{\int_{-\varrho}^{\varrho} \overline{\varrho}_2(t)\,\mathrm{d}t} < \infty,$$

condition (H_2) is satisfied.

R e m a r k 5. Similarly to Theorem 7 in [1] let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous such that $\lim\limits_{|t|\to\infty} f(t) = 0$. Then f is (μ,ν)-ergodic for all $\mu,\nu \in \mathcal{M}$.

From Remark 4.1 we have $[t \mapsto \exp(-|t|)] \in \mathcal{E}(\mathbb{R},\mathbb{H},\mu,\nu)$ for all $\mu,\nu \in \mathcal{M}$. Since $\lim\limits_{|t|\to\infty} \exp(-|t|) = 0$, it follows that $(\mathbb{A}_3)$ holds. Then

$$\begin{aligned}\theta &= \max_{1\leqslant i\leqslant 2}\biggl\{\frac{1}{(a_{i*}b_{i*})^{1-1/2}} \sum_{j=1}^{2} a^*_j(o^*_{ij}\|M^f_j\|_2 + w^*_{ij}\|M^g_j\|_2 + (d^*_{ij} + \delta^*_{ij})\overline{k}_{ij}\|M^h_j\|_2)\biggr\} \\ &= \max\{0.2026; 0.2245\} \\ &= 0.2245 < \frac{1}{2}\end{aligned}$$

and

$$\xi = \max_{i\in\Lambda}\biggl\{\frac{L^*_i}{a_{i*}b_{i*}}\biggr\} = \max\biggl\{\frac{0.6}{6}; \frac{0.31}{4}\biggr\} = 0.1.$$

We have for any $\varpi_0 \in]0,1[$:

$$\int_0^\infty k_{ij}(s)\exp(\varpi_0 s)\,\mathrm{d}s = \int_0^\infty \exp(-(1-\varpi_0)s)\,\mathrm{d}s = \frac{1}{1-\varpi_0} < \infty.$$

As a conclusion, all conditions of Theorem 1 and Theorem 2 are satisfied. Then the differential equation (12) has a unique (μ,ν)-pseudo almost automorphic solution which is globally exponentially stable (as shown in Figs. 1–3) in the region

$$\Im = \Big\{\vartheta \in \mathcal{PAA}(\mathbb{R},\mathbb{R}^n,\mu,\nu),\ \|\vartheta-\vartheta_0\|_\infty \leqslant \frac{\theta\xi}{1-\theta} \leqslant 0.0289\Big\}.$$

Remark 6. In Figures 1 and 2, the initial conditions are random for $s \in (-\infty, 0]$ we take:

- ▷ $x_1(s) = \vartheta_1(s) = -1$, $x_2(s) = \vartheta_2(s) = 1$,
- ▷ $x_1(s) = \vartheta_1(s) = 1$, $x_2(s) = \vartheta_2(s) = -1$,
- ▷ $x_1(s) = \vartheta_1(s) = -0.8$, $x_2(s) = \vartheta_2(s) = 0.8$,
- ▷ $x_1(s) = \vartheta_1(s) = 0.8$, $x_2(s) = \vartheta_2(s) = -0.8$,
- ▷ $x_1(s) = \vartheta_1(s) = 2$, $x_2(s) = \vartheta_2(s) = -2$,
- ▷ $x_1(s) = \vartheta_1(s) = -2$, $x_2(s) = \vartheta_2(s) = 2$,
- ▷ $x_1(s) = \vartheta_1(s) = -1.5$, $x_2(s) = \vartheta_2(s) = 1.5$,
- ▷ $x_1(s) = \vartheta_1(s) = -0.5$, $x_2(s) = \vartheta_2(s) = 0.5$,
- ▷ $x_1(s) = \vartheta_1(s) = 1.5$, $x_2(s) = \vartheta_2(s) = -1.5$,
- ▷ $x_1(s) = \vartheta_1(s) = 0.5$, $x_2(s) = \vartheta_2(s) = -0.5$.

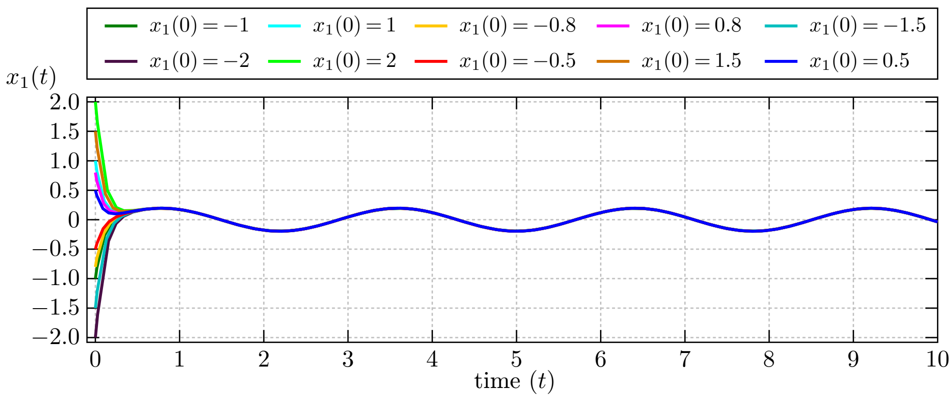

Figure 1. Curves of $x_1(t)$ with random initial conditions.

5. Conclusion

In this paper, the Cohen-Grosberg differential equations with mixed delays and time-varying coefficient have been investigated. Based on few properties of the doubly measure pseudo-almost automorphic functions and the fixed-point theorem, we established a new criterion for the existence, the uniqueness and the global exponential stability of the (μ, ν)-pseudo almost automorphic solutions. To the best of our knowledge, this is the first time that the doubly measure pseudo almost automorphic solution for differential equations with mixed delays and time varying coefficient is studied. In future works, we would like to extend our results to more general neutral-type of delayed differential equations, such as high-order neutral-type of differential equations, second-order differential equations and third-order differential equations.

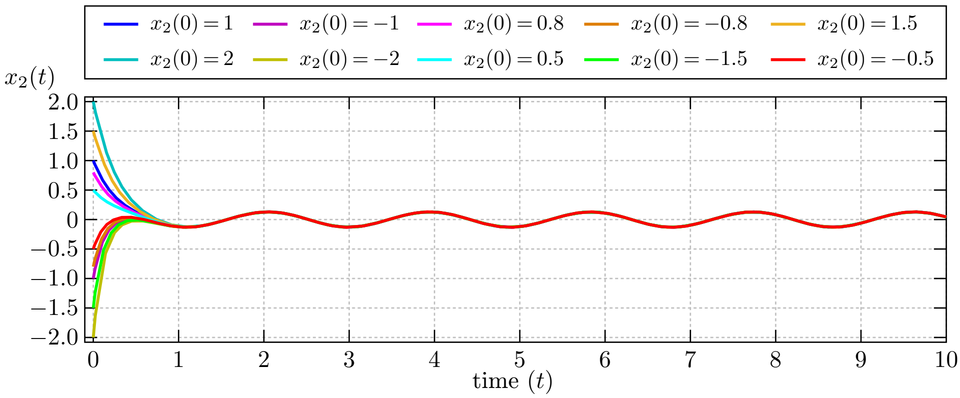

Figure 2. Curves of $x_2(t)$ with random initial conditions.

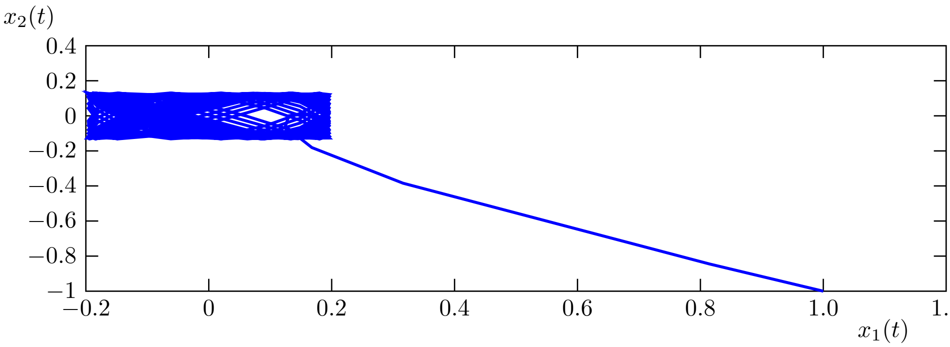

Figure 3. The orbits of $x_1(t)$ and $x_2(t)$.

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