

## A FRICTIONAL CONTACT PROBLEM WITH ADHESION FOR VISCOELASTIC MATERIALS WITH LONG MEMORY

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*Abstract.* We consider a quasistatic contact problem between a viscoelastic material with long-term memory and a foundation. The contact is modelled with a normal compliance condition, a version of Coulomb's law of dry friction and a bonding field which describes the adhesion effect. We derive a variational formulation of the mechanical problem and, under a smallness assumption, we establish an existence theorem of a weak solution including a regularity result. The proof is based on the time-discretization method, the Banach fixed point theorem and arguments of lower semicontinuity, compactness and monotonicity.

*Keywords:* viscoelastic material; long memory; adhesion; quasistatic process; Coulomb's law of dry friction; normal compliance; the time-discretization method; variational inequality

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### 1. INTRODUCTION

In rheology, connecting a spring and a dashpot in parallel leads to the Kelvin-Voigt model whereas using a combination of a spring and a dashpot in series yields the Maxwell model, where springs and dashpots represent elastic and viscous properties, respectively. However, it is well known that the Maxwell model cannot adequately describe viscoelastic behaviour in creep and the Kelvin-Voigt model cannot predict the stress relaxation. For this reason, we need to construct more complex rheological models. For example, by using two springs and a dashpot, one can build up the Zener model, the so-called standard linear solid model. It is the simplest model that can describe these two phenomena (see, e.g. [5], [6]). Taking the Maxwell model together with a spring in parallel leads to the Maxwell representation of the Zener model, see Figure 1.

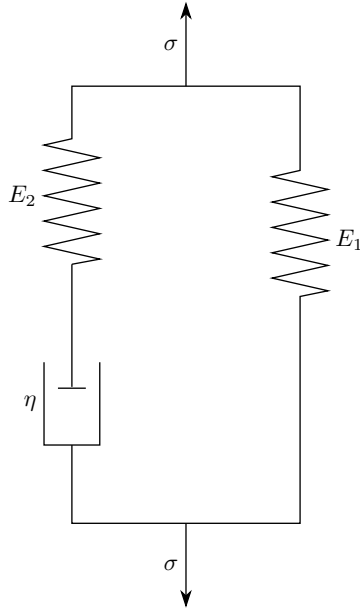


Figure 1. The Maxwell form of the Zener model.

In this model, the dashpot and the two springs obey Newton's law and Hooke's law, respectively, that is

$$(1.1) \quad (i) \quad \sigma_D = \eta \dot{\varepsilon}_D, \quad (ii) \quad \sigma_2 = E_2 \varepsilon_2, \quad (iii) \quad \sigma_1 = E_1 \varepsilon_1,$$

and, moreover, for series components, we have the relationships

$$(1.2) \quad \sigma_m = \sigma_D = \sigma_2, \quad \varepsilon_m = \varepsilon_D + \varepsilon_2,$$

whereas, for parallel components, we have

$$(1.3) \quad \sigma = \sigma_m + \sigma_1, \quad \varepsilon = \varepsilon_m = \varepsilon_1,$$

where  $\sigma_D$  is the stress applied to the dashpot,  $\sigma_1$  and  $\sigma_2$  are the stresses applied to spring 1 and spring 2, respectively,  $\varepsilon_D$  is the strain that occurs in the dashpot, an overdot denotes ordinary differentiation with respect to the time variable  $t$ ,  $\varepsilon_1$  and  $\varepsilon_2$  are the strains that occur in spring 1 and spring 2, respectively,  $\eta$  is the viscosity of the dashpot component,  $E_1$  and  $E_2$  represent Young's modulus of spring 1 and spring 2, respectively,  $\sigma_m$  is the stress applied to the Maxwell arm,  $\varepsilon_m$  is the strain that occurs in the Maxwell arm,  $\sigma$  is the total stress and  $\varepsilon$  is the total strain. It

follows from (1.1)–(1.3), that

$$(1.4) \quad \frac{E_2}{\eta} \sigma + \dot{\sigma} = \frac{E_1 E_2}{\eta} \varepsilon + (E_1 + E_2) \dot{\varepsilon},$$

which gives

$$\left( \frac{E_2}{\eta} \sigma(s) + \dot{\sigma}(s) \right) e^{E_2 s / \eta} = \frac{E_1 E_2}{\eta} \varepsilon(s) e^{E_2 s / \eta} + (E_1 + E_2) \dot{\varepsilon}(s) e^{E_2 s / \eta},$$

integrating both sides of this equation on  $(0, t)$  with the initial conditions  $\sigma(0) = \varepsilon(0) = 0$  and using integration by parts, we get

$$(1.5) \quad \sigma(t) = a \varepsilon(t) + \int_0^t b(t-s) \varepsilon(s) \, ds,$$

where

$$a = E_1 + E_2, \quad b(t-s) = -\frac{E_2^2}{\eta} e^{-E_2(t-s)/\eta}.$$

Also, taking the Kelvin-Voigt model together with a linear elastic spring in series, we obtain the Kelvin-Voigt representation of the Zener model with equations analogous to those obtained in (1.4)–(1.5). We now extend the integral law (1.5) to the  $d$ -dimensional case ( $d = 2, 3$ ) to obtain

$$(1.6) \quad \sigma(t) = \mathcal{A} \varepsilon(u(t)) + \int_0^t \mathcal{B}(t-s) \varepsilon(u(s)) \, ds,$$

where  $\sigma$  represents the stress tensor,  $u$  denotes the displacement field,  $\varepsilon(u)$  is the linearized strain tensor,  $\mathcal{A}$  is the elasticity operator and  $\mathcal{B}$  denotes the tensor of relaxation. Analysis of various boundary value problems with a constitutive equation of the form (1.6), also known as the viscoelastic law with long memory, can be found for instance in [14], [16], [22], [23] and references therein.

This paper represents a continuation of [14]. There, the contact problem with Tresca's law involving the slip dependent coefficient of friction for materials with a constitutive law of the form (1.6) was modelled and an existence result, for a friction coefficient small enough, was established. The novelty of the present paper consists in dealing with a quasistatic contact problem for viscoelastic materials with a constitutive law of the form (1.6), such that the contact is modelled with a normal compliance condition and, moreover, both friction and adhesion are taken into account.

The adhesive contact between bodies, when a glue is used to bind two separate surfaces together and prevents their relative motion, has received recently considerable attention in the mathematical literature. General models with adhesion can be found

in [9], [10]. Related adhesive contact problems can be found in [3], [4], [13], [20], [21], [24] and references therein. Following [9], [10], we introduce a surface internal variable  $\beta \in [0, 1]$ , which is a measure of the intensity of adhesion between the contact surface and the foundation. When  $\beta = 1$  all the bonds are active and there is total adhesion; when  $0 < \beta < 1$  partial adhesion takes place; when  $\beta = 0$  there is no adhesion.

The rest of this paper is organized as follows. In Section 2 we present the notation and some preliminaries we use in our study. Section 3 and 4 are dedicated to describe the mechanical problem and derive its variational formulation. In Section 5 we establish the existence of a weak solution to the problem. Our analysis is based on the time-discretization method. By using the backward Euler scheme, we construct a sequence of elliptic quasi-variational inequalities for which at each time step, under a smallness assumption, we prove the existence of a unique solution. Then, after obtaining the necessary estimates, we construct approximate solutions and prove that the limit of a subsequence of the solutions of the approximate problems is a solution of the continuous problem.

## 2. NOTATION AND PRELIMINARIES

Here we introduce the notation we will use and some preliminary materials. For further details we refer the reader to [7], [18], [21]. We use the notation  $\mathbb{N}^*$  for the set of positive integers. We denote by  $\mathbb{S}^d$  the space of second order symmetric tensors on  $\mathbb{R}^d$  ( $d = 2, 3$ ) and we define the inner products and the corresponding norms on  $\mathbb{R}^d$  and  $\mathbb{S}^d$  by

$$\begin{aligned} w \cdot v &= \sum_{i=1}^d w_i v_i, \quad |v| = \sqrt{v \cdot v} \quad \forall w, v \in \mathbb{R}^d; \\ \sigma \cdot \xi &= \sum_{1 \leq i, j \leq d} \sigma_{ij} \xi_{ij}, \quad |\sigma| = \sqrt{\sigma \cdot \sigma} \quad \forall \sigma, \xi \in \mathbb{S}^d. \end{aligned}$$

Let  $\Omega \subset \mathbb{R}^d$  ( $d = 2, 3$ ) be a bounded domain with a Lipschitz boundary  $\Gamma$  and let  $\nu$  denote the unit outer normal on  $\Gamma$ . Let  $[0, T]$ ,  $T > 0$  be the time interval of interest, let  $t \in [0, T]$  be the time variable and let  $x \in \bar{\Omega}$  be the spatial variable. We introduce the spaces

$$\begin{aligned} H &= \{v = (v_i); v_i \in L^2(\Omega), 1 \leq i \leq d\}, \\ H_1 &= \{v = (v_i); v_i \in H^1(\Omega), 1 \leq i \leq d\}, \\ \mathcal{Q} &= \{\xi = (\xi_{ij}); \xi_{ij} = \xi_{ji} \in L^2(\Omega), 1 \leq i, j \leq d\}, \\ \mathcal{Q}_1 &= \left\{ \xi \in \mathcal{Q}; \sum_{j=1}^d \frac{\partial \xi_{ij}}{\partial x_j} \in L^2(\Omega), 1 \leq i \leq d \right\}. \end{aligned}$$

Note that  $H$ ,  $\mathcal{Q}$ ,  $H_1$  and  $\mathcal{Q}_1$  are real Hilbert spaces endowed with the inner products given by

$$\begin{aligned}(w, v)_H &= \int_{\Omega} w \cdot v \, dx, \quad (\sigma, \tau)_{\mathcal{Q}} = \int_{\Omega} \sigma \cdot \tau \, dx, \\(w, v)_{H_1} &= (w, v)_H + (\varepsilon(w), \varepsilon(v))_{\mathcal{Q}}, \\(\sigma, \tau)_{\mathcal{Q}_1} &= (\text{Div } \sigma, \text{Div } \tau)_H + (\sigma, \tau)_{\mathcal{Q}},\end{aligned}$$

where  $\varepsilon: H_1 \rightarrow \mathcal{Q}$  is the deformation operator, defined by

$$\varepsilon(v) = (\varepsilon_{ij}(v)), \quad \varepsilon_{ij}(v) = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right), \quad 1 \leq i, j \leq d, \quad \forall v \in H_1,$$

$\text{Div}: \mathcal{Q}_1 \rightarrow H$  is the divergence operator for tensor functions, defined by

$$\text{Div } \sigma = (\sigma_{ij,j}), \quad \sigma_{ij,j} = \sum_{j=1}^d \frac{\partial \sigma_{ij}}{\partial x_j}, \quad 1 \leq i \leq d, \quad \forall \sigma \in \mathcal{Q}_1.$$

The associated norms on the spaces  $H$ ,  $\mathcal{Q}$ ,  $H_1$  and  $\mathcal{Q}_1$  are denoted by  $\|\cdot\|_H$ ,  $\|\cdot\|_{\mathcal{Q}}$ ,  $\|\cdot\|_{H_1}$  and  $\|\cdot\|_{\mathcal{Q}_1}$ . We note that the real-valued function  $v \mapsto \|v\|$  defined by

$$\|v\| = \left( \sum_{i=1}^d \int_{\Omega} (v_i)^2 \, dx + \sum_{i=1}^d \sum_{j=1}^d \int_{\Omega} \left( \frac{\partial v_i}{\partial x_j} \right)^2 \, dx \right)^{1/2} \quad \forall v \in H_1$$

and  $\|\cdot\|_{H_1}$  are equivalent norms on  $H_1$ . Let  $\tilde{\gamma}: H_1 \rightarrow L^2(\Gamma)^d$  be the trace map. We recall that  $\tilde{\gamma}$  is a compact operator, i.e., for any bounded sequence  $\{v_n\}$  in  $H_1$  there is a subsequence of  $\{v_n\}$  which is convergent in  $L^2(\Gamma)^d$ . For every element  $v \in H_1$  we denote by  $\tilde{\gamma}(v)$  the trace of  $v$  on  $\Gamma$  and for all  $v \in H_1$  we denote by  $v_{\nu}$  and  $v_{\tau}$  the normal and tangential components of  $v$  on the boundary  $\Gamma$ ,

$$v_{\nu} = v \cdot \nu, \quad v_{\tau} = v - v_{\nu} \nu \quad \text{on } \Gamma.$$

In a similar manner, the normal and tangential components of a regular (say  $C^1$ ) tensor field  $\sigma$  are defined by

$$\sigma_{\nu} = \sigma \nu \cdot \nu, \quad \sigma_{\tau} = \sigma \nu - \sigma_{\nu} \nu \quad \text{on } \Gamma,$$

moreover, Green's formula

$$(2.1) \quad (\text{Div } \sigma, v)_H + (\sigma, \varepsilon(v))_{\mathcal{Q}} = \int_{\Gamma} \sigma \nu \cdot v \, da \quad \forall v \in H_1$$

holds, where  $da$  is the surface measure element. For every real Banach space  $(X, \|\cdot\|_X)$ , we denote by  $C([0, T]; X)$  the space of continuous functions from  $[0, T]$  to  $X$  with the norm

$$\|v\|_{C([0, T]; X)} = \max_{t \in [0, T]} \|v(t)\|_X.$$

Also, we use the standard notation for the spaces  $L^p(0, T; X)$  and  $W^{k, p}(0, T; X)$ ,  $p \in [1, \infty]$  and  $k \geq 1$ . Finally, we conclude this section with the following Gronwall type inequality.

**Lemma 2.1.** *Assume that  $\tilde{a}$  and  $\tilde{b}: [0, T] \rightarrow \mathbb{R}$  are two functions in  $L^1(0, T)$  satisfying*

$$(2.2) \quad \tilde{a}(t) \leq \tilde{b}(t) + \tilde{c} \int_0^t \tilde{a}(s) \, ds \quad \forall t \in [0, T],$$

where  $\tilde{c}$  is a nonnegative constant. Then,

$$(2.3) \quad \tilde{a}(t) \leq \tilde{b}(t) + \tilde{c} \int_0^t e^{\tilde{c}(t-s)} \tilde{b}(s) \, ds \quad \forall t \in [0, T].$$

*Proof.* Use arguments similar to those in [8], proof of Proposition 2.1. □

### 3. PROBLEM STATEMENT

The physical setting is as follows. A deformable body occupies a bounded domain  $\Omega \subset \mathbb{R}^d$  (with  $d = 2, 3$ ). The body is assumed to obey a viscoelastic law with long memory and the process is quasistatic in the time interval of interest  $[0, T]$ . We assume that the boundary  $\Gamma$  of the domain  $\Omega$  is Lipschitz continuous and is divided into three disjoint measurable parts  $\Gamma_1, \Gamma_2, \Gamma_3$  such that  $\text{meas}(\Gamma_1) > 0$ . The body is clamped on  $\Gamma_1$  and therefore the displacement field vanishes there, while volume forces of density  $f_0$  act in  $\Omega$  and surface tractions of density  $f_2$  act on  $\Gamma_2$ . The body is supposed to be in adhesive contact over  $\Gamma_3$  with the foundation and, moreover, both normal compliance and a version of Coulomb's law of dry friction are included. To simplify the notation, we do not indicate explicitly the dependence of various functions on the spatial variable  $x \in \Omega \cup \Gamma$ . Under the above assumptions, the classical formulation of our problem is the following.

**Problem 3.1.** Find a displacement field  $u: \Omega \times [0, T] \rightarrow \mathbb{R}^d$ , a stress field  $\sigma: \Omega \times [0, T] \rightarrow \mathbb{S}^d$  and a bonding field  $\beta: \Gamma_3 \times [0, T] \rightarrow \mathbb{R}$  such that

$$(3.1) \quad \sigma(t) = \mathcal{A}\varepsilon(u(t)) + \int_0^t \mathcal{B}(t-s)\varepsilon(u(s)) \, ds \quad \text{in } \Omega \times (0, T),$$

$$(3.2) \quad \text{Div } \sigma + f_0 = 0 \quad \text{in } \Omega \times (0, T),$$

$$(3.3) \quad u = 0 \quad \text{on } \Gamma_1 \times (0, T),$$

$$(3.4) \quad \sigma\nu = f_2 \quad \text{on } \Gamma_2 \times (0, T),$$

$$(3.5) \quad -\sigma_\nu = p_\nu(u_\nu) - q_\nu(\beta)R_\nu(u_\nu) \quad \text{on } \Gamma_3 \times (0, T),$$

$$(3.6) \quad \begin{cases} |\sigma_\tau + q_\tau(\beta)R_\tau(u_\tau)| \leq p_\tau(u_\tau), \\ |\sigma_\tau + q_\tau(\beta)R_\tau(u_\tau)| < p_\tau(u_\tau) \Rightarrow \dot{u}_\tau = 0, \\ |\sigma_\tau + q_\tau(\beta)R_\tau(u_\tau)| = p_\tau(u_\tau) \Rightarrow \exists \lambda \geq 0 \\ \text{such that } \sigma_\tau + q_\tau(\beta)R_\tau(u_\tau) = -\lambda\dot{u}_\tau \quad \text{on } \Gamma_3 \times (0, T), \end{cases}$$

$$(3.7) \quad \dot{\beta} = H_{\text{ad}}(\beta, R_\nu(u_\nu)) \quad \text{on } \Gamma_3 \times (0, T),$$

$$(3.8) \quad \beta(0) = \beta_0 \quad \text{on } \Gamma_3,$$

$$(3.9) \quad u(0) = u_0 \quad \text{in } \Omega.$$

We now briefly comment on the problem (3.1)–(3.9). Equation (3.1) represents the viscoelastic law with long memory. Equation (3.2) is the equilibrium equation posed on the domain  $\Omega$ . Conditions (3.3)–(3.4) are the displacement-traction boundary conditions where  $\sigma\nu$  represents the Cauchy stress vector. Relations (3.5)–(3.6) characterize the contact boundary conditions. Here and below, the dot above a variable represents its derivative with respect to the time variable. Relation (3.5) is the normal compliance condition in which the contribution of the adhesive to the normal traction is represented by  $q_\nu(\beta)R_\nu(u_\nu)$ , where  $u_\nu$  is the normal displacement and  $q_\nu$  is a nonnegative prescribed function. A possible choice of the function  $q_\nu$  is

$$q_\nu(\beta) = \gamma_\nu\beta^2,$$

where  $\gamma_\nu$  is a given positive material parameter (see, e.g. [4], [21], [24]). The normal compliance function  $p_\nu$  is a nonnegative prescribed function which vanishes for negative arguments. An example of the normal compliance function  $p_\nu$  is

$$p_\nu(r) = c_\nu(r)_+,$$

where  $(r)_+$  denotes the positive part of  $r$ , that is  $(r)_+ = \max\{r, 0\}$ ,  $c_\nu$  is the surface stiffness coefficient. We note that an early attempt to study the quasistatic contact problem with the normal compliance model was done in [1], [15]. The relations (3.6)

represent a version of Coulomb's law of dry friction, where  $p_\tau$  is a prescribed non-negative function, the so-called friction bound. Given  $p_\nu$ , we may choose the friction bound function

$$p_\tau(r) = \mu p_\nu(r),$$

where  $\mu \geq 0$  is the coefficient of friction (for more details, see, e.g. [20]). The function  $q_\tau$  is a nonnegative function which acts as the tangential stiffness. In particular, the following form of the function  $q_\tau$  was considered for instance in [13], [24],

$$q_\tau(\beta) = c_\tau \beta^2,$$

where  $c_\tau$  is a given positive material parameter. Here and below,  $R_\nu: \mathbb{R} \rightarrow \mathbb{R}$  and  $R_\tau: \mathbb{R}^d \rightarrow \mathbb{R}^d$  are truncation functions defined by

$$(3.10) \quad R_\nu(s) = \begin{cases} 0 & \text{if } 0 \leq s, \\ -s & \text{if } -L \leq s \leq 0, \\ L & \text{if } s \leq -L, \end{cases}$$

$$(3.11) \quad R_\tau(v) = \begin{cases} v & \text{if } 0 \leq |v| \leq L, \\ L \frac{v}{|v|} & \text{if } |v| > L. \end{cases}$$

The introduction of the operators  $R_\nu$  and  $R_\tau$  is motivated by the mathematical arguments where  $L > 0$  is the characteristic length of the bond, beyond which it stretches without offering any additional resistance (see, e.g. [19], [21]). As in [3], [4], [13], equation (3.7) describes the evolution of the bonding field where  $H_{\text{ad}}$  is a general function. An example of the adhesion rate function  $H_{\text{ad}}$  is

$$(3.12) \quad H_{\text{ad}}(\beta, r) = -(\varepsilon_\nu \beta r^2 - \varepsilon_a)_+ \quad \text{on } \Gamma_3 \times (0, T),$$

where  $\varepsilon_\nu, \varepsilon_a$  are given positive material parameters. We note that in (3.12), the process is irreversible and once debonding occurs bonding cannot be reestablished (see, e.g. [20], [21]). Finally, (3.8)–(3.9) are the initial conditions.

#### 4. ASSUMPTIONS AND VARIATIONAL FORMULATION

In order to obtain the variational formulation of the mechanical problem (3.1)–(3.9), we introduce the space  $V$  defined by

$$V = \{v \in H_1, v = 0 \text{ on } \Gamma_1\}.$$



Since  $\text{meas}(\Gamma_1) > 0$ , Korn's inequality

$$(4.1) \quad C_K \|v\|_{H_1} \leq \|\varepsilon(v)\|_{\mathcal{Q}} \quad \forall v \in V$$

holds, where  $C_K > 0$  is a positive constant depending only on  $\Omega$  and  $\Gamma_1$ . A proof of Korn's inequality can be found, for instance, in [17], page 79. Over the space  $V$ , we consider the inner product and its associated norm, given by

$$(4.2) \quad (w, v)_V = (\varepsilon(w), \varepsilon(v))_{\mathcal{Q}}, \quad \|w\|_V = \|\varepsilon(w)\|_{\mathcal{Q}} \quad \forall w, v \in V.$$

It follows from Korn's inequality (4.1) that  $\|\cdot\|_{H_1}$  and  $\|\cdot\|_V$  are equivalent norms on  $V$ . Therefore,  $(V, (\cdot, \cdot)_V)$  is a real Hilbert space. Moreover, by the Sobolev trace theorem, there exists a positive constant  $c_0$  depending only on the domain  $\Omega$ ,  $\Gamma_1$  and  $\Gamma_3$  such that

$$(4.3) \quad \|v\|_{L^2(\Gamma_3)^d} \leq c_0 \|v\|_V \quad \forall v \in V.$$

In the study of the mechanical problem (3.1)–(3.9), we consider the following assumptions. We assume that  $\mathcal{A}: \Omega \times \mathbb{S}^d \rightarrow \mathbb{S}^d$  satisfies:

$$(4.4) \quad \left\{ \begin{array}{l} \text{(a)} \quad \text{There exists } m_{\mathcal{A}} > 0 \text{ such that} \\ \quad (\mathcal{A}(x, \varepsilon_1) - \mathcal{A}(x, \varepsilon_2)) \cdot (\varepsilon_1 - \varepsilon_2) \geq m_{\mathcal{A}} |\varepsilon_1 - \varepsilon_2|^2 \\ \quad \text{for a.e. } x \in \Omega \quad \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d. \\ \text{(b)} \quad \text{There exists } L_{\mathcal{A}} > 0 \text{ such that} \\ \quad |\mathcal{A}(x, \varepsilon_1) - \mathcal{A}(x, \varepsilon_2)| \leq L_{\mathcal{A}} |\varepsilon_1 - \varepsilon_2| \text{ for a.e. } x \in \Omega \quad \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d. \\ \text{(c)} \quad \text{The mapping } x \mapsto \mathcal{A}(x, \varepsilon) \text{ is Lebesgue measurable on } \Omega \text{ for any } \varepsilon \in \mathbb{S}^d. \\ \text{(d)} \quad \text{The mapping } x \mapsto \mathcal{A}(x, 0_{\mathbb{S}^d}) \text{ belongs to } \mathcal{Q}. \end{array} \right.$$

We assume that the operator  $\mathcal{B}$  satisfies

$$(4.5) \quad \mathcal{B} \in W^{1, \infty}(0, T; \mathbf{Q}_{\infty}),$$

where  $\mathbf{Q}_{\infty}$  is the space of fourth-order tensor fields defined by

$$\mathbf{Q}_{\infty} = \{\mathcal{E} = (\mathcal{E}_{ijkl}); \mathcal{E}_{ijkl} = \mathcal{E}_{jikl} = \mathcal{E}_{klij} \in L^{\infty}(\Omega) \forall i, j, k, l \in \{1, \dots, d\}\},$$

which is a real Banach space with the norm

$$(4.6) \quad \|\mathcal{E}\|_{\mathbf{Q}_{\infty}} = \max_{1 \leq i, j, k, l \leq d} \|\mathcal{E}_{ijkl}\|_{L^{\infty}(\Omega)}.$$

We assume that the function  $p_\alpha: \Gamma_3 \times \mathbb{R} \rightarrow \mathbb{R}^+$  ( $\alpha = \nu, \tau$ ) satisfies:

$$(4.7) \quad \left\{ \begin{array}{l} \text{(i)} \quad \text{There exists } L_\alpha > 0 \text{ such that} \\ \quad |p_\alpha(x, r_1) - p_\alpha(x, r_2)| \leq L_\alpha |r_1 - r_2| \quad \forall r_1, r_2 \in \mathbb{R} \text{ for a.e. } x \in \Gamma_3. \\ \text{(ii)} \quad p_\alpha(x, r) = 0 \quad \forall r \leq 0 \text{ for a.e. } x \in \Gamma_3. \\ \text{(iii)} \quad \text{The mapping } x \mapsto p_\alpha(x, r) \text{ is Lebesgue measurable on } \Gamma_3 \quad \forall r \in \mathbb{R}. \end{array} \right.$$

We assume that the function  $q_\alpha: \Gamma_3 \times \mathbb{R} \rightarrow \mathbb{R}^+$  ( $\alpha = \nu, \tau$ ) satisfies

$$(4.8) \quad \left\{ \begin{array}{l} \text{(i)} \quad \text{For all } b_1, b_2 \in \mathbb{R}, \text{ there exists } M_{\alpha b_1 b_2} > 0 \text{ such that} \\ \quad |q_\alpha(x, \zeta_1) - q_\alpha(x, \zeta_2)| \leq M_{\alpha b_1 b_2} |\zeta_1 - \zeta_2| \\ \quad \forall \zeta_1, \zeta_2 \in [b_1, b_2] \text{ for a.e. } x \in \Gamma_3. \\ \text{(ii)} \quad \text{The mapping } x \mapsto q_\alpha(x, \zeta) \text{ is Lebesgue measurable on } \Gamma_3 \\ \quad \text{for any } \zeta \in \mathbb{R}. \\ \text{(iii)} \quad \text{The mapping } x \mapsto q_\alpha(x, 0) \text{ belongs to } L^\infty(\Gamma_3). \end{array} \right.$$

The adhesion rate function  $H_{\text{ad}}: \Gamma_3 \times \mathbb{R} \times [-L, L] \rightarrow \mathbb{R}$  is assumed to satisfy:

$$(4.9) \quad \left\{ \begin{array}{l} \text{(i)} \quad \text{There exists } L_{H_{\text{ad}}} > 0 \text{ such that} \\ \quad |H_{\text{ad}}(x, \zeta_1, r) - H_{\text{ad}}(x, \zeta_2, r)| \leq L_{H_{\text{ad}}} |\zeta_1 - \zeta_2| \\ \quad \text{for a.e. } x \in \Gamma_3 \quad \forall \zeta_1, \zeta_2 \in \mathbb{R} \quad \forall r \in [-L, L]. \\ \text{(ii)} \quad \text{For all } b_1, b_2 \in \mathbb{R}, \text{ there exists } L_{b_1 b_2} > 0 \text{ such that} \\ \quad |H_{\text{ad}}(x, \zeta_1, r_1) - H_{\text{ad}}(x, \zeta_2, r_2)| \leq L_{b_1 b_2} (|\zeta_1 - \zeta_2| + |r_1 - r_2|) \\ \quad \forall \zeta_1, \zeta_2 \in [b_1, b_2] \quad \forall r_1, r_2 \in [-L, L] \text{ for a.e. } x \in \Gamma_3. \\ \text{(iii)} \quad \text{The mapping } x \mapsto H_{\text{ad}}(x, \zeta, r) \text{ is Lebesgue measurable on } \Gamma_3 \\ \quad \forall \zeta \in \mathbb{R} \quad \forall r \in [-L, L]. \\ \text{(iv)} \quad H_{\text{ad}}(x, 0, r) = 0 \quad \forall r \in [-L, L] \text{ for a.e. } x \in \Gamma_3. \\ \text{(v)} \quad H_{\text{ad}}(x, \zeta, r) \geq 0 \quad \forall \zeta \leq 0 \quad \forall r \in [-L, L] \text{ for a.e. } x \in \Gamma_3 \text{ and} \\ \quad H_{\text{ad}}(x, \zeta, r) \leq 0 \quad \forall \zeta \geq 1 \quad \forall r \in [-L, L] \text{ for a.e. } x \in \Gamma_3. \end{array} \right.$$

The densities of forces satisfy

$$(4.10) \quad \text{(i)} \quad f_0 \in W^{1,\infty}(0, T; H), \quad \text{(ii)} \quad f_2 \in W^{1,\infty}(0, T; L^2(\Gamma_2)^d).$$

Finally, we assume that the initial data satisfy

$$(4.11) \quad \beta_0 \in L^\infty(\Gamma_3), \quad 0 \leq \beta_0 \leq 1 \quad \text{for a.e. } x \in \Gamma_3,$$

$$(4.12) \quad u_0 \in V.$$

It follows from (4.10) that the function  $f: [0, T] \rightarrow V$ , defined by

$$(4.13) \quad (f(t), w)_V = \int_{\Omega} f_0(t) \cdot w \, dx + \int_{\Gamma_2} f_2(t) \cdot w \, da \quad \forall w \in V \quad \forall t \in [0, T],$$

has the regularity

$$(4.14) \quad f \in W^{1, \infty}(0, T; V).$$

In the sequel, we use the functionals  $\psi: V \times V \rightarrow \mathbb{R}$ ,  $j_{\text{ad}}: L^\infty(\Gamma_3) \times V \times V \rightarrow \mathbb{R}$  and  $\varphi: L^\infty(\Gamma_3) \times V \times V \rightarrow \mathbb{R}$  defined by

$$(4.15) \quad \psi(v, w) = \int_{\Gamma_3} p_\nu(v_\nu) w_\nu \, da + \int_{\Gamma_3} p_\tau(v_\tau) |w_\tau| \, da,$$

$$(4.16) \quad j_{\text{ad}}(\zeta, v, w) = - \int_{\Gamma_3} q_\nu(\zeta) R_\nu(v_\nu) w_\nu \, da + \int_{\Gamma_3} q_\tau(\zeta) R_\tau(v_\tau) \cdot w_\tau \, da,$$

$$(4.17) \quad \varphi(\zeta, v, w) = j_{\text{ad}}(\zeta, v, w) + \psi(v, w),$$

respectively, for all  $v, w \in V$  and for all  $\zeta \in L^\infty(\Gamma_3)$ . Using the Riesz representation theorem, we can introduce the operator  $\mathcal{F}: V \rightarrow V$  defined by

$$(4.18) \quad (\mathcal{F}v, w)_V = (\mathcal{A}\varepsilon(v), \varepsilon(w))_{\mathcal{Q}} \quad \forall v, w \in V.$$

Also, we introduce the operator  $\mathcal{G}: [0, T] \rightarrow \mathcal{L}(V)$  defined by

$$(4.19) \quad (\mathcal{G}(t)w, v)_V = (\mathcal{B}(t)\varepsilon(w), \varepsilon(v))_{\mathcal{Q}} \quad \forall w, v \in V \quad \forall t \in [0, T],$$

where  $\mathcal{L}(V)$  represents the space of linear and continuous operators from  $V$  to  $V$  with the norm  $\|\cdot\|_{\mathcal{L}(V)}$ . We turn now to derive a variational formulation of the mechanical problem (3.1)–(3.9). To do that, let us assume that  $(u, \sigma, \beta)$  are smooth functions satisfying (3.1)–(3.9). Let  $w \in V$  and let  $t \in [0, T]$ . We use (3.2) and the Green formula (2.1), to obtain

$$(4.20) \quad (\sigma(t), \varepsilon(w))_{\mathcal{Q}} - \int_{\Omega} f_0(t) \cdot w \, dx = \int_{\Gamma} \sigma(t) \nu \cdot w \, da.$$

Moreover, since  $w \in V$ , it follows from (3.4)–(3.5) that

$$(4.21) \quad \begin{aligned} \int_{\Gamma} \sigma(t) \nu \cdot w \, da &= \int_{\Gamma_2} \sigma(t) \nu \cdot w \, da + \int_{\Gamma_3} \sigma_\nu(t) w_\nu \, da + \int_{\Gamma_3} \sigma_\tau(t) \cdot w_\tau \, da \\ &= \int_{\Gamma_2} f_2(t) \cdot w \, da - \int_{\Gamma_3} p_\nu(u_\nu(t)) w_\nu \, da \\ &\quad + \int_{\Gamma_3} q_\nu(\beta(t)) R_\nu(u_\nu(t)) w_\nu \, da + \int_{\Gamma_3} \sigma_\tau(t) \cdot w_\tau \, da. \end{aligned}$$

Thus, (4.13), (4.20) and (4.21) lead us to

$$(4.22) \quad \begin{cases} (\sigma(t), \varepsilon(w) - \varepsilon(\dot{u}(t)))_{\mathcal{Q}} + \int_{\Gamma_3} p_\nu(u_\nu(t))(w_\nu - \dot{u}_\nu(t)) \, da \\ - \int_{\Gamma_3} q_\nu(\beta(t))R_\nu(u_\nu(t))(w_\nu - \dot{u}_\nu(t)) \, da \\ - \int_{\Gamma_3} \sigma_\tau(t) \cdot (w_\tau - \dot{u}_\tau(t)) \, da = (f(t), w - \dot{u}(t))_V. \end{cases}$$

On the other hand, using (3.6), we get

$$\begin{aligned} \int_{\Gamma_3} (\sigma_\tau(t) + q_\tau(\beta(t))R_\tau(u_\tau(t))) \cdot \dot{u}_\tau(t) \, da &= - \int_{\Gamma_3} |\sigma_\tau(t) + q_\tau(\beta(t))R_\tau(u_\tau(t))| |\dot{u}_\tau(t)| \, da \\ &= - \int_{\Gamma_3} p_\tau(u_\nu(t)) |\dot{u}_\tau(t)| \, da, \end{aligned}$$

and using the fact that

$$\begin{aligned} - \int_{\Gamma_3} (\sigma_\tau(t) + q_\tau(\beta(t))R_\tau(u_\tau(t))) \cdot w_\tau \, da &\leq \int_{\Gamma_3} |(\sigma_\tau(t) + q_\tau(\beta(t))R_\tau(u_\tau(t)))| |w_\tau| \, da \\ &\leq \int_{\Gamma_3} p_\tau(u_\nu(t)) |w_\tau| \, da, \end{aligned}$$

we obtain

$$(4.23) \quad - \int_{\Gamma_3} \sigma_\tau(t) \cdot (w_\tau - \dot{u}_\tau(t)) \, da \leq \int_{\Gamma_3} p_\tau(u_\nu(t)) (|w_\tau| - |\dot{u}_\tau(t)|) \, da \\ + \int_{\Gamma_3} q_\tau(\beta(t))R_\tau(u_\tau(t)) \cdot (w_\tau - \dot{u}_\tau(t)) \, da.$$

Now, from (4.22) and (4.23), we find that

$$(4.24) \quad \begin{cases} (\sigma(t), \varepsilon(w) - \varepsilon(\dot{u}(t)))_{\mathcal{Q}} + \int_{\Gamma_3} p_\nu(u_\nu(t))(w_\nu - \dot{u}_\nu(t)) \, da \\ + \int_{\Gamma_3} p_\tau(u_\nu(t)) (|w_\tau| - |\dot{u}_\tau(t)|) \, da \\ - \int_{\Gamma_3} q_\nu(\beta(t))R_\nu(u_\nu(t))(w_\nu - \dot{u}_\nu(t)) \, da \\ + \int_{\Gamma_3} q_\tau(\beta(t))R_\tau(u_\tau(t)) \cdot (w_\tau - \dot{u}_\tau(t)) \, da \geq (f(t), w - \dot{u}(t))_V. \end{cases}$$

Therefore, combine (4.24) with (3.1), (4.15), (4.16), (4.17), (4.18) and (4.19), integrate (3.7) on  $(0, t)$ , and use the initial conditions (3.8)–(3.9) to obtain the following variational formulation in terms of displacement and adhesion fields.

**Problem 4.1.** Find a displacement field  $u: [0, T] \rightarrow V$  and a bonding field  $\beta: [0, T] \rightarrow L^\infty(\Gamma_3)$  such that

$$(4.25) \quad \left\{ \begin{array}{l} (\mathcal{F}u(t), w - \dot{u}(t))_V + \left( \int_0^t \mathcal{G}(t-s)u(s) ds, w - \dot{u}(t) \right)_V \\ \quad + \varphi(\beta(t), u(t), w) - \varphi(\beta(t), u(t), \dot{u}(t)) \\ \geq (f(t), w - \dot{u}(t))_V \quad \forall w \in V \text{ and for a.e. } t \in (0, T), \end{array} \right.$$

$$(4.26) \quad \beta(t) = \int_0^t H_{\text{ad}}(\beta(s), R_\nu(u_\nu(s))) ds + \beta_0 \quad \forall t \in [0, T],$$

$$(4.27) \quad u(0) = u_0.$$

To study the problem (4.25)–(4.27), we need the additional assumption on the initial data

$$(4.28) \quad (\mathcal{A}(\varepsilon(u_0)), \varepsilon(w))_{\mathcal{Q}} + \varphi(\beta_0, u_0, w) \geq (f(0), w)_V \quad \forall w \in V$$

and we make the smallness assumption

$$(4.29) \quad L_\tau + L_\nu < \frac{m_{\mathcal{A}}}{c_0^2},$$

where  $c_0$ ,  $m_{\mathcal{A}}$  and  $L_\alpha$  ( $\alpha = \nu, \tau$ ) are given in (4.3), (4.4) and (4.7), respectively.

We end this section by presenting some properties of  $\mathcal{F}$ ,  $\mathcal{G}$ ,  $\psi$ ,  $R_\nu$ ,  $R_\tau$  and  $j_{\text{ad}}$ . It follows from (4.4) and (4.18), that the operator  $\mathcal{F}$  satisfies

$$(4.30) \quad m_{\mathcal{A}} \|w_1 - w_2\|_V^2 \leq (\mathcal{F}w_1 - \mathcal{F}w_2, w_1 - w_2)_V \quad \forall w_1, w_2 \in V,$$

$$(4.31) \quad \|\mathcal{F}w_1 - \mathcal{F}w_2\|_V \leq L_{\mathcal{A}} \|w_1 - w_2\|_V \quad \forall w_1, w_2 \in V.$$

Thanks to (4.19), (4.5), (4.2) and (4.6), we conclude that there exists  $L_{\mathcal{G}} > 0$  such that

$$(4.32) \quad \|(\mathcal{G}(t) - \mathcal{G}(s))w\|_V \leq L_{\mathcal{G}} |t - s| \|w\|_V,$$

which gives

$$(4.33) \quad \|\mathcal{G}(t)w\|_V \leq (TL_{\mathcal{G}} + \|\mathcal{G}(0)\|_{\mathcal{L}(V)}) \|w\|_V$$

for all  $w \in V$  and for all  $t, s \in [0, T]$ . Using (4.3), (4.7) and (4.15), we deduce that

$$(4.34) \quad \psi(g, v) - \psi(g, w) + \psi(z, w) - \psi(z, v) \leq c_0^2 (L_\tau + L_\nu) \|g - z\|_V \|v - w\|_V,$$

$$(4.35) \quad \psi(g, -z) - \psi(g, g - z) \leq c_0^2 (L_\tau + L_\nu) \|g\|_V^2,$$

$$(4.36) \quad \psi(g, v) - \psi(g, w) \leq \psi(g, v - w),$$

$$(4.37) \quad \psi(g, w) - \psi(z, w) \leq c_0^2 (L_\tau + L_\nu) \|g - z\|_V \|w\|_V,$$

$$(4.38) \quad |\psi(g, v) - \psi(g, w)| \leq c_0^2 (L_\tau + L_\nu) \|g\|_V \|v - w\|_V,$$

$$(4.39) \quad \psi(g, w) \leq c_0 (L_\tau + L_\nu) \|g\|_V \|w\|_{L^2(\Gamma_3)^d}$$

for all  $v, g, w, z \in V$ . Also, using (3.11), we find that

$$(4.40) \quad |R_\tau(v)| \leq |v|, \quad |R_\tau(v)| \leq L \quad \forall v \in \mathbb{R}^d.$$

Let  $w, v \in \mathbb{R}^d$ . Using the properties of the inner product, we get

$$|R_\tau(v) - R_\tau(w)|^2 - |v - w|^2 = \begin{cases} 0 & \text{if } |w| \leq L \text{ and } |v| \leq L, \\ \left(\frac{L}{|v|} - 1\right)(L|v| + |v|^2 - 2v \cdot w) & \text{if } |w| \leq L < |v|, \\ 2(L^2 - |v||w|)\left(1 - \frac{v \cdot w}{|v||w|}\right) - (|v| - |w|)^2 & \text{if } L < |w| \text{ and } L < |v|. \end{cases}$$

Therefore, using the inequality

$$v \cdot w \leq |v||w|,$$

we obtain

$$|R_\tau(v) - R_\tau(w)|^2 - |v - w|^2 \leq 0,$$

which leads to

$$(4.41) \quad |R_\tau(w) - R_\tau(v)| \leq |w - v| \quad \forall w, v \in \mathbb{R}^d.$$

Using again (3.11), we infer that

$$(R_\tau(v) - R_\tau(w)) \cdot (v - w) = \begin{cases} |v - w|^2 & \text{if } |v| \leq L \text{ and } |w| \leq L, \\ L(|v| + |w|)\left(1 - \frac{v \cdot w}{|v||w|}\right) & \text{if } L < |v| \text{ and } L < |w|, \\ (L - |v|)(|w| - |v|) + (|w||v| - v \cdot w) + L\left(|v| - \frac{v \cdot w}{|w|}\right) & \text{if } |v| \leq L < |w|, \end{cases}$$

which yields

$$(4.42) \quad (R_\tau(v) - R_\tau(w)) \cdot (v - w) \geq 0 \quad \forall w, v \in \mathbb{R}^d.$$

On the other hand, keeping in mind (3.10), it is straightforward to show that

$$(4.43) \quad |R_\nu(s)| \leq |s|, \quad |R_\nu(s)| \leq L \quad \forall s \in \mathbb{R},$$

$$(4.44) \quad |R_\nu(s_1) - R_\nu(s_2)| \leq |s_1 - s_2| \quad \forall s_1, s_2 \in \mathbb{R}.$$

Moreover, since  $R_\nu$  is a decreasing function, we get

$$(4.45) \quad (R_\nu(s_1) - R_\nu(s_2))(s_1 - s_2) \leq 0 \quad \forall s_1, s_2 \in \mathbb{R}.$$

It follows from (4.40), (4.41), (4.43), (4.44), (4.3), (4.8) and (4.16), that for each  $c > 0$ , there exists  $M_c > 0$  such that the following inequalities hold:

$$(4.46) \quad |j_{\text{ad}}(\theta, v, w) - j_{\text{ad}}(\zeta, v, w)| \leq M_c \|\theta - \zeta\|_{L^2(\Gamma_3)} \|w\|_V,$$

$$(4.47) \quad |j_{\text{ad}}(\zeta, v, w)| \leq M_c \|v\|_V \|w\|_{L^2(\Gamma_3)^d},$$

$$(4.48) \quad |j_{\text{ad}}(\zeta, v, g) - j_{\text{ad}}(\zeta, w, g)| \leq M_c \|v - w\|_V \|g\|_V,$$

$$(4.49) \quad |j_{\text{ad}}(\zeta, g, v) - j_{\text{ad}}(\zeta, g, w)| \leq M_c \|g\|_V \|v - w\|_V$$

for all  $v, g, w \in V$  and for all  $\theta, \zeta \in L^\infty(\Gamma_3)$  with  $\|\theta\|_{L^\infty(\Gamma_3)} \leq c$  and  $\|\zeta\|_{L^\infty(\Gamma_3)} \leq c$ . Finally, we use (4.16), (4.45) and (4.42) to obtain

$$(4.50) \quad j_{\text{ad}}(\zeta, v, v - w) - j_{\text{ad}}(\zeta, w, v - w) \geq 0,$$

$$(4.51) \quad j_{\text{ad}}(\zeta, w, w) \geq 0$$

for all  $v, w \in V$  and for all  $\zeta \in L^\infty(\Gamma_3)$ .

## 5. EXISTENCE OF A WEAK SOLUTION

The following theorem is the main result of this paper.

**Theorem 5.1.** *Assume that (4.4)–(4.12) and (4.28)–(4.29) are fulfilled. Then, the problem (4.25)–(4.27) has at least one solution  $\{u, \beta\}$  which satisfies*

$$(5.1) \quad u \in W^{1,\infty}(0, T; V),$$

$$(5.2) \quad \beta \in W^{1,\infty}(0, T; L^\infty(\Gamma_3)), 0 \leq \beta(t) \leq 1 \quad \text{for a.e. } x \in \Gamma_3 \quad \forall t \in [0, T].$$

We divide the proof of Theorem 5.1 into several steps.

**F i r s t** step. For each  $m \in \mathbb{N}^*$ , we introduce a uniform partition of the time interval  $[0, T]$ , denoted by  $t_i^m = ih_m$ ,  $h_m = T/m$ ,  $i = 0, \dots, m$ . For a sequence  $\{w_m^i\}_{i=0}^m$ , we put  $\delta w_m^{i+1} = (w_m^{i+1} - w_m^i)/h_m$  and for a continuous function  $z \in C([0, T]; X)$  with values in a normed space  $X$ , we use the notation  $z_i^m = z(t_i^m)$ ,  $i = 0, \dots, m$ . We consider the following incremental problems  $\mathcal{P}_m^{i+1}$ ,  $i \in \{0, \dots, m-1\}$ .

**Problem 5.2** ( $\mathcal{P}_m^{i+1}$ ). Find a function  $u_m^{i+1} \in V$  such that

$$(5.3) \quad \begin{cases} (\mathcal{F}u_m^{i+1}, w - \delta u_m^{i+1})_V + (h_m \sum_{j=0}^i \mathcal{G}_{i+1,j}^m, w - \delta u_m^{i+1})_V \\ \quad + \varphi(\beta_m^{i+1}, u_m^{i+1}, w) - \varphi(\beta_m^{i+1}, u_m^{i+1}, \delta u_m^{i+1}) \\ \geq (f_{i+1}^m, w - \delta u_m^{i+1})_V \quad \forall w \in V, \end{cases}$$

where  $u_m^j$  is the unique solution of problem  $\mathcal{P}_m^j$ ,  $j = 1, \dots, i$ ,

$$(5.4) \quad \mathcal{G}_{i+1,j}^m = \mathcal{G}(t_{i+1}^m - t_j^m)u_m^j, \quad i = 0, \dots, m-1, \quad j = 0, \dots, i,$$

$$(5.5) \quad \beta_m^{i+1} = h_m \sum_{j=0}^i H_{\text{ad}}(\beta_m^j, R_\nu(u_{m\nu}^j)) + \beta_m^0, \quad i = 0, \dots, m-1,$$

$$(5.6) \quad f_{i+1}^m = f(t_{i+1}^m), \quad i = 0, \dots, m-1,$$

$$(5.7) \quad \text{(i) } u_m^0 = u_0, \quad \text{(ii) } \beta_m^0 = \beta_0.$$

Thanks to (4.9) (i), (4.9) (iv) and (5.5), we deduce that

$$|\beta_m^{i+1} - \beta_m^i| \leq L_{H_{\text{ad}}} h_m |\beta_m^i| \quad \text{for a.e. } x \in \Gamma_3, \quad 0 \leq i \leq m-1,$$

which implies that if  $\beta_m^i \in L^\infty(\Gamma_3)$ , then  $\beta_m^{i+1} \in L^\infty(\Gamma_3)$ ,  $0 \leq i \leq m-1$ . Now, by setting  $w = (v - u_m^i)/h_m$  in (5.3), it follows that  $\mathcal{P}_m^{i+1}$  is formally equivalent to the following problem.

**Problem 5.3** ( $\mathcal{Q}_m^{i+1}$ ). Find a function  $u_m^{i+1} \in V$ , such that

$$(5.8) \quad \begin{cases} (\mathcal{F}u_m^{i+1}, v - u_m^{i+1})_V + \left( h_m \sum_{j=0}^i \mathcal{G}_{i+1,j}^m, v - u_m^{i+1} \right)_V \\ \quad + \varphi(\beta_m^{i+1}, u_m^{i+1}, v - u_m^i) - \varphi(\beta_m^{i+1}, u_m^{i+1}, u_m^{i+1} - u_m^i) \\ \geq (f_{i+1}^m, v - u_m^{i+1})_V \quad \forall v \in V, \end{cases}$$

where  $\{\mathcal{G}_{i+1,j}^m\}$ ,  $\{\beta_m^{i+1}\}$ ,  $\{f_{i+1}^m\}$ ,  $u_m^0$  and  $\beta_m^0$  are given by (5.4)–(5.7) and  $u_m^j$  is the unique solution of problem  $\mathcal{P}_m^j$ ,  $j = 1, \dots, i$ .

**Lemma 5.4.** *Problem  $\mathcal{P}_m^{i+1}$ ,  $0 \leq i \leq m-1$ , has a unique solution.*

**Proof.** Let  $A: V \rightarrow V$  be the operator defined by

$$(5.9) \quad (Av, w)_V = (\mathcal{F}v, w)_V + \left( h_m \sum_{j=0}^i \mathcal{G}_{i+1,j}^m, w \right)_V + j_{\text{ad}}(\beta_m^{i+1}, v, w)$$



for all  $v, w \in V$ . From (4.30), (4.31), (5.9), (4.48) and (4.50), we deduce that  $A$  is a strongly monotone Lipschitz continuous operator. Let  $g \in V$ , using (4.15) and (4.38), we find that the functional  $\Theta_g: V \rightarrow \mathbb{R}$ , defined by

$$\Theta_g(v) = \psi(g, v - u_m^i) \quad \forall v \in V,$$

is a proper convex continuous function. Thus, using a standard result on elliptic variational inequalities of the second kind (see [12], p. 60), we find that the problem:

Find  $u_{mg}^{i+1} \in V$ , such that

$$(5.10) \quad \begin{cases} (Au_{mg}^{i+1}, v - u_m^{i+1})_V + \psi(g, v - u_m^i) - \psi(g, u_{mg}^{i+1} - u_m^i) \\ \geq (f_{i+1}^m, v - u_m^{i+1})_V \quad \forall v \in V, \end{cases}$$

has a unique solution  $u_{mg}^{i+1} \in V$ . To continue, we define the operator  $\Psi: V \rightarrow V$  by

$$(5.11) \quad \Psi(g) = u_{mg}^{i+1} \quad \forall g \in V.$$

Let  $g_1, g_2 \in V$ . Using the notation  $u_1 = u_{mg_1}^{i+1}$  and  $u_2 = u_{mg_2}^{i+1}$ , we get by (5.10)

$$\begin{aligned} (Au_1 - Au_2, u_1 - u_2)_V &\leq \psi(g_1, u_2 - u_m^i) - \psi(g_1, u_1 - u_m^i) \\ &\quad + \psi(g_2, u_1 - u_m^i) - \psi(g_2, u_2 - u_m^i), \end{aligned}$$

which together with (4.34), (4.50), (4.30) and (5.9) implies that

$$m_{\mathcal{A}} \|u_1 - u_2\|_V^2 \leq c_0^2 (L_\tau + L_\nu) \|g_1 - g_2\|_V \|u_1 - u_2\|_V,$$

and using (5.11), we have

$$\|\Psi g_2 - \Psi g_1\|_V \leq \frac{c_0^2 (L_\tau + L_\nu)}{m_{\mathcal{A}}} \|g_1 - g_2\|_V.$$

This last inequality implies that, under the smallness assumption  $(L_\tau + L_\nu) < m_{\mathcal{A}}/c_0^2$ ,  $\Psi$  is a contraction in the Hilbert space  $V$ . Therefore, there exists a unique element  $g^* \in V$ , such that  $u_{mg^*}^{i+1} = \Psi g^* = g^*$ . We have now all the ingredients to prove Lemma 5.4. Let  $g^*$  be the unique fixed point of  $\Psi$  defined by (5.11) and let  $u_m^{i+1} = g^* = u_{mg^*}^{i+1}$  be the unique solution of the problem (5.10) for  $g = g^*$ . Keeping in mind (5.9) and (4.17), we deduce that  $u_m^{i+1}$  is a solution to problem  $\mathcal{Q}_m^{i+1}$  which is formally equivalent to problem  $\mathcal{P}_m^{i+1}$ . The uniqueness of the solution is a consequence of the uniqueness of the fixed point of the operator  $\Psi$  and of the uniqueness of the solution of the problem (5.10).  $\square$

In the rest of this paper, the same letter  $c$  will be used to denote different positive constants which depend neither on  $m \in \mathbb{N}^*$  nor on  $t \in (0, T)$ .

Second step. In this step we have the following result.

**Lemma 5.5.** *There exists  $c > 0$  such that for all  $m \in \mathbb{N}^*$ ,*

$$(5.12) \quad \|u_m^{i+1}\|_V + \|\beta_m^{i+1}\|_{L^\infty(\Gamma_3)} \leq c, \quad 0 \leq i \leq m-1,$$

$$(5.13) \quad \|\delta u_m^{i+1}\|_V \leq c, \quad 0 \leq i \leq m-1.$$

*Proof.* It follows from (5.7) that there exists  $c > 0$  such that

$$(5.14) \quad \|u_m^0\|_V + \|\beta_m^0\|_{L^\infty(\Gamma_3)} \leq c \quad \forall m \in \mathbb{N}^*.$$

Using (4.9) (i), (4.9) (iv) and (5.5), we obtain

$$\|\beta_m^{i+1}\|_{L^\infty(\Gamma_3)} \leq ch_m \sum_{j=0}^i \|\beta_m^j\|_{L^\infty(\Gamma_3)} + \|\beta_m^0\|_{L^\infty(\Gamma_3)}, \quad 0 \leq i \leq m-1.$$

Applying a discrete version of the Gronwall lemma (see, e.g. [12]) in the last inequality leads us to

$$(5.15) \quad \|\beta_m^{i+1}\|_{L^\infty(\Gamma_3)} \leq c, \quad 0 \leq i \leq m-1.$$

Taking  $v = 0_V$  in (5.8) and keeping in mind (4.17), we get

$$\begin{aligned} (\mathcal{F}u_m^{i+1}, u_m^{i+1})_V + j_{\text{ad}}(\beta_m^{i+1}, u_m^{i+1}, u_m^{i+1}) &\leq \psi(u_m^{i+1}, -u_m^i) - \psi(u_m^{i+1}, u_m^{i+1} - u_m^i)_{\mathcal{Q}} \\ &\quad - \left( h_m \sum_{j=0}^i \mathcal{G}_{i+1,j}^m, u_m^{i+1} \right)_V + (f_{i+1}^m, u_m^{i+1})_V \end{aligned}$$

and using (4.30), (4.51), (4.33), (5.4), (4.35), we arrive at

$$\begin{aligned} m_{\mathcal{A}} \|u_m^{i+1}\|_V^2 &\leq c_0^2 (L_\tau + L_\nu) \|u_m^{i+1}\|_V^2 + \left( ch_m \sum_{j=0}^i \|u_m^j\|_V \right) \|u_m^{i+1}\|_V \\ &\quad + \|f_{i+1}^m\|_V \|u_m^{i+1}\|_V + \|\mathcal{F}(0_V)\|_V \|u_m^{i+1}\|_V. \end{aligned}$$

By virtue of the assumptions (4.29) and (4.14), the last inequality becomes

$$(5.16) \quad \|u_m^{i+1}\|_V \leq ch_m \sum_{j=0}^i \|u_m^j\|_V + c.$$

Employing again a discrete version of the Gronwall lemma in (5.16) yields

$$\|u_m^{i+1}\|_V \leq c, \quad 0 \leq i \leq m-1,$$

which with (5.15) gives (5.12). To continue, using (4.9) (i), (4.9) (iv) and (5.5), one has

$$\|\beta_m^{i+1} - \beta_m^i\|_{L^\infty(\Gamma_3)} \leq ch_m \|\beta_m^i\|_{L^\infty(\Gamma_3)}, \quad 0 \leq i \leq m-1,$$

and thanks to (5.14) and (5.15), we get

$$(5.17) \quad \|\delta\beta_m^{i+1}\|_{L^\infty(\Gamma_3)} \leq c, \quad 0 \leq i \leq m-1.$$

Setting  $v = u_m^0$  in (5.8) for  $i = 0$  and  $w = u_m^1 - u_m^0$  in (4.28), and summing up the two inequalities, we obtain

$$\left\{ \begin{array}{l} (\mathcal{F}u_m^1 - \mathcal{F}u_m^0, u_m^1 - u_m^0)_V \\ \quad + j_{\text{ad}}(\beta_m^0, u_m^1, u_m^1 - u_m^0) - j_{\text{ad}}(\beta_m^0, u_m^0, u_m^1 - u_m^0) \\ \leq j_{\text{ad}}(\beta_m^0, u_m^1, u_m^1 - u_m^0) - j_{\text{ad}}(\beta_m^1, u_m^1, u_m^1 - u_m^0) \\ \quad - (h_m \mathcal{G}_{1,0}^m, u_m^1 - u_m^0)_V + \psi(u_m^0, u_m^1 - u_m^0) \\ \quad - \psi(u_m^1, u_m^1 - u_m^0) + (f_1^m - f_0^m, u_m^1 - u_m^0)_V. \end{array} \right.$$

We then use (4.30), (5.14), (5.15), (4.46), (4.50), (4.33), (5.4) and (4.37) to see that

$$\left\{ \begin{array}{l} m_{\mathcal{A}} \|u_m^1 - u_m^0\|_V^2 \leq c \|\beta_m^1 - \beta_m^0\|_{L^2(\Gamma_3)} \|u_m^1 - u_m^0\|_V \\ \quad + ch_m \|u_m^0\|_V \|u_m^1 - u_m^0\|_V + c_0^2 (L_\tau + L_\nu) \|u_m^1 - u_m^0\|_V^2 \\ \quad + \|f_1^m - f_0^m\|_V \|u_m^1 - u_m^0\|_V, \end{array} \right.$$

and thanks to (4.14), (4.29) and (5.17), we get

$$\left\| \frac{u_m^1 - u_m^0}{h_m} \right\|_V \leq c + c \left\| \frac{\beta_m^1 - \beta_m^0}{h_m} \right\|_{L^2(\Gamma_3)} + c \left\| \frac{f_1^m - f_0^m}{h_m} \right\|_V \leq c + c \|\dot{f}\|_{L^\infty(0,T;V)}.$$

Thus, we have

$$(5.18) \quad \|\delta u_m^1\|_V \leq c.$$

Taking  $w = 0_V$  in problem  $\mathcal{P}_m^{i+1}$  and  $w = (u_m^{i+1} - u_m^{i-1})/h_m$  in problem  $\mathcal{P}_m^i$ , and summing up the two inequalities, we obtain

$$\left\{ \begin{array}{l} (\mathcal{F}u_m^{i+1} - \mathcal{F}u_m^i, \delta u_m^{i+1})_V + j_{\text{ad}}(\beta_m^{i+1}, u_m^{i+1}, \delta u_m^{i+1}) - j_{\text{ad}}(\beta_m^{i+1}, u_m^i, \delta u_m^{i+1}) \\ \leq h_m \left( \sum_{j=0}^{i-1} (\mathcal{G}_{i,j}^m - \mathcal{G}_{i+1,j}^m), \delta u_m^{i+1} \right)_V - h_m (\mathcal{G}_{i+1,i}^m, \delta u_m^{i+1})_V \\ \quad + \left( \psi \left( u_m^i, \frac{u_m^{i+1} - u_m^{i-1}}{h_m} \right) - \psi \left( u_m^i, \frac{u_m^i - u_m^{i-1}}{h_m} \right) \right) - \psi \left( u_m^{i+1}, \frac{u_m^{i+1} - u_m^i}{h_m} \right) \\ \quad + j_{\text{ad}}(\beta_m^i, u_m^i, \delta u_m^{i+1}) - j_{\text{ad}}(\beta_m^{i+1}, u_m^i, \delta u_m^{i+1}) + (f_{i+1}^m - f_i^m, \delta u_m^{i+1})_V. \end{array} \right.$$

So, from (4.30), (4.50), (4.32), (4.33), (5.4), (5.15), (4.46), (4.37), (4.36), it follows that

$$\left\{ \begin{array}{l} m_{\mathcal{A}} \|u_m^{i+1} - u_m^i\|_V^2 \leq ch_m^2 \left( \sum_{j=0}^{i-1} \|u_m^j\|_V \right) \|u_m^{i+1} - u_m^i\|_V \\ + ch_m \|u_m^i\|_V \|u_m^{i+1} - u_m^i\|_V \\ + c \|\beta_m^{i+1} - \beta_m^i\|_{L^2(\Gamma_3)} \|u_m^{i+1} - u_m^i\|_V + c_0^2 (L_\tau + L_\nu) \|u_m^{i+1} - u_m^i\|_V^2 \\ + \|f_{i+1}^m - f_i^m\|_V \|u_m^{i+1} - u_m^i\|_V, \end{array} \right.$$

which gives

$$(5.19) \quad \|\delta u_m^{i+1}\|_V \leq ch_m \left( \sum_{j=0}^{i-1} \|u_m^j\|_V \right) + c \|u_m^i\|_V + c \|\delta \beta_m^{i+1}\|_{L^2(\Gamma_3)} + c \|f\|_{L^\infty(0,T;V)}.$$

Now, (5.13) is a consequence of (5.19), (5.12), (5.17) and (5.18).  $\square$

**Third step.** In this step we construct an approximate solution to the problem (4.25)–(4.27). To this end, for each  $m \in \mathbb{N}^*$ , let  $u_m^j$  be the unique solution of problem  $\mathcal{P}_m^j$ ,  $j = 1, \dots, m$ . We introduce the functions  $u_m: [0, T] \rightarrow V$ ,  $\tilde{u}_m: [0, T] \rightarrow V$ ,  $\tilde{\beta}_m: [0, T] \rightarrow L^2(\Gamma_3)$ ,  $\mathcal{G}_m: [0, T] \rightarrow V$  and  $f_m: [0, T] \rightarrow V$  defined by

$$(5.20) \quad u_m(0) = u_0, u_m(t) = u_m^i + (t - t_i^m) \delta u_m^{i+1} \quad \forall t \in (t_i^m, t_{i+1}^m], \quad 0 \leq i \leq m-1,$$

$$(5.21) \quad \tilde{u}_m(0) = u_0, \tilde{u}_m(t) = u_m^{i+1} \quad \forall t \in (t_i^m, t_{i+1}^m], \quad 0 \leq i \leq m-1,$$

$$(5.22) \quad \tilde{\beta}_m(0) = \beta_0, \tilde{\beta}_m(t) = \beta_m^{i+1} \quad \forall t \in (t_i^m, t_{i+1}^m], \quad 0 \leq i \leq m-1,$$

$$(5.23) \quad \mathcal{G}_m(0) = 0_V, \mathcal{G}_m(t) = h_m \sum_{j=0}^i \mathcal{G}_{i+1,j}^m \quad \forall t \in (t_i^m, t_{i+1}^m], \quad 0 \leq i \leq m-1,$$

$$(5.24) \quad f_m(0) = f(0), f_m(t) = f_{i+1}^m \quad \forall t \in (t_i^m, t_{i+1}^m], \quad 0 \leq i \leq m-1,$$

respectively. Here  $\{\mathcal{G}_{i+1,j}^m\}$ ,  $\{\beta_m^{i+1}\}$ ,  $\{f_{i+1}^m\}$  and  $u_m^0$  are given by (5.4)–(5.7). From (5.20), the function  $u_m$  has a derivative function which is given by

$$(5.25) \quad \dot{u}_m(t) = \delta u_m^{i+1} \quad \forall t \in (t_i^m, t_{i+1}^m], \quad 0 \leq i \leq m-1.$$

We have the following estimate results.

**Lemma 5.6.** *There exists  $c > 0$ , such that for all  $m \in \mathbb{N}^*$ ,*

$$(5.26) \quad \|\tilde{u}_m(t)\|_V \leq c \quad \forall t \in [0, T],$$

$$(5.27) \quad \|\tilde{\beta}_m(t)\|_{L^\infty(\Gamma_3)} \leq c \quad \forall t \in [0, T],$$

$$(5.28) \quad \|u_m(t)\|_V \leq c \quad \forall t \in [0, T],$$

$$(5.29) \quad \|\dot{u}_m(t)\|_V \leq c \quad \text{for a.e. } t \in [0, T],$$

$$(5.30) \quad \|\tilde{u}_m(t) - u_m(t)\|_V \leq ch_m \quad \forall t \in [0, T],$$

$$(5.31) \quad \|f_m(t) - f(t)\|_V \leq ch_m \quad \forall t \in [0, T],$$

$$(5.32) \quad \|u_m(t) - u_m(s)\|_V \leq c|t - s| \quad \forall t, s \in [0, T],$$

$$(5.33) \quad \|u_m(t) - u_m(s)\|_{L^2(\Gamma_3)^d} \leq c|t - s| \quad \forall t, s \in [0, T].$$

**Proof.** It is clear that (5.26)–(5.29) are the consequences of (5.20)–(5.22), (5.25) and Lemma 5.5. On the other hand, using (5.20)–(5.21) we get

$$\begin{aligned} \|\tilde{u}_m(t) - u_m(t)\|_V &\leq \|u_m^{i+1} - u_m^i\|_V + |t - t_i^m| \|\delta u_m^{i+1}\|_V \\ &\leq ch_m \|\delta u_m^{i+1}\|_V \quad \forall t \in (t_i^m, t_{i+1}^m], \quad 0 \leq i \leq m-1, \end{aligned}$$

which with (5.13) gives (5.30). To establish (5.31), observe that  $f_m(0) = f(0)$ , moreover, by the regularity (4.14), we have

$$\|f_m(t) - f(t)\|_V \leq \int_t^{t_{i+1}^m} \|\dot{f}(s)\|_V ds \leq ch_m \quad \forall t \in (t_i^m, t_{i+1}^m], \quad 0 \leq i \leq m-1.$$

To continue, using (5.29), we obtain

$$\|u_m(t) - u_m(s)\|_V \leq \left| \int_s^t \|\dot{u}_m(r)\|_V dr \right| \leq c|t - s|$$

for all  $t, s \in [0, T]$ . Finally, (5.33) is a direct consequence of (5.32) and (4.3).  $\square$

In the next we need the following result.

**Lemma 5.7.** *There exists  $c > 0$ , such that for all  $m, n \in \mathbb{N}^*$  with  $m > n$ ,*

$$(5.34) \quad \|\mathcal{G}_m(t) - \mathcal{G}_n(t)\|_V \leq c \int_0^t \|\tilde{u}_m(s) - \tilde{u}_n(s)\|_V ds + ch_n \quad \forall t \in [0, T].$$

**Proof.** Let  $m, n \in \mathbb{N}^*$  with  $m > n$ . It is obvious that (5.34) holds for  $t = 0$ . Now, let  $t \in (0, T]$ , then, there are three cases, (i)  $t \in (t_0^m, t_1^m] \cap (t_0^n, t_1^n]$ , (ii)  $t \in$

$(t_q^m, t_{q+1}^m] \cap (t_0^n, t_1^n]$  with  $q \in \{1, \dots, m-1\}$ , (iii)  $t \in (t_q^m, t_{q+1}^m] \cap (t_p^n, t_{p+1}^n]$  with  $q \in \{1, \dots, m-1\}$  and  $p \in \{1, \dots, n-1\}$ . Using (4.33), (5.4), (5.7) (i) and (5.23), we get

$$(5.35) \quad \begin{aligned} \|\mathcal{G}_m(t) - \mathcal{G}_n(t)\|_V &= \|h_m \mathcal{G}(t_1^m) u_m^0 - h_n \mathcal{G}(t_1^n) u_n^0\|_V \\ &\leq ch_m + ch_n \quad \forall t \in (t_0^m, t_1^m] \cap (t_0^n, t_1^n]. \end{aligned}$$

On the other hand, let  $t \in (t_q^m, t_{q+1}^m]$  with  $q \in \{1, \dots, m-1\}$ . We use (5.4), (5.23) and (5.21) to obtain

$$\mathcal{G}_m(t) = \sum_{j=1}^q \int_{t_{j-1}^m}^{t_j^m} \mathcal{G}(t_{q+1}^m - t_j^m) \tilde{u}_m(s) ds + h_m \mathcal{G}(t_{q+1}^m) u_m^0,$$

which gives

$$(5.36) \quad \begin{aligned} \mathcal{G}_m(t) &= \sum_{j=1}^q \int_{t_{j-1}^m}^{t_j^m} (\mathcal{G}(t_{q+1}^m - t_j^m) - \mathcal{G}(t_{q+1}^m - s)) \tilde{u}_m(s) ds \\ &\quad + \sum_{j=1}^q \int_{t_{j-1}^m}^{t_j^m} (\mathcal{G}(t_{q+1}^m - s) - \mathcal{G}(t - s)) \tilde{u}_m(s) ds \\ &\quad + \int_0^t \mathcal{G}(t - s) \tilde{u}_m(s) ds + \int_t^{t_q^m} \mathcal{G}(t - s) \tilde{u}_m(s) ds + h_m \mathcal{G}(t_{q+1}^m) u_m^0. \end{aligned}$$

Thus, for all  $t \in (t_q^m, t_{q+1}^m] \cap (t_0^n, t_1^n]$  with  $q \in \{1, \dots, m-1\}$ , we have

$$(5.37) \quad \begin{aligned} \|\mathcal{G}_m(t) - \mathcal{G}_n(t)\|_V &\leq c \sum_{j=1}^q \int_{t_{j-1}^m}^{t_j^m} |s - t_j^m| \|\tilde{u}_m(s)\|_V ds \\ &\quad + c \sum_{j=1}^q \int_{t_{j-1}^m}^{t_j^m} |t_{q+1}^m - t| \|\tilde{u}_m(s)\|_V ds \\ &\quad + c \int_0^{t_q^m} \|\mathcal{G}(t - s) \tilde{u}_m(s)\|_V ds \\ &\quad + \|h_m \mathcal{G}(t_{q+1}^m) u_m^0\|_V + \|h_n \mathcal{G}(t_1^n) u_n^0\|_V \\ &\leq c \sum_{j=1}^q h_m^2 + c \sum_{j=1}^q h_m^2 \\ &\quad + c \int_0^{t_q^m} \|\tilde{u}_m(s)\|_V ds + ch_m + ch_n \\ &\leq c \int_0^{t_1^n} ds + ch_m + ch_n \\ &\leq ch_m + ch_n. \end{aligned}$$

Now, it follows from (4.32), (4.33) and (5.36), that

$$\begin{aligned} \|\mathcal{G}_m(t) - \mathcal{G}_n(t)\|_V &\leq c \sum_{j=1}^q \int_{t_{j-1}^m}^{t_j^m} |s - t_j^m| \|\tilde{u}_m(s)\|_V ds \\ &+ c \sum_{j=1}^p \int_{t_{j-1}^n}^{t_j^n} |s - t_j^n| \|\tilde{u}_n(s)\|_V ds + c \sum_{j=1}^q \int_{t_{j-1}^m}^{t_j^m} |t_{q+1}^m - t| \|\tilde{u}_m(s)\|_V ds \\ &+ c \sum_{j=1}^p \int_{t_{j-1}^n}^{t_j^n} |t_{p+1}^n - t| \|\tilde{u}_n(s)\|_V ds + c \int_0^t \|\tilde{u}_m(s) - \tilde{u}_n(s)\|_V ds + ch_m + ch_n \end{aligned}$$

for all  $t \in (t_q^m, t_{q+1}^m] \cap (t_p^n, t_{p+1}^n]$  with  $q \in \{1, \dots, m-1\}$  and  $p \in \{1, \dots, n-1\}$ , and keeping in mind (5.26), we obtain

$$\begin{aligned} \|\mathcal{G}_m(t) - \mathcal{G}_n(t)\|_V &\leq c \sum_{j=1}^q h_m^2 + c \sum_{j=1}^p h_n^2 + c \sum_{j=1}^q h_m^2 + c \sum_{j=1}^p h_n^2 \\ &+ c \int_0^t \|\tilde{u}_m(s) - \tilde{u}_n(s)\|_V ds + ch_m + ch_n \\ &\leq c \int_0^t \|\tilde{u}_m(s) - \tilde{u}_n(s)\|_V ds + ch_m + ch_n, \end{aligned}$$

which, with (5.35) and (5.37), gives (5.34).  $\square$

**Lemma 5.8.** *There exists a function  $u \in W^{1,2}(0, T; V)$  and two subsequences of  $\{u_m\}$  and  $\{\tilde{u}_m\}$  again denoted by  $\{u_m\}$  and  $\{\tilde{u}_m\}$ , respectively, such that*

$$(5.38) \quad u_m \rightharpoonup u \text{ weakly in } L^2(0, T; V),$$

$$(5.39) \quad \dot{u}_m \rightharpoonup \dot{u} \text{ weakly in } L^2(0, T; V),$$

$$(5.40) \quad u_m \rightarrow u \text{ strongly in } C([0, T]; L^2(\Gamma_3)^d),$$

$$(5.41) \quad u_m \rightarrow u \text{ strongly in } C([0, T]; V),$$

$$(5.42) \quad \tilde{u}_m \rightarrow u \text{ strongly in } L^2(0, T; V).$$

**Proof.** We notice that  $L^2(0, T; V)$  is a real Hilbert space equipped with the canonical inner product

$$(w, v)_{L^2(0, T; V)} = \int_0^T (w(s), v(s))_V ds \quad \forall w, v \in L^2(0, T; V).$$

Using standard compactness arguments, see [2], it follows from (5.28) and (5.29) that there is an element  $u \in L^2(0, T; V)$  and a subsequence of  $\{u_m\}$  again denoted

by  $\{u_m\}$ , such that the convergences (5.38) and (5.39) hold. We now use (5.33) to see that  $E = \{u_m: [0, T] \rightarrow L^2(\Gamma_3)^d; m \in \mathbb{N}^*\}$ , the set of the traces of  $\{u_m\}$  on  $\Gamma_3$ , is equicontinuous. Moreover, since the trace map is a compact operator, it follows from (5.28) that  $E(t) = \{u_m(t); u_m \in E\}$  is relatively compact for all  $t \in [0, T]$ . Therefore, using a version of the Arzela-Ascoli theorem, see [11], and taking another subsequence if necessary, we obtain (5.40). We turn now to the proof of (5.41). To this end, we need to show that the subsequence  $\{u_m\}$ , obtained in (5.38)–(5.40), is a Cauchy sequence in the Banach space  $C([0, T]; V)$ . It follows from (5.8), (4.16), (4.17), (4.36), (4.28), (5.21), (5.22), (5.23) and (5.24) that  $\{\mathcal{G}_m\}$ ,  $\{\tilde{u}_m\}$ ,  $\{\tilde{\beta}_m\}$  and  $\{f_m\}$  satisfy the inequality

$$(5.43) \quad \begin{cases} (\mathcal{F}\tilde{u}_m(t), v - \tilde{u}_m(t))_V + (\mathcal{G}_m(t), v - \tilde{u}_m(t))_V \\ \quad + j_{\text{ad}}(\tilde{\beta}_m(t), \tilde{u}_m(t), v - \tilde{u}_m(t)) + \psi(\tilde{u}_m(t), v - \tilde{u}_m(t)) \\ \geq (f_m(t), v - \tilde{u}_m(t))_V \quad \forall v \in V \quad \forall t \in [0, T]. \end{cases}$$

Now, let  $m, n \in \mathbb{N}^*$ , such that  $m > n > T$ . By taking  $(\mathcal{G}_m, \tilde{u}_m, \tilde{\beta}_m, f_m, v) = (\mathcal{G}_m, \tilde{u}_m, \tilde{\beta}_m, f_m, \tilde{u}_n)$ ,  $(\mathcal{G}_m, \tilde{u}_m, \tilde{\beta}_m, f_m, v) = (\mathcal{G}_n, \tilde{u}_n, \tilde{\beta}_n, f_n, \tilde{u}_m)$  in (5.43) and adding the two inequalities, we get

$$\begin{cases} (\mathcal{F}\tilde{u}_m(t) - \mathcal{F}\tilde{u}_n(t), \tilde{u}_m(t) - \tilde{u}_n(t))_V \\ \leq \psi(\tilde{u}_m(t), \tilde{u}_n(t) - \tilde{u}_m(t)) + \psi(\tilde{u}_n(t), \tilde{u}_m(t) - \tilde{u}_n(t)) \\ \quad + j_{\text{ad}}(\tilde{\beta}_n(t), \tilde{u}_n(t), \tilde{u}_m(t) - \tilde{u}_n(t)) + j_{\text{ad}}(\tilde{\beta}_m(t), \tilde{u}_m(t), \tilde{u}_n(t) - \tilde{u}_m(t)) \\ \quad + (\mathcal{G}_n(t) - \mathcal{G}_m(t), \tilde{u}_m(t) - \tilde{u}_n(t))_V \\ \quad + (f_m(t) - f_n(t), \tilde{u}_m(t) - \tilde{u}_n(t))_V \quad \forall t \in [0, T], \end{cases}$$

which combined with (4.30), (5.26), (5.27), (4.39), (4.47) and the inequality

$$ab \leq \frac{a^2}{m_{\mathcal{A}}} + \frac{m_{\mathcal{A}}}{4}b^2 \quad \forall a, b \in \mathbb{R}$$

leads us to

$$(5.44) \quad \|\tilde{u}_m(t) - \tilde{u}_n(t)\|_V^2 \leq c\|\tilde{u}_m(t) - \tilde{u}_n(t)\|_{L^2(\Gamma_3)^d} + c\|\mathcal{G}_n(t) - \mathcal{G}_m(t)\|_V^2 \\ + c\|f_m(t) - f_n(t)\|_V^2 + c\|f_n(t) - f_m(t)\|_V^2 \quad \forall t \in [0, T].$$

Using (5.30), we get

$$(5.45) \quad \|\tilde{u}_m(t) - \tilde{u}_n(t)\|_V \leq \|u_m(t) - u_n(t)\|_V + ch_m + ch_n.$$



Also, from (5.30) and (4.3), we deduce that

$$(5.46) \quad \begin{aligned} \|\tilde{u}_m(t) - \tilde{u}_n(t)\|_{L^2(\Gamma_3)^d} &\leq \|\tilde{u}_m(t) - u_m(t)\|_{L^2(\Gamma_3)^d} + \|u_m(t) - u_n(t)\|_{L^2(\Gamma_3)^d} \\ &\quad + \|u_n(t) - \tilde{u}_n(t)\|_{L^2(\Gamma_3)^d} \\ &\leq \|u_m(t) - u_n(t)\|_{L^2(\Gamma_3)^d} + ch_m + ch_n. \end{aligned}$$

Now, it follows from (5.44), (5.45), (5.46), (5.34) and (5.31), that

$$\|\tilde{u}_m(t) - \tilde{u}_n(t)\|_V^2 \leq c\|u_m(t) - u_n(t)\|_{L^2(\Gamma_3)^d} + c \int_0^t \|u_n(s) - u_m(s)\|_V^2 ds + ch_m + ch_n.$$

This, together with the fact that

$$\|u_m(t) - u_n(t)\|_V^2 \leq c\|u_m(t) - \tilde{u}_m(t)\|_V^2 + c\|\tilde{u}_m(t) - \tilde{u}_n(t)\|_V^2 + c\|\tilde{u}_n(t) - u_n(t)\|_V^2,$$

implies that

$$\begin{aligned} \|u_m(t) - u_n(t)\|_V^2 &\leq c\|u_m(t) - u_n(t)\|_{L^2(\Gamma_3)^d} + c \int_0^t \|u_n(s) - u_m(s)\|_V^2 ds \\ &\quad + ch_m + ch_n \quad \forall t \in [0, T]. \end{aligned}$$

We now use Lemma 2.1 in the last inequality to obtain

$$\begin{aligned} \|u_m(t) - u_n(t)\|_V^2 &\leq c\|u_m(t) - u_n(t)\|_{L^2(\Gamma_3)^d} + c \int_0^t \|u_m(s) - u_n(s)\|_{L^2(\Gamma_3)^d} ds \\ &\quad + ch_m + ch_n \quad \forall t \in [0, T]. \end{aligned}$$

Therefore, we have

$$\|u_m - u_n\|_{C([0, T]; V)}^2 \leq c\|u_m - u_n\|_{C([0, T]; L^2(\Gamma_3)^d)} + ch_n,$$

which combined with (5.40) implies that  $\{u_m\}$  is a Cauchy sequence in  $C([0, T]; V)$ . Thus, using the convergence (5.38), we obtain (5.41). Finally, the convergence (5.42) is a consequence of (5.30) and (5.41).  $\square$

In the rest of this paper  $u$  is the function obtained in Lemma 5.8,  $\{u_m\}$ ,  $\{\mathcal{G}_m\}$ ,  $\{\tilde{u}_m\}$ ,  $\{\tilde{\beta}_m\}$  and  $\{f_m\}$  represent appropriate subsequences of  $\{u_m\}$ ,  $\{\mathcal{G}_m\}$ ,  $\{\tilde{u}_m\}$ ,  $\{\tilde{\beta}_m\}$  and  $\{f_m\}$  such that the convergences (5.38)–(5.42) hold. Now, consider the following problem.

**Problem 5.9.** Find a function  $\beta: [0, T] \rightarrow L^\infty(\Gamma_3)$ , such that

$$(5.47) \quad \dot{\beta}(t) = H_{\text{ad}}(\beta(t), R_\nu(u_\nu(t))) \quad \text{for a.e. } t \in (0, T),$$

$$(5.48) \quad \beta(0) = \beta_0.$$

**Lemma 5.10.** *The problem (5.47)–(5.48) has a unique solution which satisfies*

$$(5.49) \quad \begin{cases} \text{(i)} & \beta \in W^{1,\infty}(0, T; L^\infty(\Gamma_3)), \\ \text{(ii)} & 0 \leq \beta(t) \leq 1 \quad \forall t \in [0, T] \text{ for a.e. } x \in \Gamma_3. \end{cases}$$

Moreover, we have

$$(5.50) \quad \tilde{\beta}_m \rightarrow \beta \text{ strongly in } L^2(0, T; L^2(\Gamma_3)).$$

**P r o o f.** The proof of the existence and uniqueness of the solution to the problem (5.47)–(5.48) which satisfies (5.49) can be obtained by arguments similar to those used in the proof of [13], Lemma 4.3. To continue, we use (5.5), (5.7), (5.21) and (5.22) to obtain

$$(5.51) \quad \begin{aligned} \tilde{\beta}_m(t) = & \int_0^t H_{\text{ad}}(\tilde{\beta}_m(s), R_\nu(\tilde{u}_{m\nu}(s))) \, ds + \int_t^{t_i^m} H_{\text{ad}}(\tilde{\beta}_m(s), R_\nu(\tilde{u}_{m\nu}(s))) \, ds \\ & + h_m H_{\text{ad}}(\beta_0, R_\nu(u_{0\nu})) + \beta_0 \end{aligned}$$

for all  $t \in (t_i^m, t_{i+1}^m]$  with  $i \in \{0, \dots, m-1\}$ . It follows from (4.9), (4.43), (4.44), (5.27), (5.51) and (5.47)–(5.48), that

$$\|\tilde{\beta}_m(t) - \beta(t)\|_{L^2(\Gamma_3)} \leq c \int_0^t \|\tilde{\beta}_m(s) - \beta(s)\|_{L^2(\Gamma_3)} \, ds + c \int_0^t \|\tilde{u}_m(s) - u(s)\|_V \, ds + ch_m$$

for all  $t \in [0, T]$ . Using Lemma 2.1 in the last inequality, we obtain

$$\|\tilde{\beta}_m(t) - \beta(t)\|_{L^2(\Gamma_3)} \leq c \int_0^t \|\tilde{u}_m(s) - u(s)\|_V \, ds + ch_m \quad \forall t \in [0, T],$$

which gives

$$\|\tilde{\beta}_m - \beta\|_{L^2(0, T; L^2(\Gamma_3))} \leq c \|\tilde{u}_m - u\|_{L^2(0, T; V)} + ch_m.$$

Passing to the limit as  $m \rightarrow \infty$  in the last inequality by using (5.42), we get (5.50).  $\square$

**Lemma 5.11.** *The following convergences hold:*

$$(5.52) \quad \mathcal{F}\tilde{u}_m \rightarrow \mathcal{F}u \text{ strongly in } L^2(0, T; V),$$

$$(5.53) \quad f_m \rightarrow f \text{ strongly in } L^2(0, T; V),$$

$$(5.54) \quad \mathcal{G}_m \rightarrow \tilde{\mathcal{G}} \text{ strongly in } L^2(0, T; V),$$

where the function  $\tilde{\mathcal{G}}: [0, T] \rightarrow V$  is defined by

$$(5.55) \quad \tilde{\mathcal{G}}(t) = \int_0^t \mathcal{G}(t-s)u(s) \, ds \quad \forall t \in [0, T].$$

*Proof.* Obviously, (4.31) and (5.42) gives (5.52). On the other hand, (5.53) follows from (5.31). For the proof of (5.54), we use arguments similar to those in [14], Lemma 4.10.  $\square$

We have the following convergence results.

**Lemma 5.12.** *For all  $v \in L^2(0, T; V)$  we have*

$$(5.56) \quad \lim_{m \rightarrow \infty} \int_0^T \varphi(\tilde{\beta}_m(s), \tilde{u}_m(s), v(s)) \, ds = \int_0^T \varphi(\beta(s), u(s), v(s)) \, ds,$$

$$(5.57) \quad \lim_{m \rightarrow \infty} \int_0^T [\varphi(\tilde{\beta}_m(s), \tilde{u}_m(s), \dot{u}_m(s)) - \varphi(\beta(s), u(s), \dot{u}_m(s))] \, ds = 0,$$

$$(5.58) \quad \liminf_{m \rightarrow \infty} \int_0^T \varphi(\tilde{\beta}_m(s), \tilde{u}_m(s), \dot{u}_m(s)) \, ds \geq \int_0^T \varphi(\beta(s), u(s), \dot{u}(s)) \, ds.$$

*Proof.* Using the properties of the functional  $\varphi$  defined by (4.17), (4.3), (5.27) and (5.49), we deduce that

$$(5.59) \quad \left\{ \begin{array}{l} \left| \int_0^T [\varphi(\tilde{\beta}_m(s), \tilde{u}_m(s), v(s)) - \varphi(\beta(s), u(s), v(s))] \, ds \right| \\ \leq c(\|\tilde{\beta}_m - \beta\|_{L^2(0, T; L^2(\Gamma_3))} + \|\tilde{u}_m - u\|_{L^2(0, T; V)})\|v\|_{L^2(0, T; V)} \end{array} \right.$$

for all  $v \in L^2(0, T; V)$ . Therefore, the convergences (5.56)–(5.57) follow from (5.42), (5.50), (5.59) and (5.29). To continue, let  $\Phi: L^2(0, T; V) \rightarrow \mathbb{R}$  be the functional defined by

$$(5.60) \quad \Phi(v) = \int_0^T \varphi(\beta(s), u(s), v(s)) \, ds \quad \forall v \in L^2(0, T; V),$$

where  $\beta$  is the unique solution of the problem (5.47)–(5.48). We use (4.17), (4.15), (4.16), (4.38), (4.49), (5.49) and (5.60) to see that  $\Phi$  is convex and continuous. Therefore, we deduce that  $\Phi$  is a weakly lower semicontinuous function, see [2], which with (5.39) gives

$$(5.61) \quad \liminf_{m \rightarrow \infty} \Phi(\dot{u}_m) \geq \Phi(\dot{u}).$$

On the other hand, one has

$$(5.62) \quad \int_0^T \varphi(\tilde{\beta}_m(s), \tilde{u}_m(s), \dot{u}_m(s)) \, ds \\ = \int_0^T [\varphi(\tilde{\beta}_m(s), \tilde{u}_m(s), \dot{u}_m(s)) \, ds - \varphi(\beta(s), u(s), \dot{u}_m(s))] \, ds + \Phi(\dot{u}_m).$$

Thus, taking into account (5.57) and (5.61) when passing to the  $\liminf$  as  $m \rightarrow \infty$  in (5.62), we obtain (5.58).  $\square$

**Fourth step.** We have now all the ingredients to prove Theorem 5.1.

**Proof of Theorem 5.1.** It follows from (5.41) and (5.20) that  $u(0) = u_0$ . Let  $\beta$  be the unique solution of the problem (5.47)–(5.48). Let  $t \in (0, T)$ , let  $r > 0$ , such that  $t + r \in (0, T)$ . For each  $w \in V$  we define a function  $v \in L^2(0, T; V)$  by

$$v(s) = \begin{cases} w & \text{for } s \in (t, t + r), \\ \dot{u}(s) & \text{elsewhere.} \end{cases}$$

We now use (5.3), (5.21), (5.22), (5.23), (5.24), (5.25) to obtain the inequality

$$(5.63) \quad \begin{cases} \int_0^T (\mathcal{F}\tilde{u}_m(s), v(s) - \dot{u}_m(s))_V \, ds + \int_0^T (\mathcal{G}_m(s), v(s) - \dot{u}_m(s))_V \, ds \\ + \int_0^T \varphi(\tilde{\beta}_m(s), \tilde{u}_m(s), v(s)) \, ds - \int_0^T \varphi(\tilde{\beta}_m(s), \tilde{u}_m(s), \dot{u}_m(s)) \, ds \\ \geq \int_0^T (f_m(s), v(s) - \dot{u}_m(s))_V \, ds. \end{cases}$$

Passing to the  $\limsup$  as  $m \rightarrow \infty$  in (5.63), by using Lemma 5.11, Lemma 5.12 and the convergence (5.39), we obtain

$$(5.64) \quad \begin{cases} \frac{1}{r} \int_t^{t+r} (\mathcal{F}u(s), w - \dot{u}(s))_V \, ds + \frac{1}{r} \int_t^{t+r} (\tilde{\mathcal{G}}(s), w - \dot{u}(s))_V \, ds \\ + \frac{1}{r} \int_t^{t+r} [\varphi(\beta(s), u(s), w) - \varphi(\beta(s), u(s), \dot{u}(s))] \, ds \\ \geq \frac{1}{r} \int_t^{t+r} (f(s), w - \dot{u}(s))_V \, ds \quad \forall w \in V. \end{cases}$$

Letting  $r \rightarrow 0$  in (5.64) and keeping in mind (5.55), we conclude that  $\{u, \beta\}$  is a solution of the problem (4.25)–(4.27). On the other hand, using (5.32), we obtain

$$\|u(t) - u(s)\|_V \leq \|u(t) - u_m(t)\|_V + \|u_m(t) - u_m(s)\|_V + \|u_m(s) - u(s)\|_V \\ \leq \|u(t) - u_m(t)\|_V + c|t - s| + \|u_m(s) - u(s)\|_V \quad \forall t, s \in [0, T].$$

Passing to the limit as  $m \rightarrow \infty$  in the last inequality and using (5.41), we get

$$\|u(t) - u(s)\|_V \leq c|t - s| \quad \forall t, s \in [0, T].$$

Thus,  $u$  satisfies (5.1). The regularity (5.2) follows from Lemma 5.10. Finally, we notice that the uniqueness of the solution remains, as far as we know, an open question.  $\square$

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