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Tripartite Entanglement in Coupled Three-Wave Interactions

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Abstract. We study the three-mode entanglement properties in a new kind of the coupled nonlinear optical interactions consisting of two parametric down-conversion processes and one up-conversion. Approach, based on partial scaling transformation of the covariant matrix, is used to reveal tripartite entanglement. It is established that the three-mode field is in the true entangled state.

Keywords: coupled parametric interactions, multipartite entanglement, partial scaling transformation
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1. Introduction

During recent years quantum entangled states cause significant interest of researchers. Such light states find various applications: quantum teleportation, dense coding, cryptography and others (see, for example, [1, 2]). To present time bipartite, two-mode entanglement of light is investigated both theoretically and experimentally in detail. Bipartite entangled states are produced by spontaneous parametric down-conversion.

The proposed methods of forming tripartite, three-mode entanglement may be divided into two groups. The first group uses two-partite entangled states and their subsequent linear transformation. As is known [1, 2], such an entangled state for discrete variable has been for the first time suggested by Greenberger, Horne and Zeilinger (GHZ state). The second group of methods is based on the coupled or concurrent three-frequency interactions proceeding in a single nonlinear optical crystal [3–8]. These methods apply opportunity of performance of the phase or quasi-phase matching condition in single crystal for simultaneous occurring several conventional parametric processes which have, at least, one shared wave generated. Forming the three-mode entangled states by means of coupled processes of a down-

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and up-conversion is examined in [3–7]. Generation of tripartite entanglement by triplet of the concurrent down-conversion processes is analyzed in [8].

The aim of the present paper is to study formation of the three-mode entangled states in the crystal by a new method which includes two down-conversion processes and single up-conversion one. Actually the method contains both coupled processes and concurrent ones. Such a method can be put into practice in aperiodically poled nonlinear crystals at pumping by coherent laser radiation and its second harmonic.

The paper has the following structure. Section 2 contains the description of the optical scheme for realization of the tripartite entanglement, the Heisenberg equations for the coupled processes under consideration and their solution. In Section 3 calculation of variances of the quadrature components of the frequency triplet generated and the correlations between various quadratures are presented. Here, using the so-called partial scaling [9], the covariant matrix obtained for the quadratures is transformed. In Section 4 the transformed matrix is used to calculate values of its principal minors. Dependence of values of the principal minors on values of the scaling parameters is analyzed and a conclusion is drawn that the frequency triplet is genuinely tripartite entangled states. Section 5 summarizes the basic results obtained.

2. Basic Equations and their Solution

We examine coupled three-wave interactions with non-multiple frequencies proceeding in an aperiodically poled nonlinear crystal (APNC). Two intensive coherent waves with the frequencies ω_p and $\omega_p' = 2\omega_p$ enter APNC. These waves are pumps' waves for nonlinear optical processes (see Fig. 1).

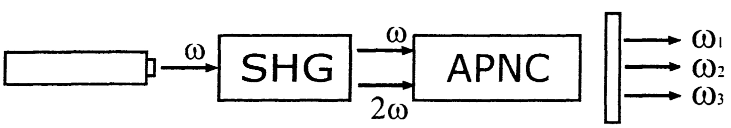


Fig. 1. Scheme for generation of tripartite entangled states with frequencies ω_1, ω_2 and ω_3 in coupled nonlinear interactions in aperiodically poled nonlinear crystal (APNC) pumped by laser radiation with frequency ω and its second harmonic 2ω. SHG is the second harmonic generator, F is the filter

In the down-conversion processes, the pumping photons with the frequency ω_p are split into two photons ω_1 and ω_2 and the pumping ones with the frequency $\omega_p' = 2\omega_p$ is split into photons with the frequencies ω_2 and ω_3. Besides, the up-conversion process takes place, namely photon with the frequency ω_p is added with photon with the frequency ω_1. As a consequence, photon with the frequency ω_3 is produced.

Thus we have got the following frequency relations:

$$\begin{aligned} \omega_p &= \omega_1 + \omega_2 , \\ \omega_3 &= \omega_p + \omega_1 , \\ \omega_p' &= 2\omega_p = \omega_2 + \omega_3 . \end{aligned} \tag{1}$$

Such processes can be implemented at the collinear quasi-phase matched wave interactions in APNL [10]. This purpose it is possible to use crystals with the Fibonacci structure [11] or crystals prepared with the help of an aperiodic grating design method [12]. Compensation of the phase mismatches of interacting waves is implemented by adjusting the thickness of structure's layers and the quasi-phase matching orders.

Simultaneous course of the considered processes is described by interaction Hamiltonian

$$\hat{H}_{(\text{int})} = i\hbar\frac{n}{c}\left(\beta_1(a_1^+a_2^+ - a_1a_2) + \beta_2(a_3^+a_1 - a_3a_1^+) + \beta_3(a_2^+a_3^+ - a_2a_3)\right) , \tag{2}$$

where a_j^+ (a_j) is the creation (annihilation) operator of a photon with frequency ω_j, β_j is the nonlinear coupling wave coefficient, n is the index of refraction of the crystal, c is the vacuum light speed.

The operators obey the commutation relations

$$[a_j, a_l^+] = \delta_{jl} , \qquad [a_j, a_l] = [a_j^+, a_l^+] = 0 \qquad (j, l = 1, 2, 3) , \tag{3}$$

where δ_{jl} is the Kronecker symbol.

The nonlinear coefficients β_1 and β_3 are related to the parametric down-conversion processes and the nonlinear coefficients β_2 to the parametric up-conversion.

Spatial evolution of the Bose operators is given by the following Heisenberg equations:

$$\frac{da_1}{dz} = -\beta_1 a_2^+ + \beta_2 a_3 , \tag{4}$$

$$\frac{da_2^+}{dz} = -\beta_1 a_1 - \beta_3 a_3 , \tag{5}$$

$$\frac{da_3}{dz} = -\beta_2 a_1 - \beta_3 a_2^+ . \tag{6}$$

Here $z = -ct/n$ is the interaction length.

It is convenient to present the solution of Eqs. (4)–(6) in the matrix form:

$$\vec{A}(z) = Q(z)\vec{A}_0 ,$$

where $\vec{A}^T(z) = (a_1(z), a_2^+(z), a_3(z))$ is the state vector of three-mode field, T means transposition and $\vec{A}_0 = \vec{A}(z=0)$ corresponds to the initial value.

$Q(z)$ is the conversion matrix:

$$Q(z) = \left\| \begin{array}{ccc} \gamma_3^2 + (1-\gamma_3^2)\cosh G & F(-\gamma_2\gamma_3; -\gamma_1) & F(\gamma_1\gamma_3; \gamma_2) \\ F(\gamma_2\gamma_3; -\gamma_1) & -\gamma_2^2 + (1+\gamma_2^2)\cosh G & F(-\gamma_1\gamma_2; -\gamma_3) \\ F(\gamma_1\gamma_3; -\gamma_2) & F(\gamma_1\gamma_2; -\gamma_3) & \gamma_1^2 + (1-\gamma_1^2)\cosh G \end{array} \right\|,$$

where

$$F(\gamma_j\gamma_k; \gamma_m) = \gamma_j\gamma_k(\cosh G - 1) + \gamma_m \sinh G\,, \quad G = (1-\varepsilon_2^2+\varepsilon_3^2)^{-1/2}\beta_1 z\,,$$

$$\gamma_j = \varepsilon_j(1-\varepsilon_2^2+\varepsilon_3^2)^{-1/2} \quad \text{and} \quad \varepsilon_j = \beta_j/\beta_1\,.$$

The expressions for the elements $Q_{jk}(z)$ of the conversion matrix are written down for the regime of monotone growth of the intensities of interacting waves which occurs under condition $\varepsilon_1^2 + \varepsilon_3^2 \geq \varepsilon_2^2$. Otherwise, the hyperbolic functions are replaced with the usual trigonometrical functions. At $\varepsilon_3 = 0$ ($\beta_3 = 0$) one parametric down-conversion process is excluded from interaction and in this case the matrix $Q_{jk}(z)$ corresponds to the coupled (interlinked) interactions considered in [3, 6].

We have studied quantum properties three-mode field applying some criteria developed for multipartite entanglement in terms of continuous variables [9, 13]. Below results connected with using the separability criterion based on partial scaling transformation of a covariant matrix [9] will be produced.

3. Covariant Matrix for Quadrature Components

Let us define quadrature components of a mode as

$$\hat{x}_j = \frac{a_j + a_j^+}{\sqrt{2}}\,, \qquad \hat{y}_j = \frac{i(a_j - a_j^+)}{\sqrt{2}}\,.$$

If modes are in the vacuum states at the input of the nonlinear crystal that is field state $|\Psi_0\rangle = |0\rangle = |0\rangle_1|0\rangle_2|0\rangle_3$ then mean values of all the quadrature components are equal to zero: $<\hat{x}_j> = <\hat{y}_j> = 0 \quad (j = 1, 2, 3)$.

Covariant (correlation) matrix for the quadrature components written down in the symmetric form

$$r_{jk} = \frac{1}{2}\langle \hat{\xi}_j\hat{\xi}_k + \hat{\xi}_\xi\hat{\xi}_j \rangle\,, \qquad \hat{\xi}_j = \hat{x}_j, \hat{y}_j$$

looks like

$$V = \left\| \begin{array}{cccccc} V_{11} & 0 & V_{13} & 0 & V_{15} & 0 \\ 0 & V_{11} & 0 & -V_{13} & 0 & V_{15} \\ V_{13} & 0 & V_{33} & 0 & V_{35} & 0 \\ 0 & -V_{13} & 0 & V_{33} & 0 & -V_{35} \\ V_{15} & 0 & V_{35} & 0 & V_{55} & 0 \\ 0 & V_{15} & 0 & -V_{35} & 0 & V_{55} \end{array} \right\|, \tag{7}$$

where

$$V_{jj} = \frac{1}{2}\sum_{k=1}^{3} Q_{jk}^2 \quad (j = 1,2,3), \qquad V_{13} = \frac{1}{2}\sum_{k=1}^{3} Q_{1k}Q_{2k},$$

$$V_{15} = \frac{1}{2}\sum_{k=1}^{3} Q_{1k}Q_{3k}, \qquad V_{35} = \frac{1}{2}\sum_{k=1}^{3} Q_{2k}Q_{3k} \quad (Q_{jk} = Q_{jk}(z)).$$

It is easy to see, that the covariant matrix (7) may be represented as a block matrix

$$V = \begin{Vmatrix} A & D & E \\ D & B & F \\ E & F & C \end{Vmatrix}, \tag{8}$$

where A, B, ... are diagonal matrices.

Following the work [9], we change the scale of quadrature components y_1, y_2 and y_3 multiplying on h_1, h_2 and h_3, respectively. Thus one obtains matrix V^s. Last is added to the block diagonal matrix (antisymmetric commutation matrix). The new matrix S is created in such a way:

$$S = V^s + \frac{i}{2}\begin{Vmatrix} \Omega & 0 & 0 \\ 0 & \Omega & 0 \\ 0 & 0 & \Omega \end{Vmatrix}, \qquad \Omega = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}. \tag{9}$$

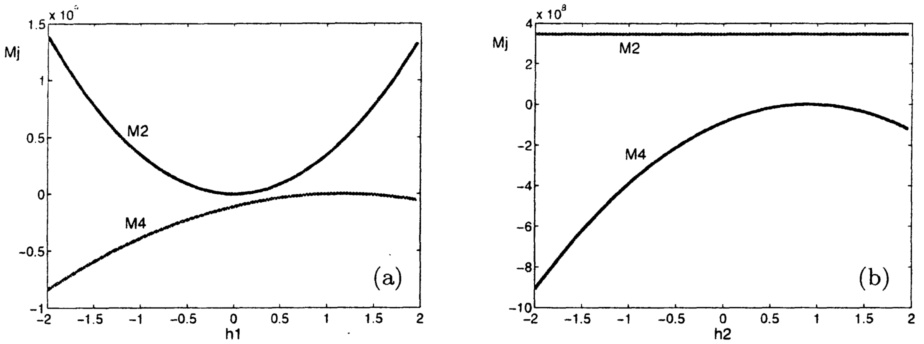
Further analysis consists in calculation of all the principal minors of the 6×6 matrix S (9). If all 6 the principal minors are greater or equal to zero for all h_j $(j = 1,2,3)$ which on the module are greater or equal to unit, then the field state is separable. It is the necessary condition for the separability [9]. If even one principal minor is negative, state of the field is an inseparable, and therefore three-mode field is in an entangled state.

4. Results and Discussion

Let us address now to results of numerical calculation of the principal minors. We focus on the regime of monotone changing the interaction wave intensities.

Firstly we check the properties of bipartite entanglement by supposing two scaling parameters are equal to unit and changing the third parameter. Figure 2 shows the dependence of the principal minors for this case. One can see that at $|h_j| \geq 1$ some minors take negative values. It should be noted that the principal minors of the first, second and third orders is always positive. The principal minors of higher orders may be both negative (see Fig. 2a,b) and positive (Fig. 2c). It follows from Fig. 2 that each of three modes is connected with two others. In other words, the entanglement of one mode with the remaining others takes place.

In order to examine the properties of tripartite entanglement we have analyzed behavior the principal minors depending on two scaling parameters.



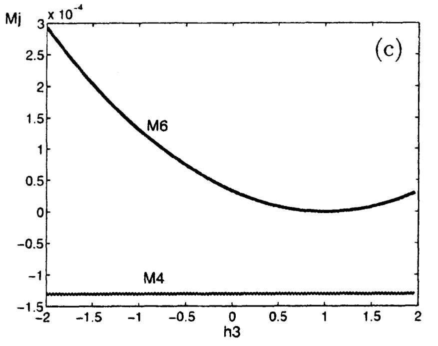


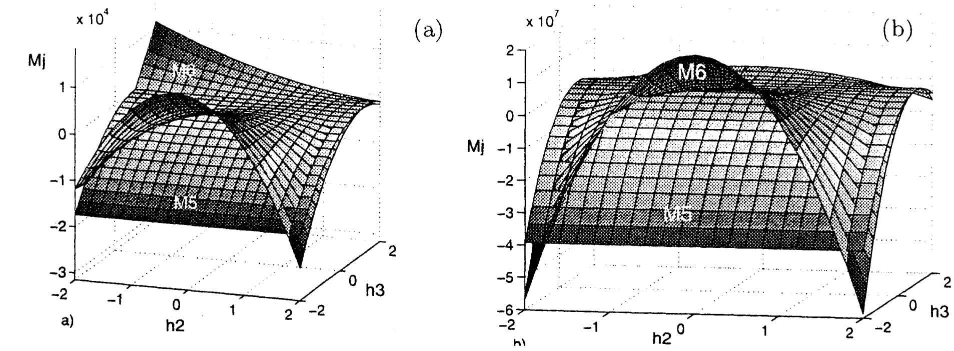
Fig. 2. Principal minors M_j as a function of scaling parameter h_1 (a), h_2 (b) and h_3 (c). Curves are calculated for value $\beta_1 z = 2.0$ and nonlinear coefficients' ratios $\beta_2/\beta_1 = 0.8$ and $\beta_3/\beta_1 = 1.2$

Figure 3 shows the dependence of the minors of the fifth and sixth orders on scaling parameters h_2 and h_3 for different ratio of the nonlinear coefficients β_3 and β_1 responsible for the parametric down-conversion processes. One can see that at $|h_2|, |h_3| \geq 1$ the principal minors in point have small negative values in comparison with unit. Thus, according to [9] the three-frequency field at the output of nonlinear crystal is in the tripartite entangled state.

In the case $\beta_3 = 0$ (Fig. 3a) there is no direct connection between photons with the frequencies ω_2 and ω_3 (see Eqs. (1)). This case actually corresponds to earlier investigated coupled processes [3, 6]. When $\beta_3 \neq 0$ (Fig. 3b,c) such a connection occurs. At the considered variation of β_3 this connection results in a reduction of the absolute value of the principal minors.

5. Conclusion

A new method of generation of the three-mode entangled states in single nonlinear optical crystal has been considered. It includes two parametric down-conversion processes and one up-conversion. Covariant matrix for the continuous variables (quadrature components) of the three-mode field at the output of the nonlinear crystal is calculated.



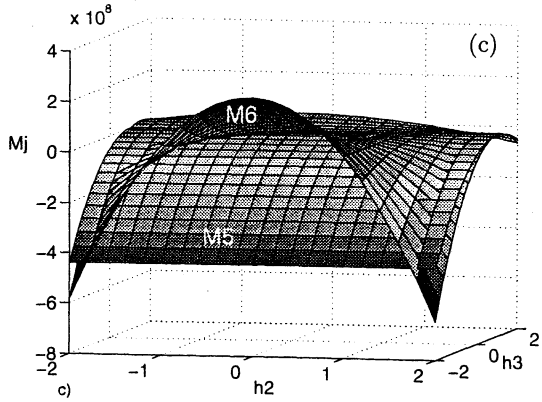


Fig. 3. Principal minors M_5 and M_6 as functions of scaling parameters h_2 and h_3 for ratio $\beta_2/\beta_1 = 0.8$ and different ratios β_3/β_1 : 0 (a), 1 (b) and 1.2 (c)

In the present paper to analyze the three-mode entanglement we have used the separability criterion based on the partial scaling transformation of the covariant matrix. The principal minors of the scaling transformed matrix depending on the nonlinear coupling coefficients and the scaling parameters were examined. The results obtained witness about the inseparability of the three-mode field, that is the three-mode field is in the true entangled state. We came to the same conclusion applying the separability criterion offered in the paper [13].

In conclusion, it should be noted that in absence of the parametric down-conversion process in the field of the second harmonic the considered method of forming the three-mode entangled states gives the results of the method [4,6]. Generation of the tripartite entanglement by last method is recently realized experimentally by Bondani with coauthors [7].

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References

1. *The Physics of Quantum Information*, eds. D. Bouwmeester, A. Ekert and A. Zeilinger, Springer, Heidelberg, 2000.
2. S.L. Braunstein and P. van Loock, *Rev. Mod. Phys.* **77** (2005) 513.
3. A.V. Rodionov and A.S. Chirkin, *Opt. Spectroscop.* **96** (2004) 721 [*Opt. Spektroskop.* **96** (2004) 790].
4. A.V. Rodionov and A.S. Chirkin, *JETP Lett.* **79** (2004) 253 [*Pisma ZhETF* **29** (2004) 311].
5. A. Allevi, A. Andreoni, M. Bondani, A. Ferraro, M.G. Paris and E. Puddu, *J. Mod. Opt.* **51** (2004) 1031.
6. A. Ferraro, M.G. Paris, M. Bondani, A. Allevi, E. Puddu and A. Andreoni, *J. Opt. Soc. Amer. B* **21** (2004) 1241.
7. M. Bondani, A. Allevi, E. Gevinti, A. Aglianti and A. Andreoni, quant-ph/0604002 (2006).
8. A.S. Bradly, M.K. Olsen, O. Pfister and R.C. Pooser, *Phys. Rev. A.* **72** (2005) 053805.
9. O.V. Man'ko, V.I. Man'ko, G. Marmo, A. Shaji, E.C.G. Sudarshan and F. Zaccaria, *Phys. Lett. A* **339** (2005) 194.
10. A.A. Novikov and A.S. Chirkin, to be published.
11. Yong-yuan Zhu and Nai-ben Ming, *Opt. Quant. Electron.* **31** (1999) 1093.
12. T. Kartağlu, Z.G. Figen and O. Aytür, *J.Opt. Soc. Am. B* **20** (2003) 343.
13. P. van Loock and A. Furusawa, *Phys. Rev. A.* **67** (2003) 052315.