A LOGARITHMIC REGULARITY CRITERION FOR 3D NAVIER-STOKES SYSTEM IN A BOUNDED DOMAIN

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Abstract. This paper proves a logarithmic regularity criterion for 3D Navier-Stokes system in a bounded domain with the Navier-type boundary condition.

Keywords: regularity criterion; Navier-Stokes system; bounded domain

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1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded simply connected domain with smooth boundary $\partial\Omega$, and n be the unit outward normal vector to $\partial\Omega$. We consider the regularity criterion of the Navier-Stokes system:

$$\operatorname{div} u = 0, \tag{1.1}$$

$$\partial_t u + u \cdot \nabla u + \nabla \pi = \Delta u \quad \text{in } \Omega \times (0,\infty), \tag{1.2}$$

$$u \cdot n = 0,\ \operatorname{rot} u \times n = 0 \quad \text{on } \partial\Omega \times (0,\infty), \tag{1.3}$$

$$u(\cdot,0) = u_0 \quad \text{in } \Omega \subset \mathbb{R}^3. \tag{1.4}$$

Here u and π denote the velocity field vector and pressure scalar of the fluid, respectively. We will denote the vorticity $\omega := \operatorname{rot} u$.

It is well-known that the problem has at least a global-in-time weak solution and a unique local-in-time strong solution [10], [13]. However, the regularity of weak solutions is still a very challenging open problem. On the other hand, the

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development of regularity criterion is of great importance for both theoretical and practical purpose. Giga [8], Berselli [6], Kim [12], and He, Ma and Wang [9] proved some regularity criteria. Very recently, Nakao and Taniuchi [16], [17] showed the regularity criterion

$$\int_0^T \frac{\|\omega\|_{L^\infty}}{\log(\mathrm{e}+\|\omega\|_{L^\infty})}\,\mathrm{d}t < \infty \tag{1.5}$$

with the no-slip boundary condition or (1.3). The aim of this note is to refine it as

$$\int_0^T \frac{\|\omega\|_{\mathrm{BMO}}}{\log(\mathrm{e}+\|\omega\|_{\mathrm{BMO}})}\,\mathrm{d}t < \infty. \tag{1.6}$$

Here BMO is the space of bounded mean oscillation whose norm is defined by

$$\|f\|_{\mathrm{BMO}} := \|f\|_{L^2} + [f]_{\mathrm{BMO}}$$

with

$$[f]_{\mathrm{BMO}} := \sup_{\substack{x\in\Omega\\ r\in(0,d)}} \frac{1}{|\Omega_r(x)|}\int_{\Omega_r(x)} |f(y) - f_{\Omega_r(x)}|\,\mathrm{d}y,$$

$$f_{\Omega_r(x)} := \frac{1}{|\Omega_r(x)|}\int_{\Omega_r(x)} f(y)\,\mathrm{d}y,$$

where $\Omega_r(x) := B_r(x)\cap\Omega$, where $B_r(x)$ is the ball with center x and radius r and d is the diameter of Ω. The symbol $|\Omega_r(x)|$ denotes the Lebesgue measure of $\Omega_r(x)$.

We will prove the following theorem.

Theorem 1.1. *Let $u_0 \in H_0^1 \cap H^3$ and $\operatorname{div} u_0 = 0$ in Ω. Let u be a local smooth solution to problem* (1.1)–(1.4). *If* (1.6) *holds with $0 < T < \infty$, then the solution u can be extended beyond $T > 0$.*

2. Preliminaries

In this section we will collect some lemmas which will be used in the proof.

Lemma 2.1 (Poincaré inequality)**.** *Let Ω be a bounded simple connected domain with smooth boundary and w be a smooth vector satisfying $w\cdot n = 0$ on the boundary $\partial\Omega$. Then*

$$\|w\|_{L^p} \leqslant C\|\nabla w\|_{L^p} \tag{2.1}$$

holds for $2 \leqslant p < \infty$.

P r o o f. If $p = 2$, then the proof was given in Lions [14], (6.47) on page 75. We assume $2 < p < \infty$. Using the Gagliardo-Nirenberg inequality and the case $p = 2$, we see that

$$\begin{aligned}\|w\|_{L^p} &\leqslant C\|w\|_{L^2}^{1-\theta}\|\nabla w\|_{L^p}^{\theta} + C\|w\|_{L^2}\\ &\leqslant C\|\nabla w\|_{L^2}^{1-\theta}\|\nabla w\|_{L^p}^{\theta} + C\|\nabla w\|_{L^2}\\ &\leqslant C\|\nabla w\|_{L^p}^{1-\theta}\|\nabla w\|_{L^p}^{\theta} + C\|\nabla w\|_{L^p} \leqslant C\|\nabla w\|_{L^p}.\end{aligned}$$

This completes the proof. □

Lemma 2.2 ([19]). *For any smooth vector w satisfying $w \cdot n = 0$ or $w \times n = 0$ on $\partial\Omega$ and $1 < p < \infty$*

$$\|\nabla w\|_{L^p} \leqslant C(\|\operatorname{div} w\|_{L^p} + \|\operatorname{rot} w\|_{L^p}). \tag{2.2}$$

Lemma 2.3 ([4]). *For any smooth vector f and $1 < p < \infty$*

$$\begin{aligned}-\int_\Omega \Delta f \cdot f|f|^{p-2}\,\mathrm{d}x = \int_\Omega |f|^{p-2}|\nabla f|^2\,\mathrm{d}x + 4\frac{p-2}{p^2}\int_\Omega |\nabla|f|^{p/2}|^2\,\mathrm{d}x\\ - \int_{\partial\Omega} |f|^{p-2}(n\cdot\nabla)f\cdot f\,\mathrm{d}S.\end{aligned} \tag{2.3}$$

Lemma 2.4 ([3], Lemma 2.2). *Assume that u is sufficiently smooth and satisfies boundary condition (1.3) on $\partial\Omega$. Then the following identity for $\omega := \operatorname{rot} u$ holds on $\partial\Omega$*

$$-\frac{\partial\omega}{\partial n}\cdot\omega = (\varepsilon_{1jk}\varepsilon_{1\beta\gamma} + \varepsilon_{2jk}\varepsilon_{2\beta\gamma} + \varepsilon_{3jk}\varepsilon_{3\beta\gamma})\omega_j\omega_\beta\partial_k n_\gamma, \tag{2.4}$$

where ε_{ijk} denotes the totally anti-symmetric tensor such that $(a\times b)_i = \varepsilon_{ijk}a_jb_k$.

Lemma 2.5 ([1], Lemma 7.44, and [15], Corollary 1.7). *For any smooth f and $1 < p < \infty$*

$$\|f\|_{L^p(\partial\Omega)} \leqslant C\|f\|_{L^p(\Omega)}^{1-1/p}\|f\|_{W^{1,p}(\Omega)}^{1/p}. \tag{2.5}$$

P r o o f. We have

$$\|f\|_{L^p(\partial\Omega)} \leqslant C\|f\|_{W^{1/p,p}(\Omega)} \leqslant C\|f\|_{L^p(\Omega)}^{1-1/p}\|f\|_{W^{1,p}(\Omega)}^{1/p}.$$

□

Lemma 2.6 ([2]). *For any smooth f and $1 \leqslant q < p < \infty$*

$$\|f\|_{L^p(\Omega)} \leqslant C\|f\|_{L^q(\Omega)}^{q/p}\|f\|_{\mathrm{BMO}(\Omega)}^{1-q/p}. \tag{2.6}$$

Lemma 2.7 ([18]). *Let $1 \leqslant q, r \leqslant \infty$ and $0 \leqslant j < m$. Then the inequality*

$$\|D^j u\|_{L^p} \leqslant C_1\|D^m u\|_{L^r}^a\|u\|_{L^q}^{1-a} + C_2\|u\|_{L^s} \tag{2.7}$$

holds for any function $u\colon \Omega \to \mathbb{R}$ defined on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^N$, where

$$\frac{1}{p} = \frac{j}{N} + a\Big(\frac{1}{r} - \frac{m}{N}\Big) + (1-a)\frac{1}{q}, \quad \frac{i}{m} \leqslant a \leqslant 1, \tag{2.8}$$

$s > 0$ is arbitrary and C_1 and C_2 depend only on Ω, m and N.

Lemma 2.8 ([11]). *Let b be a solution to the Poisson equation*

$$-\Delta b = f \quad \text{in } \Omega$$

with boundary condition

$$b \cdot n = 0, \quad \operatorname{rot} b \times n = 0 \quad \text{on } \partial\Omega.$$

Then

$$\|b\|_{H^2} \leqslant C\|f\|_{L^2} + C\|\nabla b\|_{L^2}. \tag{2.9}$$

Lemma 2.9. *Let u be a solution to the Stokes system*

$$-\Delta u + \nabla\pi = f \quad \text{and} \quad \operatorname{div} u = 0 \quad \text{in } \Omega \tag{2.10}$$

with boundary condition

$$u \cdot n = 0, \quad \operatorname{rot} u \times n = 0 \quad \text{on } \partial\Omega. \tag{2.11}$$

Then

$$\|u\|_{H^2} \leqslant C\|f\|_{L^2}. \tag{2.12}$$

P r o o f. First, using Lemma 2.8 we have

$$\|u\|_{H^2} \leqslant C\|f-\nabla\pi\|_{L^2} + C\|\nabla u\|_{L^2} \leqslant C\|f\|_{L^2} + C\|\nabla\pi\|_{L^2} + C\|\nabla u\|_{L^2}. \tag{2.13}$$

Testing (2.10) by u and using $\operatorname{div} u = 0$, we get

$$\|\operatorname{rot} u\|_{L^2}^2 = \int_\Omega f u \,\mathrm{d}x \leqslant \|f\|_{L^2}\|u\|_{L^2} \leqslant C\|f\|_{L^2}\|\operatorname{rot} u\|_{L^2},$$

which gives

$$\|\nabla u\|_{L^2} \leqslant C\|\operatorname{rot} u\|_{L^2} \leqslant C\|f\|_{L^2}. \tag{2.14}$$

Boundary condition

$$\operatorname{rot} u \times n = 0$$

leads to [5]

$$n \cdot \Delta u = n \cdot \operatorname{rot}^2 u = 0,$$

and thus

$$\frac{\partial \pi}{\partial n} = f \cdot n \quad \text{on } \partial\Omega. \tag{2.15}$$

Taking div in (2.10), we observe that

$$\Delta\pi = \operatorname{div} f \quad \text{in } \Omega. \tag{2.16}$$

Testing (2.16) by $(-\pi)$ and using (2.15), we have

$$\|\nabla\pi\|_{L^2}^2 = \int_\Omega f \cdot \nabla\pi \,\mathrm{d}x \leqslant \|f\|_{L^2}\|\nabla\pi\|_{L^2}$$

which leads to

$$\|\nabla\pi\|_{L^2} \leqslant \|f\|_{L^2}. \tag{2.17}$$

Inequalities (2.13), (2.14), and (2.17) imply that (2.12) holds true.
This completes the proof. □

Lemma 2.10 ([5], [7])**.** *Let s be a non-negative real number. If u belonging to $H^2(\Omega)$ is such that $\Delta u \in H^s(\Omega)$ and such that*

$$\operatorname{div} u = 0 \quad \textit{in } \Omega \quad \textit{and} \quad u \times n = 0 \quad \textit{on } \partial\Omega,$$
$$\textit{or such that } u \cdot n = 0,\ \operatorname{rot} u \times n = 0 \textit{ on } \partial\Omega,$$

then

$$\|u\|_{H^{s+2}(\Omega)} \leqslant C(\|\Delta u\|_{H^s(\Omega)} + \|u\|_{L^2(\Omega)}). \tag{2.18}$$

Using Lemma 2.10, we have

$$\begin{aligned}\|u\|_{H^3} &\leqslant C(\|\Delta u\|_{H^1} + \|u\|_{L^2}) = C(\|\operatorname{rot}\omega\|_{H^1} + \|u\|_{L^2})\\ &\leqslant C(\|\omega\|_{H^2} + \|u\|_{L^2}) \leqslant C(\|\Delta\omega\|_{L^2} + \|\omega\|_{L^2} + \|u\|_{L^2}),\end{aligned}$$

which will be used in proving (3.10).

3. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. Since it is easy to show the well-posedness of local strong solutions, we omit the details here and we only need to establish some a priori estimates.

First, testing (1.2) by u and using (1.1) and (1.3), we find that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |u|^2\,\mathrm{d}x + \int |\operatorname{rot} u|^2\,\mathrm{d}x = 0,$$

which gives

$$\|u\|_{L^\infty(0,T;L^2)} + \|u\|_{L^2(0,T;H^1)} \leqslant C. \tag{3.1}$$

Taking rot to (1.2) and using (1.1), we get the well-known equation

$$\partial_t\omega + u\cdot\nabla\omega - \omega\cdot\nabla u - \Delta\omega = 0. \tag{3.2}$$

Testing the above equation by $|\omega|^{p-2}\omega$ $(2 \leqslant p < \infty)$ and using (1.1), (2.3), (2.4), (2.5), and (2.6), we obtain

$$\begin{aligned}&\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\omega|^p\,\mathrm{d}x + \int_\Omega |\omega|^{p-2}|\nabla\omega|^2\,\mathrm{d}x + 4\frac{p-2}{p^2}\int_\Omega |\nabla|\omega|^{p/2}|^2\,\mathrm{d}x\\ &\quad= \int_{\partial\Omega} |\omega|^{p-2}(n\cdot\nabla)\omega\cdot\omega\,\mathrm{d}S + \int_\Omega (\omega\cdot\nabla)u\cdot|\omega|^{p-2}\omega\,\mathrm{d}x\\ &\quad= -\int_{\partial\Omega} |\omega|^{p-2}\sum_i \varepsilon_{ijk}\varepsilon_{i\beta\gamma}\omega_j\omega_\beta\partial_k n_\gamma\,\mathrm{d}S + \int_\Omega (\omega\cdot\nabla)u\cdot|\omega|^{p-2}\omega\,\mathrm{d}x\\ &\quad\leqslant C\int_{\partial\Omega} |\omega|^p\,\mathrm{d}S + \|\omega\|^p_{L^{p+1}}\|\nabla u\|_{L^{p+1}}\\ &\quad\leqslant C\int_{\partial\Omega} f^2\,\mathrm{d}S + C\|\omega\|^{p+1}_{L^{p+1}} \quad (f := |\omega|^{p/2})\\ &\quad\leqslant C\|f\|_{L^2(\Omega)}\|f\|_{H^1(\Omega)} + C\|\omega\|^{p+1}_{L^{p+1}}\\ &\quad\leqslant 2\frac{p-2}{p^2}\int_\Omega |\nabla f|^2\,\mathrm{d}x + C\|\omega\|^p_{L^p} + C\|\omega\|_{\mathrm{BMO}}\|\omega\|^p_{L^p},\end{aligned}$$

which gives

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_{L^p} &\leqslant C\|\omega\|_{L^p}(1+\|\omega\|_{\mathrm{BMO}})\\
&\leqslant C\|\omega\|_{L^p}\log(\mathrm{e}+\|\omega\|_{\mathrm{BMO}})\frac{1+\|\omega\|_{\mathrm{BMO}}}{\log(\mathrm{e}+\|\omega\|_{\mathrm{BMO}})}\\
&\leqslant C\|\omega\|_{L^p}\log(\mathrm{e}+\|u\|_{H^3})\frac{\|\omega\|_{\mathrm{BMO}}}{\log(\mathrm{e}+\|\omega\|_{\mathrm{BMO}})}
\end{aligned}
$$

and therefore,

$$
\int_\Omega |\omega|^p\,\mathrm{d}x \leqslant C(\mathrm{e}+y)^{C_0\varepsilon} \tag{3.3}
$$

provided that

$$
\int_{t_0}^{t} \frac{\|\omega\|_{\mathrm{BMO}}}{\log(\mathrm{e}+\|\omega\|_{\mathrm{BMO}})}\,\mathrm{d}s \leqslant \varepsilon \ll 1 \tag{3.4}
$$

and $y(t) := \sup\limits_{[t_0,t]} \|u\|_{H^3}$ for any $0 < t_0 \leqslant t \leqslant T$ and C_0 is an absolute constant.

Here we have used the estimate

$$
\|\omega(\cdot,t_0)\|_{L^p} \leqslant C
$$

by the standard energy estimate and we omit the details.

Testing (1.2) by $\partial_t u$ and using (1.1) and (3.3), we derive

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\operatorname{rot} u|^2\,\mathrm{d}x + \int_\Omega |\partial_t u|^2\,\mathrm{d}x\\
&= -\int (u\cdot\nabla)u\cdot\partial_t u\,\mathrm{d}x \leqslant \|u\|_{L^6}\|\nabla u\|_{L^3}\|\partial_t u\|_{L^2}\\
&\leqslant C\|\nabla u\|_{L^2}\|\nabla u\|_{L^3}\|\partial_t u\|_{L^2} \leqslant \frac{1}{2}\|\partial_t u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2\|\nabla u\|_{L^3}^2\\
&\leqslant \frac{1}{2}\|\partial_t u\|_{L^2}^2 + C\|\omega\|_{L^3}^4,
\end{aligned}
$$

which implies

$$
\int_{t_0}^{t} \|\partial_t u\|_{L^2}^2\,\mathrm{d}s \leqslant C(\mathrm{e}+y)^{C_0\varepsilon}. \tag{3.5}
$$

Here we have used the facts

$$
\int_\Omega \nabla\pi\cdot\partial_t u\,\mathrm{d}x = 0
$$

and

$$-\int_\Omega \Delta u\cdot\partial_t u\,\mathrm{d}x=\int_\Omega \operatorname{rot}^2 u\cdot\partial_t u\,\mathrm{d}x=\int_\Omega \operatorname{rot} u\cdot\operatorname{rot}\partial_t u\,\mathrm{d}x=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\operatorname{rot} u|^2\,\mathrm{d}x$$

and

$$-\Delta u=\operatorname{rot}^2 u,$$

since

$$\operatorname{div} u=0.$$

Applying ∂_t to (1.2), we have

$$\partial_t^2 u+u\cdot\nabla\partial_t u+\nabla\partial_t\pi-\Delta\partial_t u=-\partial_t u\cdot\nabla u. \tag{3.6}$$

Testing (3.6) by $\partial_t u$ and using (1.1), (3.3), and (3.5), we have

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}&\int_\Omega|\partial_t u|^2\,\mathrm{d}x+\int_\Omega|\operatorname{rot}\partial_t u|^2\,\mathrm{d}x=-\int_\Omega \partial_t u\cdot\nabla u\cdot\partial_t u\,\mathrm{d}x\\
&\leqslant \|\nabla u\|_{L^6}\|\partial_t u\|_{L^2}\|\partial_t u\|_{L^3}\leqslant C\|\omega\|_{L^6}\|\partial_t u\|_{L^2}^{3/2}\|\operatorname{rot}\partial_t u\|_{L^2}^{1/2}\\
&\leqslant \frac{1}{2}\|\operatorname{rot}\partial_t u\|_{L^2}^2+C\|\omega\|_{L^6}^{4/3}\|\partial_t u\|_{L^2}^2,
\end{aligned}$$

which gives

$$\int_\Omega|\partial_t u|^2\,\mathrm{d}x+\int_{t_0}^t\int_\Omega|\operatorname{rot}\partial_t u|^2\,\mathrm{d}x\,\mathrm{d}s\leqslant C(\mathrm{e}+y)^{C_0\varepsilon}. \tag{3.7}$$

Here we have used the fact that

$$-\int_\Omega \Delta\partial_t u\cdot\partial_t u\,\mathrm{d}x=\int_\Omega \operatorname{rot}^2\partial_t u\cdot\partial_t u\,\mathrm{d}x=\int_\Omega|\operatorname{rot}\partial_t u|^2\,\mathrm{d}x.$$

On the other hand, thanks to the H^2-theory of the Stokes system (see Lemma 2.7), it follows from (1.2), (3.3), and (3.7) that

$$\begin{aligned}
\|u\|_{H^2}&\leqslant C\|-\Delta u+\nabla\pi\|_{L^2}\leqslant C\|\partial_t u+u\cdot\nabla u\|_{L^2}\\
&\leqslant C\|\partial_t u\|_{L^2}+C\|u\|_{L^6}\|\nabla u\|_{L^3}\leqslant C\|\partial_t u\|_{L^2}+C\|\nabla u\|_{L^2}\|\nabla u\|_{L^3}\\
&\leqslant C\|\partial_t u\|_{L^2}+C\|\omega\|_{L^2}\|\omega\|_{L^3}\leqslant C(\mathrm{e}+y)^{C_0\varepsilon}.
\end{aligned} \tag{3.8}$$

Testing (3.6) by $-\Delta\partial_t u+\nabla\partial_t\pi$ and using (1.1), (3.7), and (3.8), we have

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}&\int_\Omega|\operatorname{rot}\partial_t u|^2\,\mathrm{d}x+\int_\Omega|-\Delta\partial_t u+\nabla\partial_t\pi|^2\,\mathrm{d}x\\
&=\int(-\partial_t u\cdot\nabla u-u\cdot\nabla\partial_t u)(-\Delta\partial_t u+\nabla\partial_t\pi)\,\mathrm{d}x\\
&\leqslant(\|\nabla u\|_{L^3}\|\partial_t u\|_{L^6}+\|u\|_{L^\infty}\|\nabla\partial_t u\|_{L^2})\|-\Delta\partial_t u+\nabla\partial_t\pi\|_{L^2}\\
&\leqslant C\|u\|_{H^2}\|\operatorname{rot}\partial_t u\|_{L^2}\|-\Delta\partial_t u+\nabla\partial_t\pi\|_{L^2}\\
&\leqslant\frac{1}{2}\|-\Delta\partial_t u+\nabla\partial_t\pi\|_{L^2}^2+C\|u\|_{H^2}^2\|\operatorname{rot}\partial_t u\|_{L^2}^2,
\end{aligned}$$

which leads to

$$\int_\Omega |\mathrm{rot}\,\partial_t u|^2\,\mathrm{d}x + \int_{t_0}^t \|\partial_t u\|_{H^2}^2\,\mathrm{d}s \leqslant C(\mathrm{e}+y)^{C_0\varepsilon}. \tag{3.9}$$

Here we have used the fact that

$$\begin{aligned}\int_\Omega \partial_t^2 u(-\Delta\partial_t u + \nabla\partial_t\pi)\,\mathrm{d}x &= \int_\Omega \partial_t^2 u(\mathrm{rot}^2\,\partial_t u + \nabla\partial_t\pi)\,\mathrm{d}x\\ = \int_\Omega \partial_t^2 u\cdot\mathrm{rot}^2\,\partial_t u\,\mathrm{d}x &= \int_\Omega \mathrm{rot}\,\partial_t^2 u\cdot\mathrm{rot}\,\partial_t u\,\mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\mathrm{rot}\,\partial_t u|^2\,\mathrm{d}x\end{aligned}$$

due to

$$\int_\Omega \partial_t^2 u\cdot\nabla\partial_t\pi\,\mathrm{d}x = 0.$$

Here we also have used the fact that

$$\|\mathrm{rot}\,\partial_t u(\cdot,t_0)\|_{L^2} \leqslant C$$

by the standard energy method and we omit the details.

On the other hand, it follows from (2.18), (3.2), (3.3), (3.8), and (3.9) that

$$\begin{aligned}\|u\|_{H^3} &\leqslant C(\|u\|_{L^2} + \|\omega\|_{L^2} + \|\Delta\omega\|_{L^2})\\ &\leqslant C(1 + \|\partial_t\omega + u\cdot\nabla\omega - \omega\cdot\nabla u\|_{L^2})\\ &\leqslant C + C\|\partial_t\omega\|_{L^2} + C\|u\|_{L^\infty}\|\nabla\omega\|_{L^2} + C\|\omega\|_{L^4}\|\nabla u\|_{L^4}\\ &\leqslant C + C\|\partial_t\omega\|_{L^2} + C\|u\|_{H^2}^2\\ &\leqslant C(\mathrm{e}+y)^{C_0\varepsilon},\end{aligned}$$

which gives

$$\|u\|_{L^\infty(0,T;H^3)} \leqslant C. \tag{3.10}$$

This completes the proof. □

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