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Reduction of the Density Matrix and Generalized Bell Inequalities

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Abstract. The Bell–Clauser–Horne–Shimony–Holt inequalities are considered. The right-hand side of these inequalities does not depend on the form of any two-particle spin state. In the case of the generalized Bell–Clauser–Horne–Shimony–Holt inequalities the right-hand sides depend on the form of the concrete two-particle state. The left-hand sides of these inequalities depend on four arbitrary vectors defined in three-dimensional space. They define the directions on which the spins of particles forming a correlated pair are projected. Our aim is to find such vectors that the left-hand side of the inequality should take its maximum value. In other words, by these vectors the inequality transforms into an equality. It is shown that it can be done with the help of a special reduction of the density matrix of the two-state spin state.

Keywords: Bell inequalities, entanglement, spin states, density matrix reduction
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1. Introduction

The Bell–Clauser–Horne–Shimony–Holt (Bell-CHSH) inequalities [1]

$$|E(\vec{a},\vec{b}) + E(\vec{a},\vec{c}) + E(\vec{d},\vec{b}) - E(\vec{d},\vec{c})| \leq 2\sqrt{2} \qquad (1)$$

are widely used in analyses of the entanglement property of two-particle spin states. Their left-hand sides are expressed via the mean values of the spin correlation operator $E(\vec{a},\vec{b})$. Each such value depends on two vectors, $\vec{a}$, $\vec{b}$, and the whole left-hand side depends on four vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$. In the form (1) the Bell-CHSH inequality is valid for all types of two-particle spin states: factorizable, separable and entangled. With the help of experimental verification of these inequalities one can define the type of the state, which is used in concrete situation. For the

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verification it is necessary to measure a mean value of the spin correlation operator $E(\vec{a},\vec{b})$. This value depends on two vectors, $\vec{a}$ and $\vec{b}$, that define the axis in the configuration space on which the spins of particles are projected. The aim of this work is to present a method with the help of which one can find vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$ such that the left-hand side of the generalized Bell-CHSH inequality (1) takes its maximum value.

2. The Generalized Bell-CHSH Inequalities

The generalized Bell-CHSH inequalities were constructed in work [2]. They have the form

$$|E(\vec{a},\vec{b}) + E(\vec{a},\vec{c}) + E(\vec{d},\vec{b}) - E(\vec{d},\vec{c})| \le \sqrt{2} \sup_{\vec{n}_1,\vec{n}_2} (|P(n_1)| + |P(\vec{n}_2)|) \,. \qquad (2)$$

Here

$$|\vec{n}_1| = |\vec{n}_2| = 1 \,, \qquad (\vec{n}_1,\vec{n}_2) = 0 \,, \qquad |\vec{a}| = |\vec{b}| = |\vec{c}| = |\vec{d}| = 1$$

are arbitrary unit vectors in the 3-dimensional configuration space. P is a 3×3 matrix, it is a reduction of a density matrix $\rho = \|\rho_{ij}\|$ of the two-particle spin state. It has the form

$$P = ||P_{ij}|| = \qquad (3)$$

$$\begin{pmatrix} (\rho_{14}+\rho_{23}+\rho_{32}+\rho_{41}) & i(\rho_{14}-\rho_{23}+\rho_{32}-\rho_{41}) & (\rho_{13}+\rho_{31}-\rho_{24}-\rho_{42}) \\ i(\rho_{14}+\rho_{23}-\rho_{32}-\rho_{41}) & (-\rho_{14}+\rho_{23}+\rho_{32}-\rho_{41}) & i(\rho_{13}-\rho_{31}-\rho_{24}+\rho_{42}) \\ (\rho_{12}+\rho_{21}-\rho_{34}-\rho_{43}) & i(\rho_{12}-\rho_{21}-\rho_{34}+\rho_{43}) & (\rho_{11}-\rho_{22}-\rho_{33}+\rho_{44}) \end{pmatrix}.$$

With the help of the matrix P the mean value of the spin correlation operator $E(\vec{a},\vec{b})$ can be presented in the form of scalar product in the 3-dimensional configuration space

$$E(\vec{a},\vec{b}) = \mathrm{Sp}\,(\hat{a}\otimes\hat{b}\rho) = (\vec{a},\ P\vec{b}) \,. \qquad (4)$$

The verification procedure of the Bell-CHSH inequalities (2) consists of the measurement of the spin projections for various sets of vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$. The result of measurements depends on the orientation of these vectors. In order to determine if the inequality (2) is satisfied or violated it is necessary to find four vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$ such that the left-hand side of the inequality takes its maximum value. In other words, one must find such four vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$ that the inequality (2) transforms into an equality. For this aim let us rewrite the left-hand side of the inequality (2) using expression (4) for the mean value of the spin correlation operator $E(\vec{a},\vec{b})$. As a result one can get

$$\begin{aligned} &|E(\vec{a},\vec{b}) + E(\vec{a},\vec{c}) + E(\vec{d},\vec{b}) - E(\vec{d},\vec{c})| \\ &= |(\vec{a},P\vec{b}) + (\vec{a},P\vec{c}) + (\vec{d},P\vec{b}) - (\vec{d},P\vec{c})| = |(\vec{a},P(\vec{b}+\vec{c})) + (\vec{d},P(\vec{b}-\vec{c}))| \,. \end{aligned} \qquad (5)$$

It is easy to see that expression (5) takes its maximum value when

$$|(\vec{b}+\vec{c})| = |(\vec{b}-\vec{c})| = \sqrt{2}, \quad (\vec{b},\vec{c}) = 0\,. \tag{6}$$

Besides, vector $\vec{a}$ must be parallel to vector $P(\vec{b}+\vec{c})$, and vector $\vec{d}$ must be parallel to vector $P(\vec{b}-\vec{c})$.

3. Examples

Let us consider some examples. First of all we consider the so-called scalar state
1. $\Psi_{0,0} = \frac{1}{\sqrt{2}}(|+\rangle|-\rangle - |-\rangle|+\rangle)$. For this state

$$\rho_{0,0} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad P_{0,0} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

It is easy to see that for the P matrix the maximum value of right-hand side of the inequality (1) is equal to $2\sqrt{2}$

$$RH(\vec{n}_1,\vec{n}_2) = 2\sqrt{2}\,.$$

This value can be achieved at every pair of vectors $\vec{n}_1$, $\vec{n}_2$ that are orthogonal to each other: $(\vec{n}_1,\vec{n}_2) = 0$. The four unit vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$ can be expressed via the vectors $\vec{n}_1,\vec{n}_2$ as follows

$$\vec{a} = \vec{n}_1\,, \quad \vec{b} = \frac{1}{\sqrt{2}}(\vec{n}_1+\vec{n}_2)\,, \quad \vec{c} = \frac{1}{\sqrt{2}}(\vec{n}_1-\vec{n}_2)\,, \quad \vec{d} = \vec{n}_2\,.$$

One can get the same result for the state
2. $\Psi_{1,0} = \frac{1}{\sqrt{2}}(|+\rangle|-\rangle + |-\rangle|+\rangle)$. For this state

$$\rho_{1,0} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad P_{1,0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

For the state $\Psi_{1,0}$ the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$ can be expressed via the vectors $\vec{n}_1,\vec{n}_2$ in the same manner as in the case of the state $\Psi_{0,0}$.

$$RH(\vec{n}_1,\vec{n}_2) = 2\sqrt{2}\,.$$

$$\vec{a} = \vec{n}_1\,, \quad \vec{b} = \frac{1}{\sqrt{2}}(\vec{n}_1+\vec{n}_2)\,, \quad \vec{c} = \frac{1}{\sqrt{2}}(\vec{n}_1-\vec{n}_2)\,, \quad \vec{d} = \vec{n}_2\,.$$

The vectors $\vec{n}_1$, $\vec{n}_2$ are arbitrary with the only restriction $(\vec{n}_1,\vec{n}_2) = 0$.

3. Up till now we considered only pure states. Let us consider now the Werner state [3]. It is a mixed state described by the density matrix

$$\rho_W = \frac{1}{8}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & -2 & 0 \\ 0 & -2 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{7}$$

For the state (7) the P matrix (3) has the form

$$P_W = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{8}$$

With the help of the matrix (8) one can find that for the state (7)

$$RH(\vec{n}_1, \vec{n}_2) = 2\sqrt{2},$$

and this maximum value is achieved at every pair of vectors $\vec{n}_1, \vec{n}_2$ that are orthogonal to each other: $(\vec{n}_1, \vec{n}_2) = 0$. The four unit vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$ can be expressed via the vectors $\vec{n}_1$, $\vec{n}_2$ as follows

$$\vec{a} = \vec{n}_1\,, \quad \vec{b} = \frac{1}{\sqrt{2}}(\vec{n}_1 + \vec{n}_2)\,, \quad \vec{c} = \frac{1}{\sqrt{2}}(\vec{n}_1 - \vec{n}_2)\,, \quad \vec{d} = \vec{n}_2\,.$$

4. Let us now consider a mixed state that is described by a density matrix

$$\rho_\mu = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -\mu & 0 \\ 0 & -\mu & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{9}$$

For the state (9) the P-matrix (3) has the form

$$P_\mu = \begin{pmatrix} -\mu & 0 & 0 \\ 0 & -\mu & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{10}$$

In this case

$$RH(\vec{n}_1, \vec{n}_2) = 2\sqrt{1+\mu}\,, \quad \vec{n}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{n}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix},$$

$$\vec{a} = \vec{n}_1\,, \quad \vec{b} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{c} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{d} = \vec{n}_2\,.$$

5. In all the previous cases we considered pairs of correlated spin states in which both states of the pair had the definite values of spin projections at the same axis. Let us consider now a state $\Psi_{0,0}^{\vec{k}_1,\vec{k}_2}$ that is analog to the scalar state $\Psi_{0,0}$. It is formed by two states, one state has a definite value of spin projections at the axis defined by the vector $\vec{k}_1$, and the other state has a definite value of spin projections at the axis defined by the vector $\vec{k}_2$. In the case when

$$\vec{k}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{k}_2 = \begin{pmatrix} 0 \\ \sin\theta \\ \cos\theta \end{pmatrix},$$

the density matrix of such state has a form

$$\rho_\theta = \frac{1}{4}\begin{pmatrix} 2\sin^2\frac{1}{2}\theta & \sin\theta & -\sin\theta & 2\sin^2\frac{1}{2}\theta \\ \sin\theta & 2\cos^2\frac{1}{2}\theta & -2\cos^2\frac{1}{2}\theta & \sin\theta \\ -\sin\theta & -2\cos^2\frac{1}{2}\theta & 2\cos^2\frac{1}{2}\theta & -\sin\theta \\ 2\sin^2\frac{1}{2}\theta & \sin\theta & -\sin\theta & 2\sin^2\frac{1}{2}\theta \end{pmatrix}, \tag{11}$$

For the state (11) a P matrix (3) has the form

$$P_\theta = \begin{pmatrix} -\cos\theta & 0 & -\sin\theta \\ 0 & -1 & 0 \\ \sin\theta & 0 & -\cos\theta \end{pmatrix}. \tag{12}$$

For this state

$$RH(\vec{n}_1, \vec{n}_2) = 2\sqrt{2}, \quad \vec{n}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{n}_2 = \begin{pmatrix} \cos\alpha \\ 0 \\ \sin\alpha \end{pmatrix},$$

$$\vec{a} = \vec{n}_1, \quad \vec{b} = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos\alpha \\ 1 \\ \sin\alpha \end{pmatrix}, \quad \vec{c} = \frac{1}{\sqrt{2}}\begin{pmatrix} -\cos\alpha \\ 1 \\ -\sin\alpha \end{pmatrix}, \quad \vec{d} = \vec{n}_2.$$

Here α is an arbitrary angle.

6. Let us now consider the decoherence process that transforms a state (11) into a state described by a density matrix $\rho_{\theta\mu}$

$$\rho_{\theta\mu} = \frac{1}{4}\begin{pmatrix} 2\sin^2\frac{1}{2}\theta & \mu\sin\theta & -\mu\sin\theta & 2\mu\sin^2\frac{1}{2}\theta \\ \mu\sin\theta & 2\cos^2\frac{1}{2}\theta & -2\mu\cos^2\frac{1}{2}\theta & \mu\sin\theta \\ -\mu\sin\theta & -2\mu\cos^2\frac{1}{2}\theta & 2\cos^2\frac{1}{2}\theta & -\mu\sin\theta \\ 2\mu\sin^2\frac{1}{2}\theta & \mu\sin\theta & -\mu\sin\theta & 2\sin^2\frac{1}{2}\theta \end{pmatrix}, \tag{13}$$

For the state (13) a P matrix (3) has the form

$$P_{\theta\mu} = \begin{pmatrix} -\mu\cos\theta & 0 & -\mu\sin\theta \\ 0 & -\mu & 0 \\ \mu\sin\theta & 0 & -\cos\theta \end{pmatrix}. \tag{14}$$

If the angle $\theta = \frac{1}{4}\pi$ one can show that

$$RH(\vec{n}_1, \vec{n}_2) = 2\sqrt{1+\mu}\,, \quad \vec{n}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{n}_2 = \begin{pmatrix} \cos\alpha \\ 0 \\ \sin\alpha \end{pmatrix}, \quad \tan 2\alpha = \frac{2\mu}{1+\mu}\,, \quad \mu \neq 1\,,$$

$$\vec{a} = \vec{n}_1\,, \quad \vec{b} = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos\alpha \\ 1 \\ \sin\alpha \end{pmatrix}, \quad \vec{c} = \frac{1}{\sqrt{2}}\begin{pmatrix} -\cos\alpha \\ 1 \\ -\sin\alpha \end{pmatrix}, \quad \vec{d} = \vec{n}_2\,.$$

4. Conclusions

It was shown that with the help of the matrix P, that is a reduction of a density matrix, one can find the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$ such a way that the left-hand side of the generalized Bell-CHSH inequality (1) takes its maximum value.

References

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