



Shear resistance analytical evaluation for RC beams with transverse reinforcement with two different inclinations

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Abstract An analysis-oriented mechanical model for shear strength evaluation of Reinforced Concrete (RC) beams with transverse reinforcement with two different inclinations, which required a numerical analysis, is turned into a design-oriented analytical model that can easily be utilized for practical purposes. The model assessed the shear resistance, according to the “lower-bound solution”, employing a numerical procedure that maximizes the element shear strength varying the stresses in the two sets of transverse reinforcement and the magnitude and inclination of the web concrete compressive stress field. The model is formulated with the aim of representing an extension of Eurocode 2 framework to RC beams with two orders of stirrups. In this paper, an analytical procedure is derived, substituting the former numerical maximization procedure, in order to

obtain the optimal values of the aforementioned parameters, for any layout and amount of shear reinforcement. Comparison between shear strength predictions provided by the model and test results available in the literature confirms the model’s efficiency.

Keywords Shear strength · Design-oriented analytical model · Different inclined stirrups · Variable inclination of compressive stress field

List of symbols

a	Shear span
b_w	Cross-section minimum web width
d	Cross-section depth
f_c	Compressive strength of concrete
f_{cd}	Design compressive strength of concrete
f'_{cd}	Design reduced compressive strength of concrete
f_{yd}	Design tensile strength of steel
s_{tw1}	Spacing of the first order of transverse reinforcement
s_{tw2}	Spacing of the second order of transverse reinforcement
v	Non-dimensional shear strength
x_c	Neutral axis
z	Internal lever arm, equal to $0.9 d$
A'_s	Cross-sectional area of the top longitudinal reinforcement

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A_s	Cross-sectional area of the bottom longitudinal reinforcement
A_{tw1}	Cross-sectional area of the first order of transverse reinforcement
A_{tw2}	Cross-sectional area of the second order of transverse reinforcement
1	Angle of inclination, with respect to the beam axis, of the first order of transverse reinforcement
2	Angle of inclination, with respect to the beam axis, of the second order of transverse reinforcement
	Slope of the web concrete stress field
ν'	Coefficient to be applied to the compressive strength of concrete f_c to take into account the biaxial stress state
ξ	Non-dimensional neutral axis depth, equal to x_c/z
$\tilde{\sigma}_{cw}$	Non-dimensional stress of the web concrete
$\tilde{\sigma}_{lw}$	Non-dimensional stress of the web longitudinal reinforcement
$\tilde{\sigma}_{tw1}$	Non-dimensional stress of the first order of transverse reinforcement
$\tilde{\sigma}_{tw2}$	Non-dimensional stress of the second order of transverse reinforcement
ω_{lw}	Mechanical ratio of the web longitudinal reinforcement
ω'_s	Mechanical ratio of the top longitudinal reinforcement
ω_s	Mechanical ratio of the bottom longitudinal reinforcement
ω_{tw1}	Mechanical ratio of the first order of transverse reinforcement
ω_{tw2}	Mechanical ratio of the second order of transverse reinforcement

1 Introduction

Shear failure in RC elements is one of the most undesirable modes of failure due to its rapid progression. Diagonal cracks are the warning signs of incipient shear failure. Usually, the inclined shear cracks start at the middle height of the beam or at the location of the longitudinal reinforcement, and extend towards the compression zone. In order to prevent shear cracking or reduce its width, transverse

reinforcement has to be provided. Since the principal tensile stresses act in an inclined direction, the most effective configuration is obtained when the shear reinforcement is inclined along the direction of the principal tensile force. However, in order to control shear cracking and to provide adequate beam shear strength, stirrups are the most commonly used shear reinforcement, for their simplicity in fabrication and installation. Stirrups are spaced closely at the beam end, aiming to provide both strength in the high shear region, and concrete confinement in the zone of possible plastic hinge activation. However, reinforcement congestion near the support of the RC beams due to the presence of a large amount of longitudinal reinforcement and closely spaced stirrups increases the cost and time required for installation, and calls for the study of other alternatives.

In this regard, many of the innovative solutions, as well as the construction practice of the past, used inclined transverse reinforcement [1–15]. One of the most interesting attempts, characterized by the presence of inclined transverse reinforcement, consists in the use of continuous spirals [1–4]. It is well known that RC elements with rectangular or circular cross-section reinforced using continuous spirals show an enhanced response both in terms of strength and ductility if compared to members provided with normal stirrups [1–4]. Shear capacity of RC elements having continuous spirals as transverse reinforcement has been found to be higher than that provided by beams with traditional shear reinforcement [5–7]. Among the several transverse reinforcement layouts developed over the last years that use inclined bars, the swimmer bar system is described in detail in [8]. It is constituted by inclined bars having both ends bent parallel to beam axis and anchored to top and bottom flexural reinforcement using welds or bolts, obtaining inclined shear reinforcing bars in addition to or to replace the classic vertical stirrups, similar to the bent-up bars used until the seventies in RC frames. In old RC building construction practice, shear cracking and shear strength were controlled by adding bent reinforcing bars to the traditional stirrups. Where all the tensile reinforcement at the top chord was not needed to carry the bending moment leaving the beam-to-column connection zone, some of the tensile bars were bent-down in the high shear region to form the inclined legs of shear reinforcement. This practice was also extended to slabs due to the efficiency of bent bars



both as shear reinforcement and as integrity reinforcement, as reported by Tassinari et al. [9], Mohamadyan-Yasouj et al. [10], and Saravanakumar and Govindaraj [11], performing shear tests on slender and wide beams, having several typologies of transverse reinforcement, demonstrate that specimens with both vertical and inclined stirrups showed not only an increment in shear capacity, but also stiffer behavior and more gradual failure compared to beams only reinforced with vertical or inclined stirrups.

Currently, a novel strategy for shear reinforcement of RC beams is gaining in popularity, constituted by two orders of transverse reinforcement arranged with two different inclinations. Several structural elements use this configuration, i.e. deep beams typical of bridges, in which transverse reinforcement are constituted by vertical and inclined stirrups. Furthermore, semi-precast Hybrid Steel-Trussed Concrete Beams (HSTCBs), which consist in a factory-made steel truss completed with cast-in situ concrete, adopt a transverse reinforcement arranged with two different inclinations as well [12–15].

In the American code and past European codes, where shear strength was evaluated on the basis of the additive contributions due to concrete and steel reinforcement, the strength of multiple inclinations of reinforcement could easily have been taken into account by adding their contributions. Currently, the European design codes (e.g. [16]) contain no specific provisions for the abovementioned structural cases, and their design can be performed only by adjusting the existing models developed for other structural typologies. Recently, in Colajanni et al. [17] a physical model for evaluation of the shear capacity in beams containing two sets of stirrups with different inclinations is derived. Formulated by means of a suitable modification of a model proposed in previous papers [18, 19], it is a generalization of the classical model currently proposed in Eurocode 2. In both these two models, and in those derived from them [20], evaluation of shear strength is obtained, according to the “lower-bound solution”, by means of a numerical procedure that maximizes the element shear capacity by varying the stress in the two orders of transverse reinforcement and the value and inclination of the web concrete compressive stress field.

Here, the procedure of maximization is analysed, and analytical expressions of the optimal values of the aforementioned parameters are derived, for different

arrangements and amounts of transverse reinforcement. Thus, the analysis model is turned into a design model, and the implication of different layout and amounts of reinforcement is analysed. Moreover, the analytical results prove that the model represents an extension of Eurocode 2 model to RC beams with two orders of stirrups.

2 Mechanical model

Colajanni et al. [17] developed a model able to assess the shear resistance of RC beams having two orders of transverse reinforcement. Here, the assumptions on which the mechanical model was based, are quoted verbatim from [17]: “the model is derived assuming that at the Ultimate Limit State (ULS) the resistant mechanism can be represented (Fig. 1) by:—two chords, the top compressed chord formed by the concrete and its reinforcement, and the bottom tensile one formed by the bottom longitudinal reinforcement and the prestressing reinforcement (if any); and the web, carrying the shear action, formed by the concrete, longitudinal reinforcement (if any), and stirrups. Other assumptions are the following: (1) both the stirrups and the longitudinal web reinforcement (if any) are subjected only to axial force (i.e. dowel action is considered elsewhere, as explained below); (2) compared to the size of the structural members, the spacing of the stirrups and of the web longitudinal bars is so small that their actions can be modelled via different uniform stress fields; (3) the concrete stress field in the web is inclined by the angle θ to the longitudinal axis, which may differ from $\beta \sim 45^\circ$, which is the alignment of the first cracks in a structural member subjected merely to bending and shear (like a beam at the Service Limit State, SLS); the maximum shear capacity is achieved for $\cot \theta$ varying in the range $1 \leq \cot \theta \leq 2.5$; more severe limitations must be imposed in elements where flexural ductility is demanded; (4) the constitutive laws of the materials are consistent with the theory of plasticity; (5) the contributions to the shear capacity of dowel action and aggregate interlock are indirectly taken into account by introducing (through the angle θ) different orientations for the principal directions of the stress fields and the cracks; (6) the contribution due to the tensile strength of the concrete (V_c) is neglected; (7) the arch action, which plays a remarkable role in the

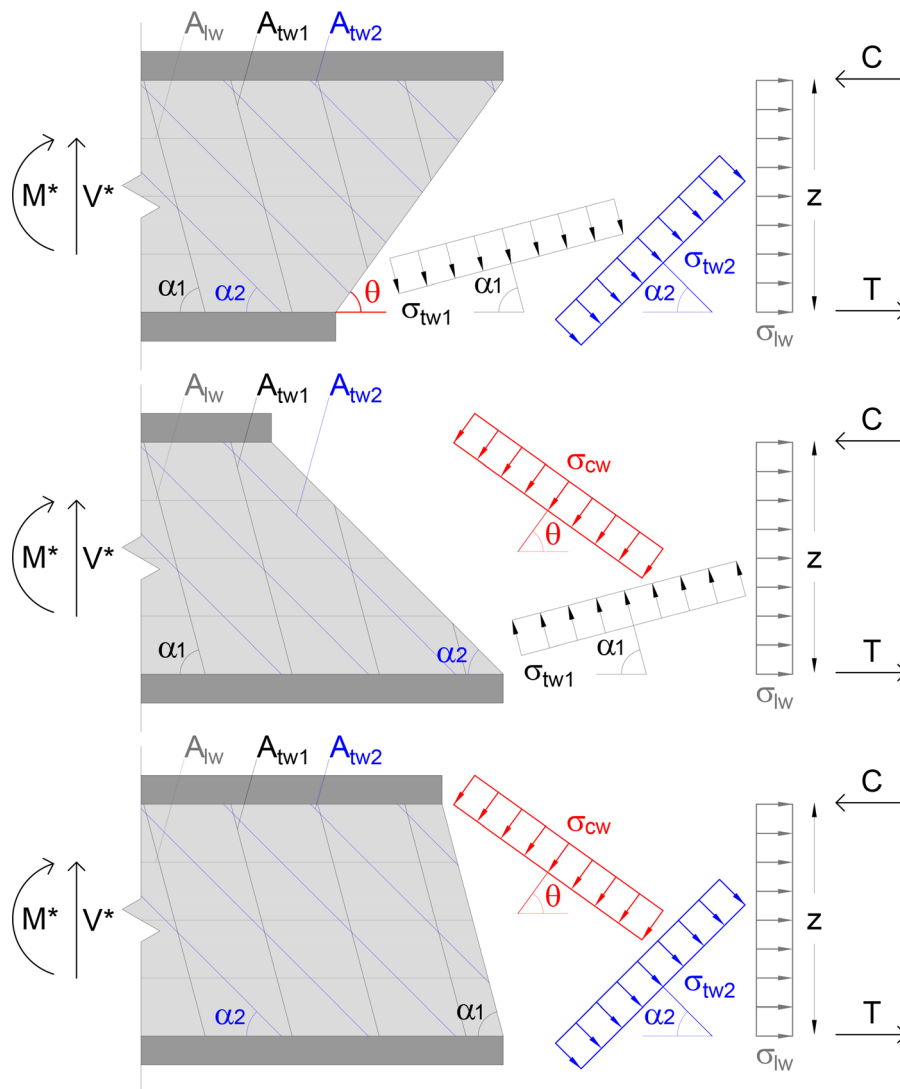


Fig. 1 Distinct beam segments obtained through three differently-oriented sections parallel to either one of the two sets of transverse reinforcement or concrete stress field direction

D (Disturbed) regions, is neglected; hence, the validity of the model is limited to *B* (Bernoulli) regions. It has to be pointed out that according to [21], assumption (ii) may be used for beams with a transverse minimum shear reinforcement mechanical ratio of $0.16/f_c^{0.5}$, being f_c the concrete strength in compression”.

The model was derived on the basis of a model formulated in [18, 19], and was extended to beams with two orders of transverse reinforcement arranged with two different inclinations α_1 and α_2 . Each order of stirrups can experience both tension or compression, on the basis of the arrangement and the amount of

reinforcement. Four stress field are used to represent the internal forces acting in the beam web, being two representative of the two sets of transverse reinforcement inclined by the angles α_1 and α_2 respectively, one representative of the concrete strut inclined by the angle θ and one representative of the longitudinal web reinforcement.

The mechanical model was developed using the static theorem of the theory of plasticity [21, 22], through which the shear resistance of a RC beam can be calculated using the commonly-named “lower-bound solution”. The mechanical model was

formulated by using the following notation: A_{tw1} , A_{tw2} , and s_{tw1} , s_{tw2} are the cross-sectional areas of the transversal web reinforcements with inclinations α_1 and α_2 and their spacings, respectively; b_w and h are the web minimum width and the cross-section depth, respectively; f_{yd} and f'_{cd} the design tensile strength of steel and the design reduced compressive strength of concrete, respectively. Therefore, assuming A_{twi} the cross-sectional area of the generic order of stirrups, the respective mechanical ratios are: $\omega_{twi} = A_{twi}/(b_w s_{twi} \sin \alpha_i) (f_{yd}/f'_{cd})$ ($i = 1, 2$). Similarly, the mechanical ratio of the web longitudinal reinforcement is equal to $\omega_{lw} = A_{lw}/(b_w h) (f_{yd}/f'_{cd})$, in which A_{lw} is the cross-sectional area of the web longitudinal bars. It should be reminded that, being the web concrete subjected to a biaxial state of stress and cracked in shear, the design compressive strength of concrete f_{cd} has to be multiplied by an efficiency coefficient v' (≤ 1), obtaining the reduced design compressive strength $f'_{cd} = v' f_{cd}$. The values of v' recommended by Eurocode 2 [16] or by Italian Construction Technical Code [23], namely $v' = 0.6(1 - f_{ck}/250)$ or $v' = 0.5$, respectively, can be used.

In order to assess the shear strength of a RC beam, three distinct beam segments are obtained through three differently-oriented sections parallel to either one of the two sets of transverse reinforcement or concrete stress field direction (Fig. 1). The equilibrium equations along the vertical axis for each of the three segments read:

$$v = \tilde{\sigma}_{tw1} \omega_{tw1} (\cot \theta + \cot \alpha_1) \sin^2 \alpha_1 + \tilde{\sigma}_{tw2} \omega_{tw2} (\cot \theta + \cot \alpha_2) \sin^2 \alpha_2 \quad (1)$$

$$v = \tilde{\sigma}_{cw} (\cot \theta + \cot \alpha_2) \sin^2 \theta + \tilde{\sigma}_{tw1} \omega_{tw1} (\cot \alpha_1 - \cot \alpha_2) \sin^2 \alpha_1 \quad (2)$$

$$v = \tilde{\sigma}_{cw} (\cot \theta + \cot \alpha_1) \sin^2 \theta + \tilde{\sigma}_{tw2} \omega_{tw2} (\cot \alpha_2 - \cot \alpha_1) \sin^2 \alpha_2 \quad (3)$$

in which $\tilde{\sigma}_{tw1}$, $\tilde{\sigma}_{tw2}$ and $\tilde{\sigma}_{cw}$ are the stresses of the two orders of reinforcement and of the web concrete respectively, made non-dimensional using the design strength of steel f_{yd} and the reduced design compressive strength of concrete f'_{cd} respectively, v the shear made non-dimensional by dividing by $b_w z f'_{cd}$. As already said, the mechanical model assesses the shear resistance of a RC beam employing the static theorem of the theory of plasticity, which provides an

evaluation of the shear capacity as the maximum value among the possible solutions validating the equilibrium conditions (1)–(3) and satisfying the following conditions of plastic admissibility:

$$0 \leq \tilde{\sigma}_{cw}, |\tilde{\sigma}_{tw1}|, |\tilde{\sigma}_{tw2}| \leq 1 \quad (4)$$

By combining (1) and (4), the following inequalities, representing the plastic admissible condition for the stress fields of the transverse reinforcement, are derived:

$$0 \leq (\tilde{\sigma}_{tw1} \omega_{tw1} \sin^2 \alpha_1 + \tilde{\sigma}_{tw2} \omega_{tw2} \sin^2 \alpha_2) (1 + \cot^2 \theta) \leq 1 \quad (5)$$

(5) elucidates the interaction between the inclination of the concrete strut and the stress fields of the two orders of stirrups. With the aim of assessing the shear resistance of RC beams via the “lower-bound solution”, the shear capacity obtained through (1) (or (2) and (3)) has to be maximized, by varying $\tilde{\sigma}_{tw1}$, $\tilde{\sigma}_{tw2}$, and $\cot \theta$ in the ranges given in (4) and (5). This operation constitutes the main drawback of the mechanical model. For this reason, in the following section an analytical procedure is derived, replacing the numerical maximization procedure which characterizes the mechanical model, with the purpose of obtaining equations able to provide the optimal values of the three above-mentioned parameters (e.g.: $\tilde{\sigma}_{tw1}$, $\tilde{\sigma}_{tw2}$, and $\cot \theta$), for any configuration and amount of transverse reinforcement, limited only to the absence of web longitudinal reinforcement ($\omega_{lw} = 0$).

3 Analytical evaluation of shear strength

In order to derive the analytical expression of the values of the three aforementioned parameters, preliminarily it has to be recognized that, since the truss model is one time redundant, according to the Nielsen’s limit analysis application to the concrete members [21], the collapse condition is attained when at least two of the three web stress fields reach their normalized stress limit values ± 1 . Two different cases are now considered, depending on the inclination of the two transverse reinforcement orders, namely the first case in which both α_1 , and α_2 are $\leq 90^\circ$, and the second case where $\alpha_1 \leq 90^\circ$, and $\alpha_2 > 90^\circ$. The former is the more frequent, and is recurrent when more effectiveness of reinforcement



placed along the inclined direction of the principal tensile stress is exploited; the second one is distinctive of over-reinforced sections, where the shear strength is limited by the capacity of the concrete web, as in thin-walled bridge sections.

3.1 $\alpha_1, \alpha_2 \leq 90^\circ$

The stress limit of the web concrete is reached when $\tilde{\sigma}_{cw} = 1$, i.e. (5) provides:

$$\tilde{\sigma}_{cw} = (\tilde{\sigma}_{tw1}\omega_{tw1} \sin^2 \alpha_1 + \tilde{\sigma}_{tw2}\omega_{tw2} \sin^2 \alpha_2)(1 + \cot^2 \theta) = 1 \tag{6}$$

From (6) the expression of $\cot \theta$ can be derived as follows:

$$\cot \theta = \sqrt{\frac{1}{\tilde{\sigma}_{tw1}\omega_{tw1} \sin^2 \alpha_1 + \tilde{\sigma}_{tw2}\omega_{tw2} \sin^2 \alpha_2} - 1} \tag{7}$$

If the normalized stress limit is reached both in the compressed concrete and in the two reinforcement orders in tension, i.e. $\tilde{\sigma}_{tw1} = \tilde{\sigma}_{tw2} = 1$ and $\tilde{\sigma}_{cw} = 1$, the slope of the web concrete stress field can be evaluated as:

$$\cot \theta = \sqrt{\frac{1}{\omega_{tw1} \sin^2 \alpha_1 + \omega_{tw2} \sin^2 \alpha_2} - 1} \tag{8}$$

Three cases can be considered, depending on the value of $\cot \theta$ provided by (8):

- $1 \leq \cot \theta \leq 2.5$: in this case, the shear strength can be easily evaluated by one of (1)–(3), assuming $\tilde{\sigma}_{cw} = 1$, and $\tilde{\sigma}_{tw1} = \tilde{\sigma}_{tw2} = 1$ and $\cot \theta$ provided by (8), since all provide the same value;
- $\cot \theta > 2.5$: the shear strength is reached at the attainment of the stress limit in the two tensile transverse reinforcement orders ($\tilde{\sigma}_{tw1} = \tilde{\sigma}_{tw2} = 1$), the limit value ($\cot \theta = 2.5$) has to be assumed, and the dimensionless design shear strength v is evaluated by (1); the normalized concrete stress can be derived from the right-hand side of (6) assuming $\tilde{\sigma}_{tw1} = \tilde{\sigma}_{tw2} = 1$ and $\cot \theta = 2.5$;
- $\cot \theta < 1$: when (8) provides $\cot \theta < 1$, the stress limit in the web concrete is attained, and one of the transverse reinforcement can be in the elastic

range. Assuming $\alpha_1 < \alpha_2$, and setting $\cot \theta = 1$, (5) reads:

$$(\tilde{\sigma}_{tw1}\omega_{tw1} \sin^2 \alpha_1 + \tilde{\sigma}_{tw2}\omega_{tw2} \sin^2 \alpha_2) \leq 0.5 \tag{9}$$

By direct inspection of (2) and (3), it can be stated that, since $\cot \alpha_1 > \cot \alpha_2$, the maximum shear strength is obtained as the minimum value given by the above Eqs. (2) and (3), being:

$$\tilde{\sigma}_{tw1} = 1 \quad (a) \quad \tilde{\sigma}_{tw2} = -1 \quad (b) \tag{10}$$

In order to detect which of the two reinforcements yields, i.e. which of (2) and (3) provides the minimum shear strength and which of (10a) and (10b) is true, (10a) and (10b) are assumed, and the inequality (2) < (3) can be rearranged in the following form:

$$\omega_{tw1} \sin^2 \alpha_1 \leq 0.5 + \omega_{tw2} \sin^2 \alpha_2 \tag{11}$$

Thus, if (11) is true, the first order of transverse reinforcement yields in tension ($\tilde{\sigma}_{tw1} = 1$), the shear strength is given by (2), while the stress in the second order of stirrups is:

$$\tilde{\sigma}_{tw2} = (0.5 - \omega_{tw1} \sin^2 \alpha_1) / (\omega_{tw2} \sin^2 \alpha_2) \tag{12}$$

If inequality (11) is false, the second order of transverse reinforcement yields in compression ($\tilde{\sigma}_{tw2} = -1$), the shear strength is given by (3), while the stress in the first order of stirrups is:

$$\tilde{\sigma}_{tw1} = (0.5 + \omega_{tw2} \sin^2 \alpha_2) / (\omega_{tw1} \sin^2 \alpha_1) \tag{13}$$

In order to represent the above conditions, the Cartesian plane of transverse reinforcement ratios $\omega_{tw1} - \omega_{tw2}$ is considered, in which the following regions are detected:

Region 1 By means of (8), assuming $\cot \theta > 2.5$, the following relation can be derived:

$$\omega_{tw1} \sin^2 \alpha_1 + \omega_{tw2} \sin^2 \alpha_2 \leq 7.25^{-1} \tag{14}$$

In this region, $\cot \theta = 2.5$, $\tilde{\sigma}_{tw1} = \tilde{\sigma}_{tw2} = 1$ and the shear strength is developed at the attainment of the stress limit in the two reinforcement orders;

Region 2 By means of (8), assuming $1 \leq \cot \theta \leq 2.5$, the following condition is obtained:

$$7.25^{-1} \leq \omega_{tw1} \sin^2 \alpha_1 + \omega_{tw2} \sin^2 \alpha_2 \leq 0.5 \tag{15}$$

In this region $\tilde{\sigma}_{tw1} = \tilde{\sigma}_{tw2} = 1$ and $\cot \theta$ is given by (8); web concrete and the two reinforcement orders reach the stress limit at the same time;



Region 3 By means of (8), assuming $\cot \theta < 1$, the following conditions are obtained:

$$\begin{aligned} \omega_{tw1} \sin^2 \alpha_1 + \omega_{tw2} \sin^2 \alpha_2 &> 0.5 \\ \omega_{tw1} \sin^2 \alpha_1 - \omega_{tw2} \sin^2 \alpha_2 &\leq 0.5 \end{aligned} \quad (16)$$

In this region $\tilde{\sigma}_{tw1} = 1$, $\cot \theta = 1$, and $\tilde{\sigma}_{tw2}$ is provided by (12);

Region 4 By means of (11) the following condition is obtained:

$$\omega_{tw1} \sin^2 \alpha_1 - \omega_{tw2} \sin^2 \alpha_2 > 0.5 \quad (17)$$

In this region $\tilde{\sigma}_{tw2} = -1$, $\cot \theta = 1$ and $\tilde{\sigma}_{tw1}$ is provided by (13).

It should be emphasized that (1) and (8) constitute a direct extension of the equations contained in Eurocode 2 for evaluation of the shear capacity of RC beams with a single order of transverse reinforcement.

$$3.2 \quad \alpha_1 \leq 90^\circ, \alpha_2 > 90^\circ$$

First of all, this layout is analysed considering the transverse reinforcement with lower inclination (α_1) yielding in tension. If the attainment of the stress limit in the concrete and the reinforcement order with lower inclination is assumed, i.e. $\tilde{\sigma}_{cw} = \tilde{\sigma}_{tw1} = 1$, by (5) the following analytical expression of the stress in the second order of transverse stirrups $\tilde{\sigma}_{tw2}$ as a function of the concrete stress field slope θ is obtained:

$$\tilde{\sigma}_{tw2} = \frac{\sin^2 \theta - \omega_{tw1} \sin^2 \alpha_1}{\omega_{tw2} \sin^2 \alpha_2} \quad (18)$$

By replacing (18) into (1) the following expression of the normalized shear strength is obtained:

$$\begin{aligned} v = \tilde{\sigma}_{tw1} \omega_{tw1} (\cot \theta + \cot \alpha_1) \sin^2 \alpha_1 \\ + \frac{(\sin^2 \theta - \omega_{tw1} \sin^2 \alpha_1) \omega_{tw2} (\cot \theta + \cot \alpha_2) \sin^2 \alpha_2}{\omega_{tw2} \sin^2 \alpha_2} \end{aligned} \quad (19)$$

The value of shear strength provided by (19) has to be maximized with respect to the inclination of the web concrete stress field θ . Therefore, taking the derivative with respect to θ , and setting it equal to zero, the following equation is obtained:

$$dv(x)/d\theta = 2 \sin \theta \cos \theta (\cot \theta + \cot \alpha_2) = 0 \quad (20)$$

(20) can be rearranged in the following form:

$$\cot^2 \theta + 2 \cot \theta \cot \alpha_2 - 1 \quad (21)$$

Thus, the positive solution of (21) is:

$$\cot \theta = -\cot \alpha_2 + \sqrt{\cot^2 \alpha_2 + 1} \quad (22)$$

If inclinations of the second order of reinforcement are only considered in the range $90^\circ < \alpha_2 \leq 135^\circ$, $\cot \theta$ will be found in the range $1 \leq \cot \theta \leq 2.5$. By replacing (22) into (18), two different cases can be found, namely:

- $-1 \leq \tilde{\sigma}_{tw2} \leq 1$: i.e. the stress satisfies the condition of plastic admissibility (4); thus the shear strength can be evaluated employing (1)–(3), assuming $\tilde{\sigma}_{cw} = \tilde{\sigma}_{tw1} = 1$ and calculating $\tilde{\sigma}_{tw2}$ and $\cot \theta$ by means of (18) and (22) respectively;
- $\tilde{\sigma}_{tw2} > 1$ or $\tilde{\sigma}_{tw2} < -1$: since the solution would violate the plastic admissibility condition, $\tilde{\sigma}_{tw2} = \pm 1$ is assumed (with the sign chosen depending on which of conditions (4) is violated by (18)), and $\cot \theta$ is evaluated exploiting the following expression:

$$\cot \theta = \sqrt{\frac{1}{\omega_{tw1} \sin^2 \alpha_1 \pm \omega_{tw2} \sin^2 \alpha_2} - 1} \quad (23)$$

The result of (23) will be comprised in one of the three following ranges:

- $1 \leq \cot \theta \leq 2.5$: in this case, the shear strength is given by any of (1)–(3), which all provide the same value;
- $\cot \theta > 2.5$: in this case, $\cot \theta = 2.5$ is assumed, and v is evaluated through (1). The web concrete stress field can be calculated by (6);
- $\cot \theta < 1$: when $\cot \theta$ provided by (23) is less than one, the collapse is due to the attainment of the stress limit in the web concrete and one of the web reinforcements, and thus the other web reinforcement can be in the elastic range. Analyzing (2) and (3), it is observed that the maximum shear strength is obtained in each of the two equations by assuming respectively:

$$\tilde{\sigma}_{tw1} = 1 \quad (a) \quad \tilde{\sigma}_{tw2} = -1 \quad (b) \quad (24)$$

Evidently, only one of (24a) and (24b) can be true, while the other stress has to ensure coincidence between the strength values provided by (2) and (3). Thus, the actual shear strength is equal to the minimum provided by (2) and (3) in which (24a) and (24b) respectively are assumed. In order to recognize

the reinforcement ratio amount for which (24a) or (24b) is true, it is necessary to evaluate whether (2) or (3) gives the minimum shear strength when (24a) and (24b) are assumed. Thus, (24a) holds, i.e. the first order of web reinforcement yields in tension, if the following inequality is true:

$$\omega_{tw1} \sin^2 \alpha_1 \leq 0.5 + \omega_{tw2} \sin^2 \alpha_2 \quad (25)$$

In this case the shear strength can be easily evaluated by (2) where $\cot \theta = 1$, and (24a) is assumed, and the stress in the second order of web reinforcement can be evaluated employing (12). By contrast, if (25) is false, it turns out that failure is due to the attainment of the stress limit in the web concrete and the second order of stirrups at the same time, both in compression. Thus, the shear strength is provided by (3), where $\cot \theta = 1$ and (24b) have to be assumed. The stress in the first order of web reinforcement can be evaluated exploiting (13).

Thus, as done for the previous case, in the Cartesian plane of the transverse reinforcement ratios ω_{tw1} – ω_{tw2} the regions characterized by the previously evaluated solutions are:

Region 1 by means of imposing $\cot \theta > 2.5$ in (23), the following condition is derived:

$$\omega_{tw1} \sin^2 \alpha_1 + \omega_{tw2} \sin^2 \alpha_2 \leq 7.25^{-1} \quad (26)$$

In this region $\cot \theta = 2.5$, and $\tilde{\sigma}_{tw1} = \tilde{\sigma}_{tw2} = 1$ have to be assumed, and the failure is due to yielding of the two web reinforcement orders;

Region 2 in this region, the transverse reinforcement orders still both yield in tension ($\tilde{\sigma}_{tw1} = \tilde{\sigma}_{tw2} = 1$) and the web concrete stress field inclination is provided by (23). The upper border of this region is determined by imposing the condition that $\cot \theta$ must reach the value provided by (22). Thus, by equating (8) and (22), the following expression of the upper boundary of region 2 is obtained:

$$\omega_{tw1} \sin^2 \alpha_1 + \omega_{tw2} \sin^2 \alpha_2 = \frac{1}{2}(1 + \cos \alpha_2) \quad (27)$$

Therefore, the region within $\tilde{\sigma}_{tw1} = \tilde{\sigma}_{tw2} = 1$ and $\cot \theta$ provided by (23) is bounded by the following conditions:

$$\frac{1}{7.25} < \omega_{tw1} \sin^2 \alpha_1 + \omega_{tw2} \sin^2 \alpha_2 \leq \frac{1}{2}(1 + \cos \alpha_2) \quad (28)$$

In this region both the web concrete and two orders of web reinforcement reach their maximum normalized stress at the same time.

Region 3 this region is characterized by $\tilde{\sigma}_{tw1} = 1$, a fixed value of $\cot \theta$ given by (22) and the elastic behaviour of the second order of web reinforcement. Its stress can be evaluated by (18) once (22) is retained, as follows:

$$\tilde{\sigma}_{tw2} = \frac{0.5(1 + \cos \alpha_2) - \omega_{tw1} \sin^2 \alpha_1}{\omega_{tw2} \sin^2 \alpha_2} \quad (29)$$

The boundaries of the region are determined by the second order web reinforcement yielding in tension (27) or in compression, i.e. $\tilde{\sigma}_{tw2} = -1$; replacing the latter in (29) provides:

$$\omega_{tw1} \sin^2 \alpha_1 - \omega_{tw2} \sin^2 \alpha_2 = 0.5(1 + \cos \alpha_2) \quad (30)$$

Thus, the region within which $\tilde{\sigma}_{tw1} = 1$, $\tilde{\sigma}_{tw2}$ given by (29) and $\cot \theta$ given by (22) is bounded by:

$$\omega_{tw1} \sin^2 \alpha_1 + \omega_{tw2} \sin^2 \alpha_2 > 0.5(1 + \cos \alpha_2) \quad (31)$$

$$\omega_{tw1} \sin^2 \alpha_1 - \omega_{tw2} \sin^2 \alpha_2 \leq 0.5(1 + \cos \alpha_2) \quad (32)$$

Region 4 in this region the two orders of web reinforcement yield, the first one in tension ($\tilde{\sigma}_{tw1} = 1$) and the second one in compression ($\tilde{\sigma}_{tw2} = -1$) and $\cot \theta$ is given by (23). The upper bound is found by imposing the condition that (23) has to provide the value $\cot \theta = 1$. The boundaries of this region are defined by the following inequalities:

$$0.5(1 + \cos \alpha_2) \leq \omega_{tw1} \sin^2 \alpha_1 - \omega_{tw2} \sin^2 \alpha_2 \leq 0.5 \quad (33)$$

Region 5 further increment of transverse reinforcement beyond the upper limit of region 4, i.e. when:

$$\omega_{tw1} \sin^2 \alpha_1 - \omega_{tw2} \sin^2 \alpha_2 > 0.5 \quad (34)$$

means that the tensile web reinforcement $\tilde{\sigma}_{tw1}$ is in the elastic range and its stress is given by (13), and $\tilde{\sigma}_{tw2} = -1$, $\cot \theta = 1$ are the other parameter values. It has to be emphasized that for both the two aforementioned cases, namely $\alpha_1, \alpha_2 \leq 90^\circ$, and $\alpha_1 \leq 90^\circ, \alpha_2 > 90^\circ$, region 1 and region 2 are those of major practical interest, while the other regions describe the behaviour of beams over-reinforced in shear, and are only of practical interest in a few special cases.



3.3 Tensile and compressive chord failure

The model described in Colajanni et al. [17] is able to detect premature failure of either the compressive or the tensile chord due to shear-flexure interaction. To this aim, the following two equations are proposed to

$$\cot^2 \theta + \frac{2a}{z} \cot \theta + \frac{\tilde{\sigma}_{tw1} \omega_{tw1} \sin^2 \alpha_1 \cot \alpha_1 \left(\frac{2a}{z} - \cot \alpha_1 \right) + \tilde{\sigma}_{tw2} \omega_{tw2} \sin^2 \alpha_2 \cot \alpha_2 \left(\frac{2a}{z} - \cot \alpha_2 \right) - 2\tilde{T}(x)}{\tilde{\sigma}_{tw1} \omega_{tw1} \sin^2 \alpha_1 + \tilde{\sigma}_{tw2} \omega_{tw2} \sin^2 \alpha_2} = 0 \quad (40)$$

$$\cot^2 \theta - \frac{2a}{z} \cot \theta - \frac{\tilde{\sigma}_{tw1} \omega_{tw1} \sin^2 \alpha_1 \cot \alpha_1 \left(\frac{2a}{z} - \cot \alpha_1 \right) + \tilde{\sigma}_{tw2} \omega_{tw2} \sin^2 \alpha_2 \cot \alpha_2 \left(\frac{2a}{z} - \cot \alpha_2 \right) + 2\tilde{C}(x)}{\tilde{\sigma}_{tw1} \omega_{tw1} \sin^2 \alpha_1 + \tilde{\sigma}_{tw2} \omega_{tw2} \sin^2 \alpha_2} = 0 \quad (41)$$

calculate the internal forces in the top and bottom chords:

$$\tilde{T}(x) = \tilde{m}(x) + 0.5 \left[\omega_{tw1} \tilde{\sigma}_{tw1} (\cot^2 \theta - \cot^2 \alpha_1) \sin^2 \alpha_1 + \omega_{tw2} \tilde{\sigma}_{tw2} (\cot^2 \theta - \cot^2 \alpha_2) \sin^2 \alpha_2 \right] \quad (35)$$

$$\tilde{C}(x) = \tilde{m}(x) - 0.5 \left[\omega_{tw1} \tilde{\sigma}_{tw1} (\cot^2 \theta - \cot^2 \alpha_1) \sin^2 \alpha_1 + \omega_{tw2} \tilde{\sigma}_{tw2} (\cot^2 \theta - \cot^2 \alpha_2) \sin^2 \alpha_2 \right] \quad (36)$$

in which the non-dimensional bending moment is equal to:

$$\tilde{m}(x) = v \frac{x}{z} \quad 0 \leq x \leq a \quad (37)$$

(35) and (36) are consistent with the evaluation of the additional tensile force in the longitudinal reinforcement due to shear required by Eurocode 2. The strength of the two chords has to satisfy the two following conditions of “plastic admissibility”:

$$\tilde{T}(x) \leq \omega_s \quad (38)$$

$$-\omega'_s \leq \tilde{C}(x) \leq \xi/v' + \omega'_s \quad (39)$$

where $\xi = x_c/z$ is the non-dimensional neutral axis depth, $A'_s, \omega'_s = (A'_s f_{yd}) / (b_w z f'_{cd})$ and $A_s, \omega_s = (A_s f_{yd}) / (b_w z f'_{cd})$ are the areas and the mechanical ratios of the longitudinal reinforcement in the compression and tension chords, respectively.

If the optimal parameters determined as described in the previous section do not satisfy either (38) or (39), the beam shear strength is ruled by the chord strength.

Substituting v with (1), (35) and (36) can be arranged as follows:

The optimal value of the three variables appearing in (40) and (41), i.e.: $\tilde{\sigma}_{tw1}$, $\tilde{\sigma}_{tw2}$ and $\cot \theta$, should be determined according the amount of transverse mechanical ratios ω_{tw1} and ω_{tw2} . For instance, if the beam belongs to “Case 1, Region 2”, the three parameter values are: $\tilde{\sigma}_{tw1} = \tilde{\sigma}_{tw2} = 1$ and $\cot \theta$ variable. Consequently, $\tilde{\sigma}_{tw1} = \tilde{\sigma}_{tw2} = 1$ are assumed and $\cot \theta$ is the only variable parameter in (40) and (41). According to this procedure, for each of the aforementioned cases/regions of ω_{tw1} – ω_{tw2} plane, the optimal values of two of the three parameters is known, and (40) or (41) can be solved to determine the third optimal value. In order to clarify the procedure in case of chord failure, the example below elucidates the flow chart of the strength evaluation. Once the shear capacity has been calculated by means of (1)–(3), the internal forces acting on the tension and compression chords are computed using (35) and (36). Subsequently, the plastic admissibility conditions regarding the two chords are checked employing (38) and (39). If one of the two inequalities is not verified (e.g. $\tilde{T}(x) > \omega_s$) the limit is assumed (e.g. $\tilde{T}(x) = \omega_s$) and the $\cot \theta$ related to the flexural failure is computed using (40), where $\tilde{\sigma}_{tw1} = \tilde{\sigma}_{tw2} = 1$.

The minimum amount of the bottom longitudinal reinforcement that ensures the shear failure of the beam can be calculated equating the external bending moment associated to the shear resistance of the beam (i.e. (37)) and the non-dimensional bending moment resistance associated to the tensile chord failure. The

Table 1 Geometrical and mechanical details of the investigated beams (TM: test method, 3P: three point bending, 4P: four point bending)

ID	b_w (mm)	d (mm)	a (mm)	f_c (Mpa)	$f_{y,s}$ (Mpa)	$f_{y,t}$ (Mpa)	A_s (mm ²)	A'_s (mm ²)	$A_{tw,1}$ (mm ²)	$A_{tw,2}$ (mm ²)	s_1 (mm)	s_2 (mm)	α_1 (°)	α_2 (°)	TM
[10] WB6	750	210	700	29	620	466	2211	678	170	157	300	150	90	50	3P
[13] A-1.2	300	248	600	24	509	395	1500	1407	226	226	300	300	54	126	3P
A-2.1	300	248	600	16	509	415	1902	1407	226	226	300	300	54	126	3P
A-2.2	300	248	600	16	509	415	1902	1407	226	226	300	300	54	126	3P
B-1	300	212	600	16	509	489	1407	1500	226	226	300	300	54	126	3P
B-2	300	212	600	24	509	489	1407	1500	226	226	300	300	54	126	3P
[14] R0-B-B	500	308	1338	39	551	397	8000	5024	402	402	450	450	30	90	3P
R0-B-S	500	308	1338	39	402	397	8000	5024	402	402	450	450	30	90	3P
R0-S-B	500	308	1338	39	551	397	8000	5024	402	402	450	450	30	90	3P
R3-B-B	500	298	1338	39	551	423	11,768	5024	402	402	450	450	30	90	3P
R3-B-S	500	298	1338	39	402	423	11,768	5024	402	402	450	450	30	90	3P
R3-S-B	500	299	1338	39	551	402	11,768	5024	402	402	450	450	30	90	3P
R5-B-B	500	294	1338	39	551	433	14,280	5024	402	402	450	450	30	90	3P
R5-B-S	500	294	1338	39	402	433	14,280	5024	402	402	450	450	30	90	3P
R5-S-B	500	295	1338	39	551	404	14,280	5024	402	402	450	450	30	90	3P
[26] 284.1	203	254	610	18	439	436	791	0	36	36	152	152	60	90	4P
284.5	203	254	610	20	383	447	791	0	81	81	152	152	60	90	4P
290.3	203	254	610	19	444	414	645	0	81	81	152	152	60	90	4P
290.5	203	254	610	19	403	392	645	0	41	41	152	152	60	90	4P



latter is computed by imposing the equality in (38) and substituting it in (35), as follows:

$$\begin{aligned} \tilde{m}(x) = & \omega_s - 0.5 [\omega_{tw1} \tilde{\sigma}_{tw1} (\cot^2 \theta - \cot^2 \alpha_1) \sin^2 \alpha_1 \\ & + \omega_{tw2} \tilde{\sigma}_{tw2} (\cot^2 \theta - \cot^2 \alpha_2) \sin^2 \alpha_2] \end{aligned} \quad (42)$$

By equating (37) and (42) the mechanical ratio of the longitudinal reinforcement in the tensile chord that ensures the concurrent shear and flexural failure can be calculated as follows:

$$\begin{aligned} \omega_s = & 0.5 [\omega_{tw1} \tilde{\sigma}_{tw1} (\cot^2 \theta - \cot^2 \alpha_1) \sin^2 \alpha_1 \\ & + \omega_{tw2} \tilde{\sigma}_{tw2} (\cot^2 \theta - \cot^2 \alpha_2) \sin^2 \alpha_2] + v \frac{x}{z} \end{aligned} \quad (43)$$

The non-dimensional shear resistance is computed using the procedures described in the previous paragraphs, thus the only variable to be calculated in (43) is ω_s . If the mechanical ratio calculated by means of the above equation is greater than or equal to the mechanical ratio of the longitudinal reinforcement in

the tensile chord of a generic beam, the RC member experiences flexural failure, otherwise the beam is shear critical. In order to elucidate the design implications related to the proposed model, in the following section some numerical analyses are carried out.

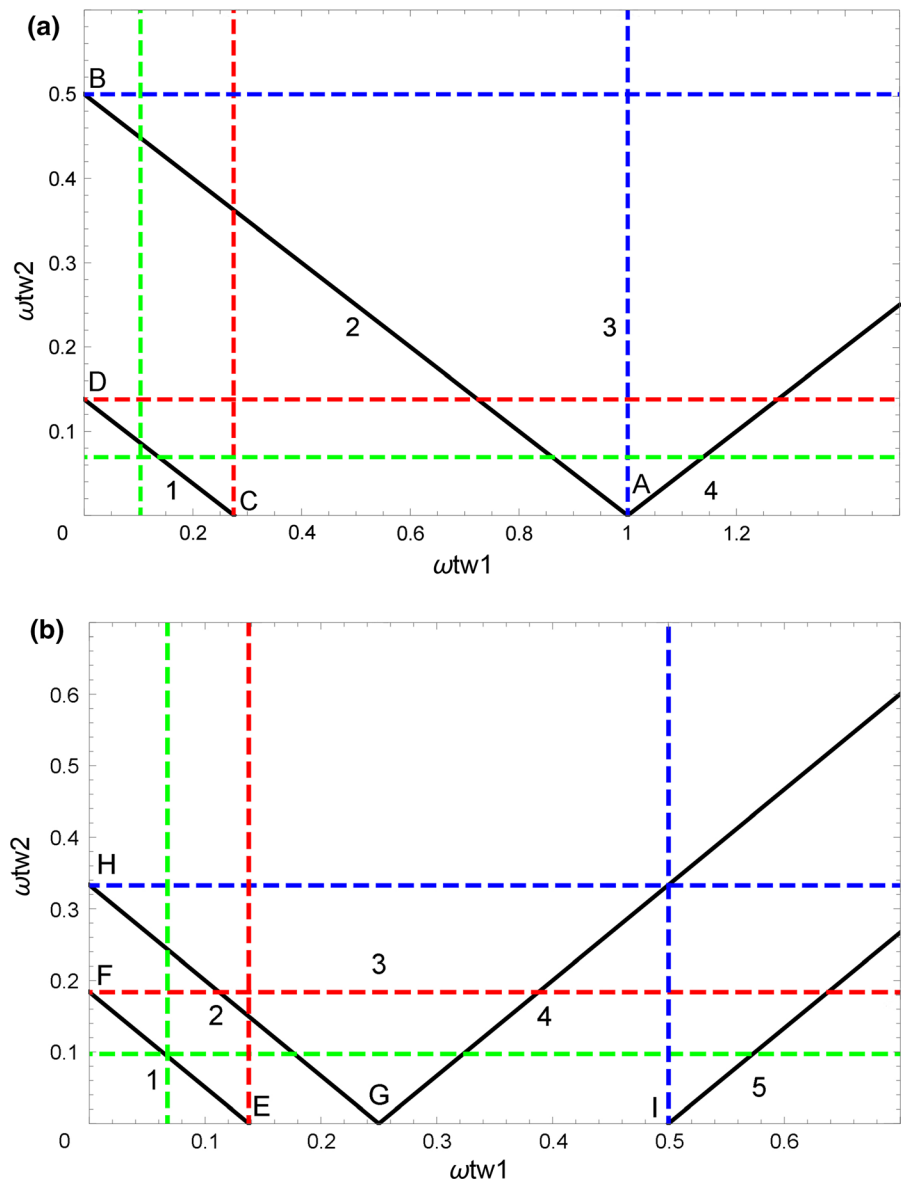
4 Model validation and numerical analysis

The numerical model proposed in Colajanni et al. [17] was validated there, and in subsequent papers [24, 25]. Here, in order to demonstrate the effectiveness of the proposed procedure for evaluation of parameter optimal values, prediction of some experimental results available in the literature is performed. The test results reported in [10, 13, 14, 26] are employed to carry out the validation. Geometrical and mechanical characteristics as well as loading conditions of these specimens are reported in Table 1. In the beam strength evaluation, the mean values of the material resistances reported in the papers describing the

Table 2 Comparison between theoretical and experimental results

	ID	V_{exp} [kN]	Model proposed ($\cot \theta_{max} = 2.5$)	Model proposed ($\cot \theta_{max} = 3.0$)	Failure	$\tilde{\sigma}_{tw1}$	$\tilde{\sigma}_{tw2}$	$\cot \theta$	Reg.
[10]	WB6	317.5	0.92	0.92	TC	1	1	1.33	1
[13]	A-1.2	288.87	1.15	1.15	TC	1	1	1.56	2
	A-2.1	211.15	0.85	0.99	S	1	0.56	1.96	3
	A-2.2	230.65	1.16	1.16	S	1	0.56	1.96	3
	B-1	167.97	0.91	1.06	S	1	0.56	1.96	3
	B-2	259.77	0.77	0.89	TC	1	1	1.75	2
[14]	R0-B-B	538.88	1.15	1.15	TC	1	1	2.44	1
	R0-B-S	541.38	0.85	0.99	S	1	1	Max	1
	R0-S-B	535.75	1.16	1.16	TC	1	1	2.44	1
	R3-B-B	674.69	0.91	1.06	S	1	1	Max	1
	R3-B-S	581.44	0.77	0.89	S	1	1	Max	1
	R3-S-B	582.69	1.06	1.23	S	1	1	Max	1
	R5-B-B	655.92	0.92	1.07	S	1	1	Max	1
	R5-B-S	610.23	0.72	0.84	S	1	1	Max	1
	R5-S-B	642.77	0.95	1.10	S	1	1	Max	1
[26]	284.1	118.84	0.94	0.94	TC	1	1	2.28	1
	284.5	122.62	1.11	1.11	TC	1	1	1.3	1
	290.3	96.16	1.15	1.15	TC	1	0.69	1	2
	290.5	87.48	1.03	1.03	TC	1	1	1.7	1
	Avg.		0.97	1.03					
	CoV		0.15	0.13					

Fig. 2 Regions for evaluation of shear strength versus ω_{tw1} and ω_{tw2} : **a** Case 1: $\alpha_1 = 45^\circ$ and $\alpha_2 = 90^\circ$; **b** Case 2: $\alpha_1 = 90^\circ$ and $\alpha_2 = 120^\circ$



experimental results were considered, without the use of partial safety factors. The results shown in Table 2 highlight that the model reproduces the experimental data well, with an acceptable underestimation in the case of a concrete stress field inclination limited to 21.8° ($\cot \theta_{\max} = 2.5$), while a slight overestimation is registered when $\cot \theta_{\max}$ is equal to 3.0 instead of 2.5, as required by Eurocode 2. This result is consistent with those reported in [17]. In general, the model predictions are accurate both when shear or chord failure occurs. In the case of shear failure, when the beams are in region 1, two different shear capacities

are computed depending on the maximum value of $\cot \theta$ that is assumed. Conversely, when chord failure occurs, the shear strength provided by the model does not change whatever $\cot \theta_{\max}$ is employed, because the shear resistance is limited by the chord capacity. Moreover, it has to be noticed that specimen 290.3 from [26], having high transverse mechanical ratios ω_{tw1} and ω_{tw2} belonging to region 2, achieves tensile chord failure, while the second order of transverse reinforcement is in the elastic range.

Below some numerical analyses are carried out with the aim of illustrating the effect of different

Table 3 Values of the coordinates of the characteristic points highlighted in Fig. 2

Points	ω_{tw1}	ω_{tw2}
A	$(2 \sin^2 \alpha_1)^{-1}$	0
B	0	$(2 \sin^2 \alpha_2)^{-1}$
C = E	$(7.25 \sin^2 \alpha_1)^{-1}$	0
D = F	0	$(7.25 \sin^2 \alpha_2)^{-1}$
G	$(1 + \cos \alpha_2)/(2 \sin^2 \alpha_1)$	0
H	0	$[2(1 - \cos \alpha_2)]^{-1}$
I	$(2 \sin^2 \alpha_1)^{-1}$	0

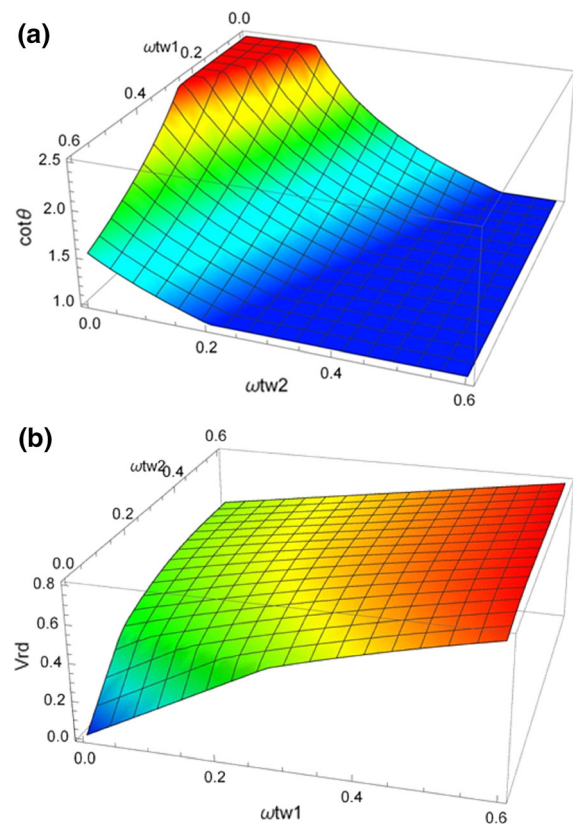


Fig. 3 Case 1: inclination of web concrete stress field ($\cot \theta$) (a) and non-dimensional shear strength (b) versus ω_{tw1} and ω_{tw2}

amounts of transverse reinforcement with two different inclinations in common RC beams of framed structures. A beam having a cross-section with dimensions $b \times h = 300 \times 500$ mm is considered, with the reduced design compressive strength of the concrete $f'_{cd} = 7.93$ MPa and the tensile yield strength

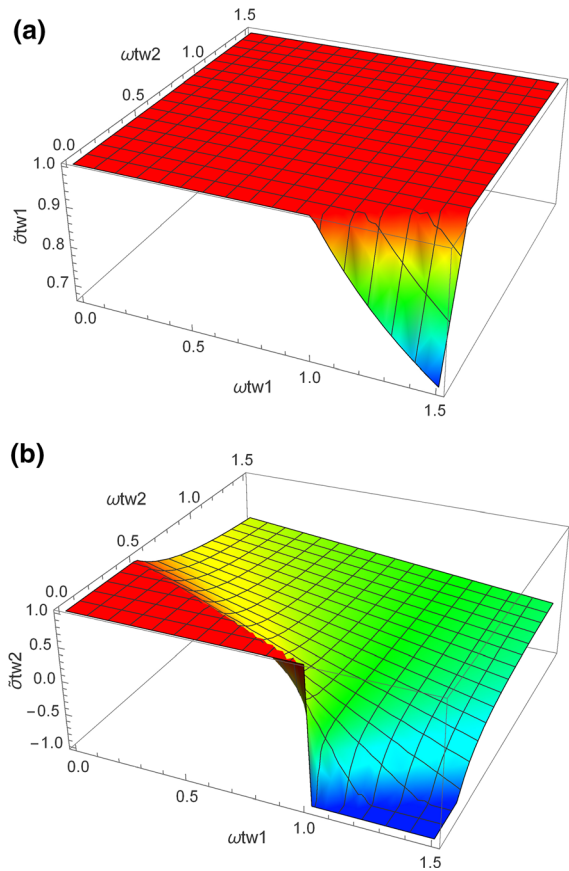


Fig. 4 Case 1: non-dimensional stress (first order (a), second order (b) of transverse reinforcement versus ω_{tw1} and ω_{tw2}

of the steel $f_y = 391$ MPa. In a first case, two transverse stirrup orders with inclinations $\alpha_1 = 45^\circ$ and $\alpha_2 = 90^\circ$ are considered, while in the second one the inclinations are $\alpha_1 = 90^\circ$ and $\alpha_2 = 120^\circ$. In Fig. 2, in the Cartesian plane of the mechanical transverse reinforcement ratios ω_{tw1} and ω_{tw2} the boundaries of the four/five regions are represented. The values of the coordinates of the characteristic points are reported in Table 3. In Figs. 3a and 3b, for the first case, the values of $\cot \theta$ and non-dimensional shear strength versus the amount of transverse reinforcement ratios ω_{tw1} and ω_{tw2} are represented in the range $0 \leq \omega_{twi} \leq 0.6$ ($i = 1, 2$), showing the greater efficiency of the first order of stirrups placed with a slope of $\alpha_1 = 45^\circ$ with respect to the vertical one $\alpha_2 = 90^\circ$. In Figs. 4a and 4b, the non-dimensional stresses for the two orders of stirrups varying the amount of the transverse reinforcement ratios ω_{tw1} and ω_{tw2} respectively are shown. It can be observed that the first order of shear

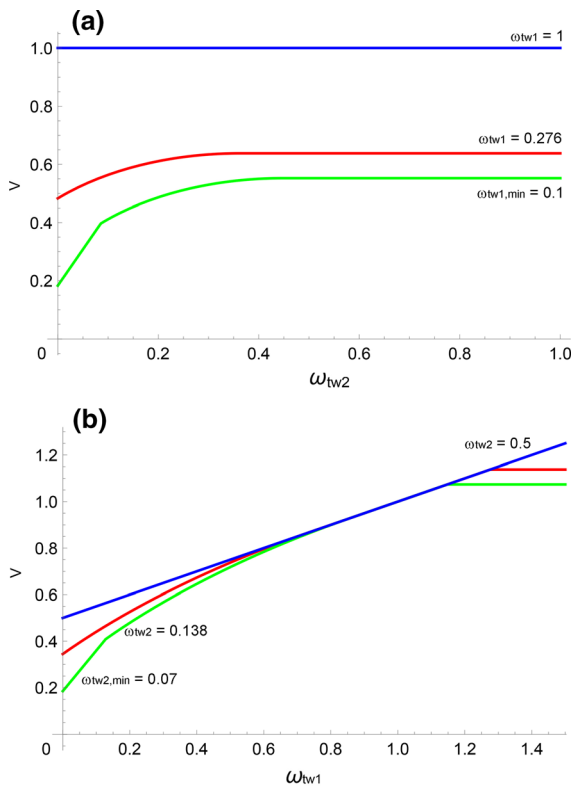


Fig. 5 Case 1: non-dimensional shear strength versus: ω_{tw2} for characteristic values of ω_{tw1} (a), ω_{tw1} for characteristic values of ω_{tw2} (b)

reinforcement always yields in tension, except for $\omega_{tw1} \geq 1$, corresponding to region 4, in which the bars are in the elastic range. Conversely, the second order of stirrups yields in tension only when a small amount of reinforcement is employed, i.e. regions 1 and 2. Incrementing ω_{tw2} over the upper boundary of region 2 leads the second order to have a stress in the elastic range. Lastly, if ω_{tw1} is also increased, the second order will yield in compression. In Fig. 5a the curves of the non-dimensional shear strength versus ω_{tw2} (amount of vertical stirrups) for the three characteristic values of ω_{tw1} (inclined stirrups), represented in Fig. 2a with a dashed line of the same colour as used in Fig. 5a, are shown. The green curve represents the case in which the amount of inclined stirrups is equal to the minimum value of stirrups required by the Italian code, $\omega_{twi,min} = 1.5f_{yd}/(1000 \sin \alpha_i) = \{0.1 (i = 1), 0.07 (i = 2)\}$. The red curve refers to the case in which the inclined stirrups alone are able to provide the condition of failure of two stirrup orders and web concrete at the same time, while the blue one

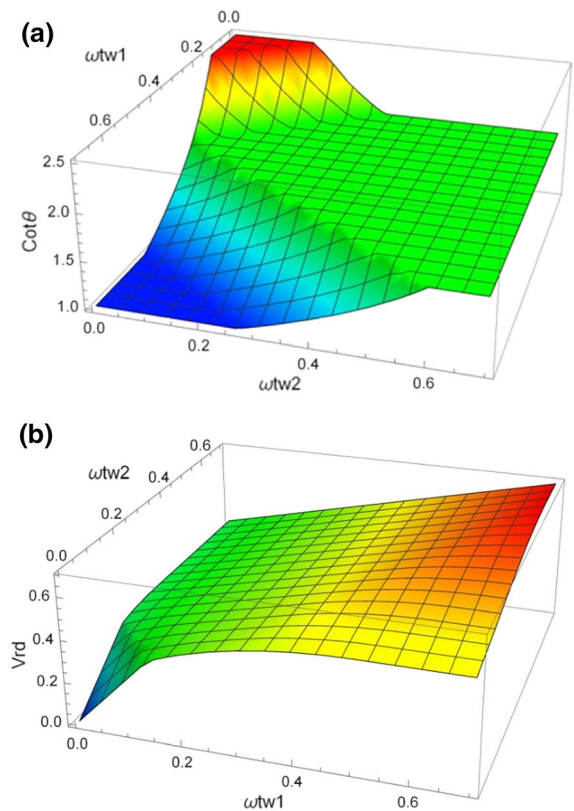


Fig. 6 Case 2: inclination of web concrete stress field ($\cot \theta$) (a) and non-dimensional shear strength (b) versus ω_{tw1} and ω_{tw2}

corresponds to the maximum shear strength that can be obtained with a single order of stirrups. The curves show that vertical stirrups are only effective when a small amount of inclined stirrups are placed in the beam ($\omega_{tw1,min} = 0.1$). In Fig. 5b the corresponding curves of the non-dimensional shear strength versus ω_{tw1} for a fixed value of ω_{tw2} are reported. They show that increasing the amount of inclined stirrups, the shear strength increases unless the mechanical ratio ω_{tw1} is more than 1. Above the latter value, the shear resistance remains constant, because the failure in compression of both the vertical stirrups and the web concrete, i.e. $\tilde{\sigma}_{tw2} = -1$ and $\cot \theta = 1$. Only an increment of vertical stirrups, which in this over-reinforced configuration are compressed, is able to increase the shear strength, allowing the concrete strut to withstand the compressive forces of the truss mechanism. Figures 6 and 7 refer to the second case, where the first order of reinforcement represents the traditional vertical stirrups ($\alpha_1 = 90^\circ$), while the second order has $\alpha_2 = 120^\circ$. The stress behaviour of



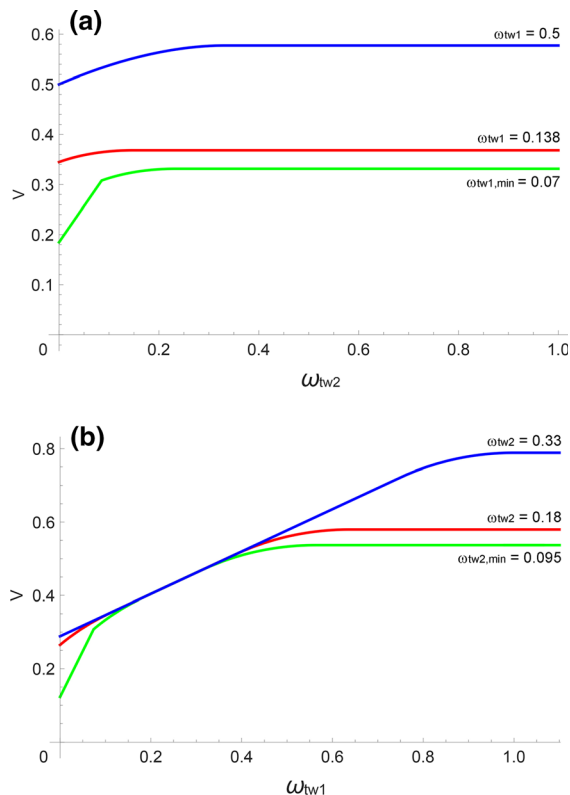


Fig. 7 Case 2: non-dimensional shear strength versus: ω_{tw2} for characteristic values of ω_{tw1} (a), ω_{tw1} for characteristic values of ω_{tw2} (b)

the two orders of transverse reinforcement is comparable to that described for case 1; thus it is not reported here. It can be noticed that inclined reinforcements with slope $\alpha_2 > 90^\circ$ are only effective in small amounts for beam with a very small amount of vertical stirrups (green line in Fig. 7a) or for over-reinforced beams (Fig. 7b, $\omega_{tw1} > 0.5$).

5 Conclusions

A design-oriented model able to predict the shear capacity of RC beams having transverse reinforcement arranged in two different inclinations has been presented. The analytical procedure depends on whether both the orders of stirrups have angles of inclination, with respect to the beam axis, in the range $45^\circ \leq \alpha_i \leq 90^\circ$ ($i = 1, 2$), or only the first-order inclination is in the range $45^\circ \leq \alpha_1 \leq 90^\circ$ while the second-order inclination is in the range $90^\circ < \alpha_2 \leq 135^\circ$. For each of the two aforementioned

cases, and for any amount of reinforcement, equations for evaluation of the parameters influencing the shear resistance, namely slope of the web concrete stress field and the non-dimensional stresses of the two orders of transverse reinforcement, have been proposed. In the Cartesian plane of transverse reinforcement ratios $\omega_{tw1}-\omega_{tw2}$ regions characterized by homogeneous behaviour of the three abovementioned parameters were detected, and equations for evaluating the boundaries of these regions were determined. The comparison carried out between the analytical predictions provided by the model and experimental results of shear critical beams shows the model's reliability. The main equations constituting the model [e.g. (1) and (8)] prove that the proposed model represents a direct extension of the Eurocode 2 model for shear assessment of beams with two order of transverse reinforcement, in which the effect of the two transverse reinforcements can be added. Major issues deserving further research include investigation of design effectiveness of different transverse reinforcement amounts and inclinations, and the use of a larger database covering all the regions identified by the model in order to prove its reliability thoroughly.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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