THE NEW ITERATION METHODS FOR SOLVING ABSOLUTE VALUE EQUATIONS

Rashid Ali, Kejia Pan, Changsha

Received March 15, 2021. Published online November 11, 2021.

Abstract. Many problems in operations research, management science, and engineering fields lead to the solution of absolute value equations. In this study, we propose two new iteration methods for solving absolute value equations $Ax - |x| = b$, where $A \in \mathbb{R}^{n\times n}$ is an M-matrix or strictly diagonally dominant matrix, $b \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$ is an unknown solution vector. Furthermore, we discuss the convergence of the proposed two methods under suitable assumptions. Numerical experiments are given to verify the feasibility, robustness and effectiveness of our methods.

Keywords: absolute value equation; iteration method; matrix splitting; linear complementarity problem; numerical experiment

MSC 2020: 65F10, 65H10, 90C30

1. Introduction

Consider the absolute value equation (AVE):

$$Ax - |x| = b, \tag{1.1}$$

where the coefficient matrix $A \in \mathbb{R}^{n\times n}$ is an M-matrix or strictly diagonally dominant matrix, $b \in \mathbb{R}^n$ and $|x| = (|x_1|, |x_2|, \ldots, |x_n|)^\top$. Another generalized AVE is in the form of

$$Ax + B|x| = b, \tag{1.2}$$

where $B \in \mathbb{R}^{n\times n}$, $B \neq 0$. When $B = -I$, where I stands for the identity matrix, then the AVE (1.2) reduces to the special form (1.1).

DOI: 10.21136/AM.2021.0055-21

The AVEs are one of the important nonlinear and non-differentiable systems which arise in optimization, such as the linear and quadratic programming, the journal bearings, the network prices, the linear complementarity problems (LCPs) and the contact problems; see [27], [28], [3], [7], [31], [25], [23], [38] and the references therein. Therefore, the research of efficient numerical algorithms and theories for AVEs has significant theoretical importance, broad application prospects, and high economic value.

Numerical methods for AVEs are concerned with the structure of solutions, mathematical theories, algebraic structures, and unique implementations of large-quality preconditioners and high performance numerical algorithms. In recent years, numerical methods of AVEs have gained a lot of concentration, and a large number of papers have proposed numerous methods such as Salkuyeh [41] suggested the Picard-HSS method for solving (1.1). Mangasarian [24] proposed an approximated generalized Newton (GN) method for solving (1.1) and showed that this algorithm converges linearly from any initial point to the unique solution under the condition that $\|A^{-1}\| < \frac{1}{4}$. Cruz et al. [4] established an inexact semi-smooth Newton algorithm for the AVE (1.1) and showed that the method is globally convergent under the condition that $\|A^{-1}\| < \frac{1}{3}$. Hu and Huang [17] reformulated the AVE system as a standard LCP without any premise and provided some existence and convexity outcomes for the solution of the AVE (1.1). Zhang et al. [46] introduced a new algorithm that relaxed the AVE into a convex optimization problem. They discovered the sparsest solution of the AVE through the minimum l_∞-norm. Feng and Liu [12], [13] presented an improved GN method and two-step iterative method for solving (1.1). Caccetta et al. [5] studied a smoothing Newton method for solving (1.1) and proved that the method is globally convergent with condition that $\|A^{-1}\| < 1$. Haghani [15] suggested the generalized Traub's method, which is better than the Mangasarian's method. Saheya et al. [40] studied smoothing type algorithms for solving (1.1) and proved that their algorithms have local and global quadratic convergence. Edalatpour et al. [11] described the generalized Gauss-Seidel (GGS) method for solving (1.1). Ke and Ma [19] suggested an SOR-like method to solve the AVE (1.1). Chen et al. [6] modified the idea of [19], and presented the SOR-like method with optimal parameters for solving (1.1). Nguyen et al. [35] presented unified smoothing functions associated with the second-order cone for solving (1.1). Gu et al. [14] suggested the nonlinear CSCS-like method and the Picard-CSCS method for solving (1.1), which involves the Toeplitz matrix. Hashemi and Ketabchi [16] introduced the numerical comparisons of smoothing functions for AVE (1.1). Wu and Li [44] introduced the special shift splitting iterative method to solve AVE (1.1) and proved the new convergence conditions of the proposed iterative method. Moosaei et al. [34] showed that the AVE (1.1) is equivalent to the bilinear programming problem. They solved AVE (1.1) by the principle of simulated annealing, and then found the AVE (1.1) solution with the minimum norm, and others, see [1], [9], [10], [20], [30], [26], [32], [36], [37], [43], [45] and the references therein.

Recently, Miao and Zhang [33], Li et al. [21], Mao et al. [29] and Dehghan and Hajrian [8] presented different approaches using the fixed-point principle to solve the LCPs. This research aims to extend this approach to AVEs using the fixed point principle and formulate efficient iterative methods for solving AVE (1.1). The main contributions of this paper are given as follows: we split matrix A into different parts (diagonal, strictly upper and lower-triangular parts) and add two additional parameters (ψ parameter for Method I, and λ parameter for Method II), which can speed up the convergence of the suggested iteration methods. Furthermore, we discuss the convergence of the proposed two methods.

This research is structured as follows. In Section 2, we discuss the proposed methods and their convergence for solving AVE (1.1). Numerical results and concluding remarks are given in Sections 3 and 4, respectively.

2. Proposed methods

In this section, we organize the proposed iteration methods for solving (1.1). We discuss some results that will be used in the following analysis.

Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, we write $A \geqslant 0$ if $a_{ij} \geqslant 0$ holds for all $1 \leqslant i,\ j \leqslant n$. We express the norm, spectral radius and absolute value of A as $\| A \|_\infty$, $\varrho(A)$ and $|A| = (|a_{ij}|)$, respectively.

Definition 2.1. Let $A \in \mathbb{R}^{n\times n}$. It is called an

(1) Z-matrix if $a_{ij} \leqslant 0$ for $i \neq j$, $i, j = 1, 2, \ldots, n$,
(2) M-matrix if it is a nonsingular Z-matrix and with $A^{-1} \geqslant 0$.

Lemma 2.1 ([42]). *Let z and x be the two points in $\mathbb{R}^n$. Then $\||z|-|x|\| \leqslant \|z-x\|$.*

To propose and analyze the algorithms, we split the A matrix as

$$A = \widehat{D}_A - \widehat{L}_A - \widehat{U}_A, \tag{2.1}$$

where $\widehat{D}_A$, $\widehat{L}_A$ and $\widehat{U}_A$ are diagonal, strictly lower and upper-triangular parts of A, respectively. The AVE (1.1) is equivalent to the fixed-point problem of solving

$$x = F(x)$$

such that

$$F(x) = x - E[Ax - |x| - b], \tag{2.2}$$

where $E \in \mathbb{R}^{n\times n}$ is a positive diagonal matrix (then by choice of $E = \widehat{D}_A^{-1}$, see for more detail [2], [22]). Let $\overline{K}$ be either a strictly lower-triangular or a strictly upper-triangular matrix. Using the splitting (2.1), we suggest the following methods for solving the AVEs:

Method I.

Step 1: Select a parameter $0 < \psi < 2$, an initial vector $x^0 \in \mathbb{R}^n$ and set $i = 0$.

Step 2: Calculate

$$x^{i+1} = x^i - E[-\psi\overline{K}x^{i+1} + (\psi(2-\psi)A + \psi\overline{K})x^i - \psi(2-\psi)(|x^i| + b)]. \tag{2.3}$$

Step 3: If $x^{i+1} = x^i$, then stop. Otherwise set $i = i + 1$ and go back to Step 2.

Method II.

Step 1: Select a parameter $0 < \lambda \leqslant 1$, an initial vector $x^0 \in \mathbb{R}^n$ and set $i = 0$.

Step 2: Calculate

$$x^{i+1} = \lambda[x^i - E((Ax^i - |x^i| - b) - \overline{K}(x^{i+1} - x^i))] + (1-\lambda)x^i. \tag{2.4}$$

Step 3: If $x^{i+1} = x^i$, then stop. Otherwise put $i = i + 1$ and return to Step 2.

It is necessary to recall the following significant results for the convergence of the proposed methods.

Theorem 2.1. *If $\{x^{i+1}\}$ and $\{x^i\}$ are the sequences generated by Method* I, *then*

$$|x^{i+1} - x^\star| \leqslant G^{-1}J|x^i - x^\star|,$$

where $G = I - \psi E|\overline{K}|$ and $J = \psi(2-\psi)E + |I - E(\psi(2-\psi)A + \psi\overline{K})|$. Furthermore, if $\varrho(G^{-1}J) < 1$, then the sequence $\{x^i\}$ converges to the unique solution $x^\star$ of AVE (1.1).

P r o o f. Let $x^\star$ be a solution of (1.1). Then

$$x^\star = x^\star - E[-\psi\overline{K}x^\star + (\psi(2-\psi)A + \psi\overline{K})x^\star - \psi(2-\psi)(|x^\star| + b)]. \tag{2.5}$$

After subtracting (2.5) from (2.3), we obtain

$$x^{i+1}-x^\star = (I-E(\psi(2-\psi)A+\psi\overline{K}))(x^i-x^\star)+\psi E\overline{K}(x^{i+1}-x^\star)+\psi(2-\psi)E(|x^i|-|x^\star|).$$

Taking absolute values on both sides and using Lemma 2.1, we have

$$\begin{aligned}
|x^{i+1} - x^\star| &\leqslant |I - E(\psi(2-\psi)A + \psi\overline{K})||x^i - x^\star| \\
&\quad + \psi E|\overline{K}||x^{i+1} - x^\star| + \psi(2-\psi)E||x^i| - |x^\star|| \\
&\leqslant |I - E(\psi(2-\psi)A + \psi\overline{K})||x^i - x^\star| \\
&\quad + \psi E|\overline{K}||x^{i+1} - x^\star| + \psi(2-\psi)E|x^i - x^\star|, \\
|x^{i+1} - x^\star| - \psi E|\overline{K}||x^{i+1} - x^\star| &\leqslant (\psi(2-\psi)E + |I - E(\psi(2-\psi)A + \psi\overline{K})|)|x^i - x^\star|, \\
(I - \psi E|\overline{K}|)|x^{i+1} - x^\star| &\leqslant (\psi(2-\psi)E + |I - E(\psi(2-\psi)A + \psi\overline{K})|)|x^i - x^\star|.
\end{aligned}$$

Since $\overline{K}$ is a strictly lower or upper-triangular matrix, $(I - \psi E|\overline{K}|)$ is invertible. Thus, $(I - \psi E|\overline{K}|)^{-1}$ exists and is non-negative, we have

$$|x^{i+1} - x^\star| \leqslant G^{-1}J|x^i - x^\star|.$$

Note that the matrix $G^{-1}J$ is non-negative. We know that in [2], Theorem 4.1, if $\varrho(G^{-1}J) < 1$, then the sequence $\{x^i\}$ of Method I converges to the solution $x^\star$ of AVE.

For uniqueness of the solution, let $z^\star$ be another solution of AVE. From the equations

$$\begin{aligned}
Ax^\star - |x^\star| &= b, \\
Az^\star - |z^\star| &= b,
\end{aligned}$$

written as

$$\begin{aligned}
x^\star &= x^\star - E[-\psi\overline{K}x^\star + (\psi(2-\psi)A + \psi\overline{K})x^\star - \psi(2-\psi)(|x^\star| + b)], \\
z^\star &= x^\star - E[-\psi\overline{K}z^\star + (\psi(2-\psi)A + \psi\overline{K})z^\star - \psi(2-\psi)(|z^\star| + b)],
\end{aligned}$$

we obtain

$$|x^\star - z^\star| \leqslant G^{-1}J|x^\star - z^\star|,$$

where $G = I - \psi E|\overline{K}|$ and $J = \psi(2-\psi)E + |I - E(\psi(2-\psi)A + \psi\overline{K})|$. Since $\varrho(G^{-1}J) < 1$, we have

$$x^\star = z^\star.$$

This completes the proof. □

Theorem 2.2. *If* $\{x^{i+1}\}$ *and* $\{x^i\}$ *are the sequences generated by Method* II, *then*

$$|x^{i+1} - x^\star| \leqslant \mathbb{R}^{-1}\overline{S}|x^i - x^\star|,$$

where $R = I - \lambda E|\overline{K}|$ *and* $\overline{S} = \lambda E + |I - \lambda E(A + \overline{K})|$. *Furthermore, if* $\varrho(\mathbb{R}^{-1}\overline{S}) < 1$, *then the sequence* $\{x^i\}$ *converges to the unique solution* $x^\star$ *of AVE* (1.1).

Proof. Let $x^\star$ be a solution of (1.1). Then

$$x^\star = \lambda[x^\star - E((Ax^\star - |x^\star| - b) - \overline{K}(x^\star - x^\star))] + (1-\lambda)x^\star. \tag{2.6}$$

After subtracting (2.6) from (2.4), we obtain

$$x^{i+1} - x^\star = (I - \lambda E(A + \overline{K}))(x^i - x^\star) + \lambda E(|x^i| - |x^\star|) + \lambda E\overline{K}(x^{i+1} - x^\star).$$

By taking absolute values on both sides and using Lemma 2.1, we have

$$\begin{aligned}
|x^{i+1} - x^\star| &\leqslant |I - \lambda E(A + \overline{K})||x^i - x^\star| + \lambda E||x^i| - |x^\star|| + \lambda E|\overline{K}||x^{i+1} - x^\star| \\
&\leqslant |I - \lambda E(A + \overline{K})||x^i - x^\star| + \lambda E|x^i - x^\star| + \lambda E|\overline{K}||x^{i+1} - x^\star|, \\
|x^{i+1} - x^\star| - \lambda E|\overline{K}||x^{i+1} - x^\star| &\leqslant (\lambda E + |I - \lambda E(A + \overline{K})|)|x^i - x^\star|, \\
(I - \lambda E|\overline{K}|)|x^{i+1} - x^\star| &\leqslant (\lambda E + |I - \lambda E(A + \overline{K})|)|x^i - x^\star|.
\end{aligned}$$

Since $\overline{K}$ is a strictly lower or upper-triangular matrix, $(I - \lambda E|\overline{K}|)$ is invertible. Thus, $(I - \lambda E|\overline{K}|)^{-1}$ exists and is non-negative, we have $|x^{i+1} - x^\star| \leqslant \mathbb{R}^{-1}\overline{S}|x^i - x^\star|$. Evidently, if $\varrho(\mathbb{R}^{-1}\overline{S}) < 1$, the iteration sequence $\{x^i\}$ created by Method II is convergent. $\square$

The proof of the uniqueness is similar to the proof of Theorem 2.1 and is omitted here.

3. Numerical experiments

In this section, we experimentally investigate the effectiveness of the novel methods to solve the AVEs. All numerical tests were conducted on a personal computer with 1.80 GHz CPU (Intel (R) Core (TM) i5-3337U) and 4 GB memory using Matlab 2016a. Furthermore, we take the matrix $\overline{K} = \widehat{L}_A$, the zero vector is the initial vector, the termination condition and formula about ERR are given by

$$\text{RES} := \|b + |x^i| - Ax^i\|_2 \leqslant 10^{-8} \quad \text{and} \quad \text{ERR} := \|x^i - x^\star\|_2,$$

respectively, where $x^\star$ is the exact solution.

Example 3.1. Let

$$A = \text{tridiag}(-1, 8, -1) = \begin{pmatrix} 8 & -1 & & & \\ -1 & 8 & -1 & & \\ & \ddots & 8 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 8 \end{pmatrix} \in \mathbb{R}^{n\times n}, \quad x^\star = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \\ \vdots \end{pmatrix} \in \mathbb{R}^n$$

and $b = Ax^\star - |x^\star| \in \mathbb{R}^n$. The results are discussed in Table 1. In Examples 3.1 and 3.2, we compare the proposed methods with the SOR-like approximate optimal parameter (SORLaopt) method [6].

Methods	n	1000	2000	3000	4000	5000
SORLaopt	Iter	20	20	20	20	20
	Time	2.7458	12.8679	34.8767	82.4866	137.9194
	RES	4.13e − 09	5.84e − 09	7.15e − 09	8.26e − 09	9.23e − 09
	ERR	6.11e − 10	8.64e − 10	1.05e − 09	1.22e − 09	1.36e − 09
Method I	Iter	14	14	15	15	15
	Time	1.4417	8.7177	21.1915	38.7743	82.3002
	RES	6.42e − 09	9.09e − 09	1.82e − 09	2.11e − 09	2.36e − 09
	ERR	8.82e − 10	1.25e − 09	3.01e − 10	3.46e − 10	3.87e − 10
Method II	Iter	17	18	18	18	18
	Time	2.1650	11.5989	27.9001	75.7294	98.9503
	RES	9.76e − 09	3.60e − 09	5.10e − 09	3.75e − 09	5.70e − 09
	ERR	1.66e − 09	6.16e − 10	7.54e − 10	8.71e − 10	9.74e − 10

Table 1. Numerical results of Example 3.1 with $\psi = \lambda = 0.9$.

In Table 1, we report the number of iterations (Iter), the CPU times in seconds (Time), the 2-norm of residual vectors (RES), and the ERR of all methods. From Table 1, we observe that the number of iterations and the Time of the proposed methods are better than the SORLaopt method.

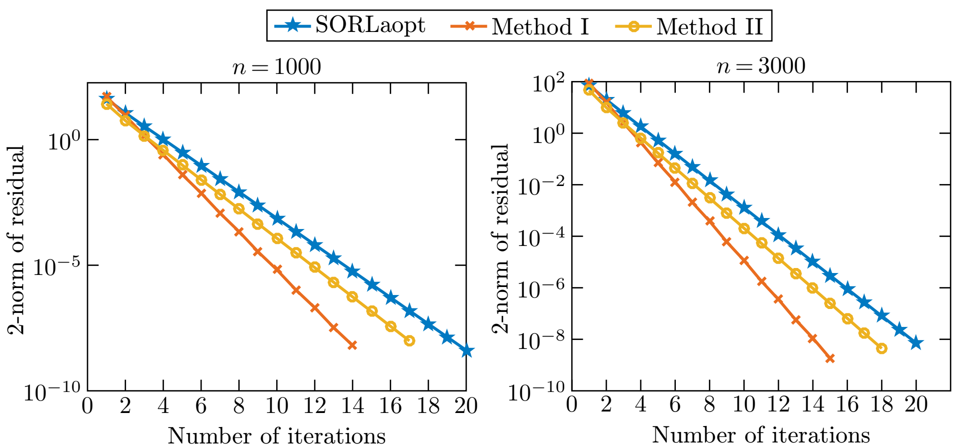

Figure 1. Convergence curves of Example 3.1 with different methods.

The convergence curves of Figure 1 show the effectiveness of the given methods. Graphical representation illustrates that the convergences of the suggested methods are better than the other known method.

Example 3.2. Let $A = M + 4I \in \mathbb{R}^{n\times n}$ and $b = Ax^\star - |x^\star| \in \mathbb{R}^n$ such that

$$M = \operatorname{tridiag}(-I, S, -I) = \begin{pmatrix} S & -I & & & \\ -I & S & -I & & \\ & \ddots & S & \ddots & \\ & & \ddots & \ddots & -I \\ & & & -I & S \end{pmatrix} \in \mathbb{R}^{n\times n}, \quad x^\star = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ \vdots \end{pmatrix} \in \mathbb{R}^n,$$

where $S = \operatorname{tridiag}(-1, 4, -1) \in \mathbb{R}^{v\times v}$, $I \in \mathbb{R}^{v\times v}$ is the identity matrix and $n = v^2$. The numerical results are reported in Table 2. For this example, we take $n = 64$ and $n = 4096$ (problem sizes) and compare all methods graphically. The graphical results are represented in Figure 2.

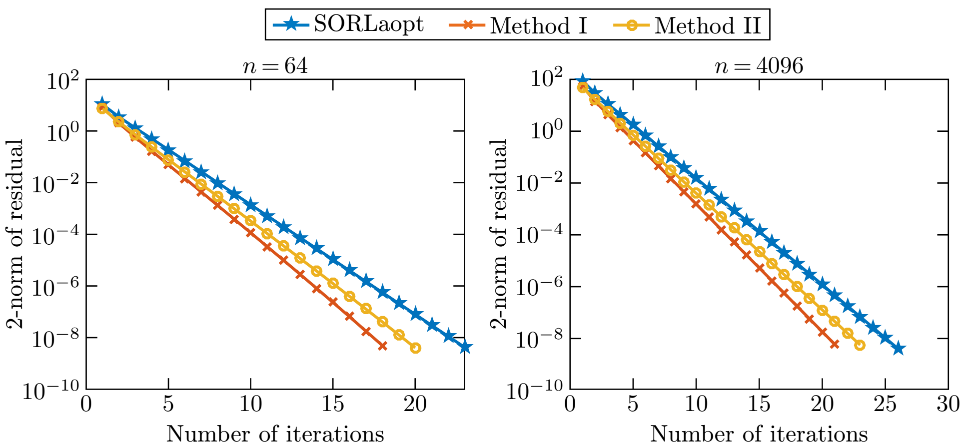

Figure 2. Convergence curves of Example 3.2 with different methods.

In Table 2, we present the numeric outcomes of the SORLaopt method, Method I, and Method II, respectively. From these results, we can recognize that our suggested methods are more effective than the SORLaopt method. Furthermore, we represent the convergence curves of all methods using $n = 64$ and $n = 4096$ (problem sizes). Consequently, the convergence curves of Figure 2 illustrate that the recommended methods are better than the SORLaopt method.

Methods	n	64	256	1024	4096
SORLaopt	Iter	23	24	25	26
	Time	0.4185	0.5135	4.1116	118.7930
	RES	4.24e − 09	5.32e − 09	4.92e − 09	4.08e − 09
	ERR	8.78e − 10	1.12e − 09	1.04e − 09	8.71e − 10
Method I	Iter	18	19	20	21
	Time	0.2797	0.3601	3.9667	102.5239
	RES	4.98e − 09	9.25e − 09	8.04e − 09	5.89e − 09
	ERR	1.07e − 09	2.26e − 09	2.03e − 09	1.50e − 09
Method II	Iter	20	21	22	23
	Time	0.2943	0.3860	4.0426	115.0780
	RES	4.02e − 09	7.47e − 09	7.04e − 09	5.64e − 09
	ERR	8.78e − 10	1.83e − 09	1.77e − 09	1.43e − 09

Table 2. Numerical results of Example 3.2 with $\psi = \lambda = 0.95$.

Example 3.3. Let the matrix A be given by

$$A = \begin{cases} 1000 + i & \text{for } j = i, \\ 1 & \text{for } \begin{cases} j = i + 1, & i = 1, 2, \ldots, n - 1, \\ j = i - 1, & i = 2, \ldots, n, \end{cases} \\ 0 & \text{otherwise.} \end{cases}$$

Compute $b = Ax^\star - |x^\star| \in \mathbb{R}^n$ with $x^\star = (x_1, x_2, x_3, \ldots, x_n)^\top \in \mathbb{R}^n$ such that $x_i = (-1)^i$. In Examples 3.3 and 3.4, we compare the proposed methods with the method presented in [18] (exposed by NM), the SORLaopt method [6] and the method presented in [39] (indicated by Picard). The results are examined in Table 3.

In Table 3, we find that all methods can efficiently and precisely solve the problem. From the numerical outcomes in Table 3, we observe that the 'Iter' and 'Time' of Method II are better than NM and SORLaopt methods. Furthermore, the 'Time' of Method II is best compared to the Picard method. On the other hand, Method I shows much higher computational performance than other known methods.

Example 3.4. Let $A = I \otimes Q + P \otimes I \in \mathbb{R}^{V \times V}$, where $I \in \mathbb{R}^{V \times V}$ stands for the identity matrix, and $\otimes$ denotes the Kronecker product. Furthermore, Q and P are $n \times n$ tridiagonal matrices given by

$$\begin{cases} Q = \operatorname{tridiag}\big[\frac{1}{8}(2 + h), 8, \frac{1}{8}(2 - h)\big], \\ P = \operatorname{tridiag}\big[\frac{1}{4}(1 + h), 4, \frac{1}{4}(1 - h)\big], \\ h = 1/n; \quad V = n^2. \end{cases}$$

The right-hand side vector $b = Ax^{\star} - |x^{\star}| \in \mathbb{R}^V$, where $x^{\star} = \text{ones}(V,1) \in \mathbb{R}^V$. The results are reported in Table 4.

In Table 4, all methods examine the solution $x^{\star}$ for different values of V. From the numerical outcomes in Table 4, we can recognize that our suggested methods are more effective than NM, SORLaopt and Picard methods under certain conditions from the point of view of the 'Iter' and 'Time'. Consequently, we conclude that our novel methods are effective and feasible for AVEs.

Methods	n	1000	2000	3000	4000	5000
NM	Iter	17	18	18	18	18
	Time	1.9831	10.5160	28.6587	63.6419	117.3205
	RES	7.38e − 09	2.60e − 09	3.19e − 09	3.68e − 09	4.11e − 09
	ERR	5.21e − 12	1.50e − 12	1.59e − 12	1.64e − 12	1.68e − 12
SORLaopt	Iter	15	15	15	15	15
	Time	1.9281	9.1987	25.5426	56.0288	102.4080
	RES	1.99e − 09	3.62e − 09	7.58e − 09	3.68e − 09	9.88e − 09
	ERR	1.33e − 12	1.79e − 12	2.14e − 12	2.43e − 12	2.69e − 12
Picard	Iter	5	5	5	5	5
	Time	0.7741	3.1126	8.5872	17.8581	32.0921
	RES	1.34e − 11	1.68e − 11	2.38e − 11	3.73e − 11	3.13e − 11
	ERR	1.12e − 14	1.18e − 14	1.25e − 14	1.31e − 14	1.37e − 04
Method I	Iter	4	4	4	4	4
	Time	0.3736	2.9140	6.6380	15.3359	28.7372
	RES	6.67e − 11	6.69e − 11	6.70e − 11	6.78e − 11	6.85e − 11
	ERR	5.90e − 14	5.91e − 14	5.91e − 14	5.91e − 14	5.91e − 14
Method II	Iter	5	5	5	5	5
	Time	0.3391	2.8898	7.0875	16.1323	30.3901
	RES	6.70e − 11	6.72e − 11	6.73e − 11	6.98e − 11	6.73e − 11
	ERR	5.91e − 14	5.92e − 14	5.92e − 14	5.92e − 14	5.93e − 14

Table 3. Numerical results of Example 3.3 with $\psi = \lambda = 0.98$.

4. Conclusion

We have presented two new iteration methods for solving the AVE in (1.1) and showed that the proposed methods converge to the AVE solution in (1.1) under suitable choices of the involved parameters. Lastly, numerical tests were also implemented so as to check the effectiveness of the proposed methods. The theoretical analysis and numerical tests have shown that the two algorithms seem promising for solving the AVEs.

Methods	V	256	1296	2401	4096
NM	Iter	13	13	13	14
	Time	1.6492	3.1458	14.5518	82.6151
	RES	2.51e − 09	5.75e − 09	7.86e − 09	1.58e − 09
	ERR	2.10e − 10	4.80e − 10	6.56e − 10	1.32e − 10
SORLaopt	Iter	17	18	18	18
	Time	0.3707	4.5734	22.7075	117.6810
	RES	6.80e − 09	3.73e − 09	5.07e − 09	6.61e − 09
	ERR	5.70e − 10	3.12e − 10	4.23e − 10	5.52e − 10
Picard	Iter	10	10	10	10
	Time	0.3862	2.5027	11.3991	58.7786
	RES	1.46e − 09	3.20e − 09	4.33e − 09	5.64e − 09
	ERR	1.23e − 10	2.68e − 10	3.62e − 10	4.71e − 10
Method I	Iter	9	9	9	9
	Time	0.2378	1.2979	11.0035	55.6473
	RES	9.41e − 09	1.26e − 09	1.38e − 09	1.49e − 09
	ERR	8.79e − 10	1.19e − 10	1.29e − 10	1.40e − 10
Method II	Iter	9	9	9	9
	Time	0.2569	2.2081	11.9317	57.3148
	RES	4.50e − 09	5.74e − 09	6.31e − 09	6.87e − 09
	ERR	4.23e − 10	5.38e − 10	5.89e − 10	6.40e − 10

Table 4. Numerical results of Example 3.4 with $\psi = 0.9$ and $\lambda = 1$.

This paper successfully examined the two new iteration methods for solving AVE when the coefficient matrix is an M-matrix or strictly diagonally dominant matrix. The cases for more general coefficient matrices are the next issue to be considered.

Appendix

In this appendix, we explain how to implement the proposed methods.

Method I.

$$x^{i+1} = x^i - E[-\psi\overline{K}x^{i+1} + (\psi(2-\psi)A + \psi\overline{K})x^i - \psi(2-\psi)(|x^i| + b)].$$

Method II.

$$x^{i+1} = \lambda[x^i - E((Ax^i - |x^i| - b) - \overline{K}(x^{i+1} - x^i))] + (1-\lambda)x^i.$$

In both methods, the right-hand side also contains x^{i+1}, which is the unknown. From $Ax - |x| = b$ we have

$$x = A^{-1}(|x| + b).$$

Thus, we can approximate x^{i+1} as

$$x^{i+1} \approx A^{-1}(|x^i| + b).$$

The above method is called the Picard iteration method [39]. Now, we discuss the algorithm of Method I,

Algorithm for Method I.

$$\begin{gathered} y^i = x^{i+1} = A^{-1}(|x^i| + b), \\ x^{i+1} = x^i - E[-\psi\overline{K}y^i + (\psi(2-\psi)A + \psi\overline{K})x^i - \psi(2-\psi)(|x^i| + b)]. \end{gathered}$$

Similarly for Method II.

In addition, Mao et al. [29] and Dehgan and Hajrian [8] used parameter ψ for LCPs in this way. In this article, we apply this idea to AVEs and speed up the convergence of the proposed iteration method by using ψ in this way.

A c k n o w l e d g m e n t s. The authors are grateful to the editors and the anonymous referees for their helpful comments and suggestions.

References

[1] *L. Abdallah, M. Haddou, T. Migot*: Solving absolute value equation using complementarity and smoothing functions. J. Comput. Appl. Math. *327* (2018), 196–207. zbl MR doi
[2] *B. H. Ahn*: Solution of nonsymmetric, linear complementarity problems by iterative methods. J. Optim. Theory Appl. *33* (1981), 175–187. zbl MR doi
[3] *Z.-Z. Bai*: Modulus-based matrix splitting iteration methods for linear complementarity problems. Numer. Linear Algebra Appl. *17* (2010), 917–933. zbl MR doi
[4] *J. Y. Bello Cruz, O. P. Ferreira, L. F. Prudente*: On the global convergence of the inexact semi-smooth Newton method for absolute value equation. Comput. Optim. Appl. *65* (2016), 93–108. zbl MR doi
[5] *L. Caccetta, B. Qu, G. Zhou*: A globally and quadratically convergent method for absolute value equations. Comput. Optim. Appl. *48* (2011), 45–58. zbl MR doi
[6] *C. Chen, D. Yu, D. Han*: Optimal parameter for the SOR-like iteration method for solving the system of absolute value equations. Available at `https://arxiv.org/abs/2001.05781` (2021), 23 pages.
[7] *R. W. Cottle, J.-S. Pang, R. E. Stone*: The Linear Complementarity Problem. Classics in Applied Mathematics 60. SIAM, Philadelphia, 2009. zbl MR doi
[8] *M. Dehghan, M. Hajarian*: Convergence of SSOR methods for linear complementarity problems. Oper. Res. Lett. *37* (2009), 219–223. zbl MR doi
[9] *M. Dehghan, A. Shirilord*: Matrix multisplitting Picard-iterative method for solving generalized absolute value matrix equation. Appl. Numer. Math. *158* (2020), 425–438. zbl MR doi
[10] *X. Dong, X.-H. Shao, H.-L. Shen*: A new SOR-like method for solving absolute value equations. Appl. Numer. Math. *156* (2020), 410–421. zbl MR doi

[11] *V. Edalatpour, D. Hezari, D. Khojasteh Salkuyeh*: A generalization of the Gauss-Seidel iteration method for solving absolute value equations. Appl. Math. Comput. *293* (2017), 156–167. zbl MR doi

[12] *J. Feng, S. Liu*: An improved generalized Newton method for absolute value equations. SpringerPlus *5* (2016), Article ID 1042, 10 pages. doi

[13] *J. Feng, S. Liu*: A new two-step iterative method for solving absolute value equations. J. Inequal. Appl. *39* (2019), Article ID 39, 8 pages. MR doi

[14] *X.-M. Gu, T.-Z. Huang, H.-B. Li, S.-F. Wang, L. Li*: Two CSCS-based iteration methods for solving absolute value equations. J. Appl. Anal. Comput. *7* (2017), 1336–1356. zbl MR doi

[15] *F. K. Haghani*: On generalized Traub's method for absolute value equations. J. Optim. Theory Appl. *166* (2015), 619–625. zbl MR doi

[16] *F. Hashemi, S. Ketabchi*: Numerical comparisons of smoothing functions for optimal correction of an infeasible system of absolute value equations. Numer. Algebra Control Optim. *10* (2020), 13–21. zbl MR doi

[17] *S.-L. Hu, Z.-H. Huang*: A note on absolute value equations. Optim. Lett. *4* (2010), 417–424. zbl MR doi

[18] *Y. Ke*: The new iteration algorithm for absolute value equation. Appl. Math. Lett. *99* (2020), Article ID 105990, 7 pages. zbl MR doi

[19] *Y.-F. Ke, C.-F. Ma*: SOR-like iteration method for solving absolute value equations. Appl. Math. Comput. *311* (2017), 195–202. zbl MR doi

[20] *C.-X. Li*: A preconditioned AOR iterative method for the absolute value equations. Int. J. Comput. Methods *14* (2017), Article ID 1750016, 12 pages. zbl MR doi

[21] *S.-G. Li, H. Jiang, L.-Z. Cheng, X.-K. Liao*: IGAOR and multisplitting IGAOR methods for linear complementarity problems. J. Comput. Appl. Math. *235* (2011), 2904–2912. zbl MR doi

[22] *O. L. Mangasarian*: Solution of symmetric linear complementarity problems by iterative methods. J. Optim. Theory Appl. *22* (1977), 465–485. zbl MR doi

[23] *O. L. Mangasarian*: Absolute value equation solution via concave minimization. Optim. Lett. *1* (2007), 3–8. zbl MR doi

[24] *O. L. Mangasarian*: A generalized Newton method for absolute value equations. Optim. Lett. *3* (2009), 101–108. zbl MR doi

[25] *O. L. Mangasarian*: Linear complementarity as absolute value equation solution. Optim. Lett. *8* (2014), 1529–1534. zbl MR doi

[26] *O. L. Mangasarian, R. R. Meyer*: Absolute value equations. Linear Algebra Appl. *419* (2006), 359–367. zbl MR doi

[27] *A. Mansoori, M. Erfanian*: A dynamic model to solve the absolute value equations. J. Comput. Appl. Math. *333* (2018), 28–35. zbl MR doi

[28] *A. Mansoori, M. Eshaghnezhad, S. Effati*: An efficient neural network model for solving the absolute value equations. IEEE Trans. Circuits Syst., II Exp. Briefs *65* (2017), 391–395. doi

[29] *X. Mao, X. Wangi, S. A. Edalatpanah, M. Fallah*: The monomial preconditioned SSOR method for linear complementarity problem. IEEE Access *7* (2019), 73649–73655. doi

[30] *F. Mezzadri*: On the solution of general absolute value equations. Appl. Math. Lett. *107* (2020), Article ID 106462, 6 pages. zbl MR doi

[31] *F. Mezzadri, E. Galligani*: Modulus-based matrix splitting methods for horizontal linear complementarity problems. Numer. Algorithms *83* (2020), 201–219. zbl MR doi

[32] *X.-H. Miao, J.-T. Yang, B. Saheya, J.-S. Chen*: A smoothing Newton method for absolute value equation associated with second-order cone. Appl. Numer. Math. *120* (2017), 82–96. zbl MR doi

[33] *S.-X. Miao, D. Zhang*: On the preconditioned GAOR method for a linear complementarity problem with an M-matrix. J. Inequal. Appl. *2018* (2018), Article ID 195, 12 pages. MR doi

[34] *H. Moosaei, S. Ketabchi, H. Jafari*: Minimum norm solution of the absolute value equations via simulated annealing algorithm. Afr. Mat. *26* (2015), 1221–1228. zbl MR doi

[35] *C. T. Nguyen, B. Saheya, Y.-L. Chang, J.-S. Chen*: Unified smoothing functions for absolute value equation associated with second-order cone. Appl. Numer. Math. *135* (2019), 206–227. zbl MR doi

[36] *M. A. Noor, K. I. Noor, S. Batool*: On generalized absolute value equations. Sci. Bull., Ser. A, Appl. Math. Phys., Politeh. Univ. Buchar. *80* (2018), 63–70. zbl MR

[37] *O. Prokopyev*: On equivalent reformulations for absolute value equations. Comput. Optim. Appl. *44* (2009), 363–372. zbl MR doi

[38] *J. Rohn*: A theorem of the alternatives for the equation $Ax+B|x|=b$. Linear Multilinear Algebra *52* (2004), 421–426. zbl MR doi

[39] *J. Rohn, V. Hooshyarbakhsh, R. Farhadsefat*: An iterative method for solving absolute value equations and sufficient conditions for unique solvability. Optim. Lett. *8* (2014), 35–44. zbl MR doi

[40] *B. Saheya, C.-H. Yu, J.-S. Chen*: Numerical comparisons based on four smoothing functions for absolute value equation. J. Appl. Math. Comput. *56* (2018), 131–149. zbl MR doi

[41] *D. K. Salkuyeh*: The Picard-HSS iteration method for absolute value equations. Optim. Lett. *8* (2014), 2191–2202. zbl MR doi

[42] *R. S. Varga*: Matrix Iterative Analysis. Prentice-Hall Series in Automatic Computation. Prentice-Hall, Englewood Cliffs, 1962. zbl MR doi

[43] *H. J. Wang, D. X. Cao, H. Liu, L. Qiu*: Numerical validation for systems of absolute value equations. Calcolo *54* (2017), 669–683. zbl MR doi

[44] *S. L. Wu, C. X. Li*: A special shift splitting iteration method for absolute value equation. AIMS Math. *5* (2020), 5171–5183. MR doi

[45] *M. Zamani, M. Hladík*: A new concave minimization algorithm for the absolute value equation solution. Optim. Lett. *15* (2021), 2141–2154. zbl MR doi

[46] *M. Zhang, Z.-H. Huang, Y.-F. Li*: The sparsest solution to the system of absolute value equations. J. Oper. Res. Soc. China *3* (2015), 31–51. zbl MR doi

Authors' address: *Rashid Ali* (corresponding author), *Kejia Pan*, School of Mathematics and Statistics, HNP-LAMA, Central South University, Changsha 410083, P. R. China, e-mail: `rashidali0887@gmail.com`, `kejiapan@csu.edu.cn`.