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Coherent States on von Neumann Lattice and Some of their Applications

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Abstract. Coherent states on von Neumann lattice, which form complete but not overcomplete set, are orthogonalized using Löwdin procedure and corresponding coefficients arising in orthogonalization are tabulated. So obtained states may be interpreted as eigenstates of operators of optimal unsharp measurement of coordinate and momentum, in a slightly modified von Neumann sense. Using the obtained results the most important statistical characteristics of these operators are calculated on coherent states and compared with corresponding values of standard quantum mechanical operators of coordinate and momentum.

Keywords: coherent states, von Neumann lattice, unsharp measurements
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1. Introduction

A partitioning of phase space in rectangular pieces of the size of the Planck constant h — now known as the von Neumann lattice — was for the first time introduced by von Neumann himself in 1932 [1]. He used it as a ground to define quantum mechanical operators, completely in accord with the rigorous quantum mechanical rules for operators and their interpretation, for simultaneous measurement of quantum observables $\hat{q}$ and $\hat{p}$, with maximal accuracy allowed by the laws of quantum mechanics.

His idea was to bridge an enormous gap between quantum and classical mechanics present in the context of noncommutativity of these observables. Namely, in quantum mechanics it is assumed that it is impossible to measure noncommuting observables simultaneously and also, in an extreme interpretation, that they do not

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exist before observation and measurement and are really created in the act of the measurement itself. On the other hand, in the classical mechanics it is assumed that all quantities exist independently of whatever measurement and observation and, if they can be measured at all, they can be measured simultaneously with unlimited accuracy.

Von Neumann argued that in fact the accuracy of classical measurements never reaches the region where quantum mechanical uncertainty relations are relevant and that, due to this the classical standpoint may be based on strictly quantum mechanical foundations in the following way. Quantum mechanical simultaneously measurable operators $\hat{Q}$ and $\hat{P}$, generically related to the operators $\hat{q}$ and $\hat{p}$ may be defined so that their simultaneous measurement represents in some sense simultaneous measurement of their originators $\hat{q}$ and $\hat{p}$. None of them, of course, with absolute accuracy, but with maximal accuracy allowed in quantum mechanics for their simultaneous measurement. These unsharp measurements of $\hat{q}$ and $\hat{p}$ but exact measurement of $\hat{Q}$ and $\hat{P}$, in the world of objects of classical physics and accuracies of corresponding measurements, will be perceived as a simultaneous and exact measurement of classical coordinate and momentum.

Coherent states of a harmonic oscillator minimize uncertainty relations. In this sense in these states both coordinate and momentum are simultaneously defined with the best accuracy allowed in quantum mechanics. This is the reason why they are the best candidates for construction of eigenstates of operators for unsharp measurement of coordinate and momentum. Taken as they stand they cannot fulfill this role because of the two following reasons: they form not a complete but an overcomplete set in Hilbert space, and they are not orthogonal.

To remove the first obstacle von Neumann introduced the mentioned lattice in phase space and took from each cell in phase space one coherent state claiming that it was easy to see that so obtained set is complete but not overcomplete. Then he orthogonalized this set by the Gramm–Schmidt procedure.

Von Neumann's claim about completeness, considered as obvious by himself was for the first time rigorously proved almost forty years later by Perelomov, who was obliged to consider many related mathematical subtleties [2].

It seems that von Neumann's approach to simultaneous coordinate and momentum unsharp measurements is known less broadly so that for example in the well-known book from Omnes, about the interpretation of quantum mechanics, these results are not mentioned [3].

We considered it worthwhile to revisit the problem and to reaffirm von Neumann's ideas and in our first related paper we treated this problem on a general level and improving some particular points where it seemed desirable [4]. So, instead of the Gramm–Schmidt orthogonalization procedure used by von Neumann we used Löwdin's procedure in which all the states to be normalized enter the procedure on equal footing [5].

In the present work we further elaborate and concretize our earlier results. In the next section we describe shortly the mathematical background of our analysis. In Section 3 we give our more concrete results. So, we present tabulated coefficients obtained in the process of orthonormalization of coherent states on the lattice, using Löwdin's procedure. With their help, we analyze some main characteristics of the operators $\hat{Q}$ and $\hat{P}$ whose eigenfunctions are vectors of the orthogonalized basis obtained in this procedure. In the last section we shortly discuss our results.

2. Mathematical Background

Phase space is a natural framework for all our further considerations. The most convenient representation of coherent states in which they can be easily related to the q, p phase plane is

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \,.$$

In units $\hbar = m\omega = 1$, it holds

$$\langle\alpha|\hat{q}|\alpha\rangle = \sqrt{\frac{1}{2}}(\alpha + \alpha^*)\,,$$

$$\langle\alpha|\hat{p}|\alpha\rangle = -i\sqrt{\frac{1}{2}}(\alpha - \alpha^*)\,.$$

Hereafter, we identify the phase plane $q + ip$ with complex plane α, so that apart from the factor $\sqrt{2}$ average values of coordinate and momentum are equal to the real and complex part of α, respectively.

Consider two complex numbers ω_1 and ω_2 which in complex plane represent sides of a parallelogram with an area S. Perelomov [2] rigorously proved that the system of coherent states

$$|\alpha_{kl}\rangle = |k\omega_1 + l\omega_2\rangle\,,$$

where k and l are integers, depending on the value of S, fulfill the following conditions:

a) if $S < \pi$, the system $|\alpha_{kl}\rangle$ is overcomplete,
b) if $S > \pi$, the system is noncomplete,
c) if $S = \pi$, after exclusion of the one, and only one whichever vector from the system, the system becomes complete.

It is convenient to choose ω_1 and ω_2 such that $|\alpha_{kl}\rangle = |\sqrt{\pi}(k + \mathrm{i}l)\rangle$ and exclude $|\alpha = 0\rangle$. Let N be the matrix of scalar products

$$N_{IJ} = \langle\alpha_I|\alpha_J\rangle\,,$$

then, by Löwdin's procedure we have

$$|\tilde{\alpha}_I\rangle = \sum_J |\alpha_J\rangle (N^{-1/2})_{JI}\,,$$

$$\langle\tilde{\alpha}_K|\tilde{\alpha}_L\rangle = \delta_{K,L}\,.$$

Matrix elements of $N^{-1/2}$ can be calculated as

$$N = I + N', \qquad N^{-1/2} = I - \frac{1}{2}N' + \frac{3}{8}N'^2 - \dots$$

if the Taylor expansion converges.

Following von Neumann, apart from the normalization procedure, we define operators for unsharp measurement of coordinate and momentum

$$\hat{Q} = \sqrt{2}\mathrm{Re}\alpha_{kl} \sum_{\substack{kl \\ (k,l)\neq(0,0)}} |\tilde{\alpha}_{kl}\rangle\langle\tilde{\alpha}_{kl}| ,$$
$$\hat{P} = \sqrt{2}\mathrm{Im}\alpha_{kl} \sum_{\substack{kl \\ (k,l)\neq(0,0)}} |\tilde{\alpha}_{kl}\rangle\langle\tilde{\alpha}_{kl}| .$$

They commute, $[\hat{Q}, \hat{P}] = 0$. It is interesting to compare mean values of $\hat{q}$, $\hat{p}$, $\hat{q}^2$ and $\hat{p}^2$ for coherent states on the lattice against orthonormalized states $|\tilde{\alpha}_{kl}\rangle$. This is done in the next section together with the presentation of our other numerical results.

3. Results

We have calculated $|\tilde{\alpha}_{kl}\rangle$ for a finite square where $2 \leq k \leq 22$ and $2 \leq l \leq 22$. The most precise calculated $|\tilde{\alpha}_{k,l}\rangle$ is for $(k,l) = (12,12)$ since it is the furthest from the edges. The result for this vector is presented below:

$$|\tilde{\alpha}_{12,12}\rangle = \sum_{k=2}^{22}\sum_{l=2}^{22} c_{k,l}|\alpha_{k,l}\rangle =$$

$$\begin{aligned}
&+0.000390824|\alpha_{22,22}\rangle - 0.00159337|\alpha_{21,21}\rangle - 0.000788677|\alpha_{22,21}\rangle + 0.00369123|\alpha_{20,20}\rangle \\
&-0.0024213|\alpha_{21,20}\rangle + 0.00119618|\alpha_{22,20}\rangle - 0.00686279|\alpha_{19,19}\rangle - 0.00501711|\alpha_{20,19}\rangle \\
&-0.0032763|\alpha_{21,19}\rangle - 0.00161428|\alpha_{22,19}\rangle + 0.011458|\alpha_{18,18}\rangle - 0.00881945|\alpha_{19,18}\rangle \\
&+0.00639478|\alpha_{20,18}\rangle - 0.00415227|\alpha_{21,18}\rangle + 0.00203909|\alpha_{22,18}\rangle - 0.0181655|\alpha_{17,17}\rangle \\
&-0.0143018|\alpha_{18,17}\rangle - 0.0108608|\alpha_{19,17}\rangle - 0.00779765|\alpha_{20,17}\rangle - 0.00502953|\alpha_{21,17}\rangle \\
&-0.00246041|\alpha_{22,17}\rangle + 0.0284228|\alpha_{16,16}\rangle - 0.0224095|\alpha_{17,16}\rangle + 0.017272|\alpha_{18,16}\rangle \\
&-0.0129134|\alpha_{19,16}\rangle + 0.00917008|\alpha_{20,16}\rangle - 0.00587215|\alpha_{21,16}\rangle + 0.00286085|\alpha_{22,16}\rangle \\
&-0.0456351|\alpha_{15,15}\rangle - 0.0351894|\alpha_{16,15}\rangle - 0.0268201|\alpha_{17,15}\rangle - 0.0201845|\alpha_{18,15}\rangle \\
&-0.0148443|\alpha_{19,15}\rangle - 0.0104248|\alpha_{20,15}\rangle - 0.00662823|\alpha_{21,15}\rangle - 0.00321641|\alpha_{22,15}\rangle \\
&+0.07996|\alpha_{14,14}\rangle - 0.0577968|\alpha_{15,14}\rangle + 0.0420692|\alpha_{16,14}\rangle - 0.0309143|\alpha_{17,14}\rangle \\
&+0.0227285|\alpha_{18,14}\rangle - 0.0164639|\alpha_{19,14}\rangle + 0.0114492|\alpha_{20,14}\rangle - 0.00723513|\alpha_{21,14}\rangle \\
&+0.00349912|\alpha_{22,14}\rangle - 0.179714|\alpha_{13,13}\rangle - 0.107065|\alpha_{14,13}\rangle - 0.0691597|\alpha_{15,13}\rangle \\
&-0.0475942|\alpha_{16,13}\rangle - 0.0339262|\alpha_{17,13}\rangle - 0.0245031|\alpha_{18,13}\rangle - 0.0175565|\alpha_{19,13}\rangle \\
&-0.0121257|\alpha_{20,13}\rangle - 0.00763067|\alpha_{21,13}\rangle - 0.00368206|\alpha_{22,13}\rangle \underline{\underline{+1.21707|\alpha_{12,12}\rangle}} \\
&-0.264391|\alpha_{13,12}\rangle + 0.122833|\alpha_{14,12}\rangle - 0.0741996|\alpha_{15,12}\rangle + 0.0497779|\alpha_{16,12}\rangle
\end{aligned}$$

$-0.0350505|\alpha_{17,12}\rangle+0.0251447|\alpha_{18,12}\rangle-0.017944|\alpha_{19,12}\rangle+0.0123628|\alpha_{20,12}\rangle$
$-0.0077683|\alpha_{21,12}\rangle+0.00374546|\alpha_{22,12}\rangle\ +\ \text{s.t.}\,,$

where the symmetrical terms, s.t. for short, can be found from the symmetry properties

$$
\begin{aligned}
&c_{12-a,12-b} = c_{12-a,12+b} = c_{12+a,12-b} = c_{12+a,12+b}\,,\\
&c_{12-b,12-a} = c_{12-b,12+a} = c_{12+b,12-a} = c_{12+b,12+a}\,,\\
&-10 \le a, b \le 10\,.
\end{aligned}
$$

If one needs a orthonormalized vector for another node in the lattice, then another square should be set. Such a node should be in the center of the square. Orthonormalization should be performed for the new square again.

Using the above development we calculated average values in this state for the operators $\hat{q}$, $\hat{p}$, $\hat{q}^2$, $\hat{p}^2$ and compared them with the average values of the same operators in the corresponding coherent state. The results are presented below.

$$
\begin{aligned}
\langle\alpha_{12,12}|\hat{q}|\alpha_{12,12}\rangle &= 30.079539295572\\
\langle\tilde{\alpha}_{12,12}|\hat{q}|\tilde{\alpha}_{12,12}\rangle &= 30.079539295572\\
\hline
\langle\alpha_{12,12}|\hat{p}|\alpha_{12,12}\rangle &= 30.079539295572\\
\langle\tilde{\alpha}_{12,12}|\hat{p}|\tilde{\alpha}_{12,12}\rangle &= 30.079539295572\\
\hline
\langle\alpha_{12,12}|\hat{q}^2|\alpha_{12,12}\rangle &= 905.27868423386\\
\langle\tilde{\alpha}_{12,12}|\hat{q}^2|\tilde{\alpha}_{12,12}\rangle &= 905.95240153106\\
\hline
\langle\alpha_{12,12}|\hat{p}^2|\alpha_{12,12}\rangle &= 905.27868423386\\
\langle\tilde{\alpha}_{12,12}|\hat{p}^2|\tilde{\alpha}_{12,12}\rangle &= 905.95240153106.
\end{aligned}
$$

We see that up to our precision, which is very high, $\hat{q}$ and $\hat{p}$ have the same average values both in $|\alpha_{12,12}\rangle$ and $|\tilde{\alpha}_{12,12}\rangle$. We guess that this equality should be exact, but were not able to prove this yet. On the other hand, average values of squares of the operators are somewhat different in α-s and $\tilde{\alpha}$-s but are very close still. Also, average values of $\hat{p}$ and $\hat{q}$ are the same as average values of $\hat{P}$ and $\hat{Q}$ in $\tilde{\alpha}$ states.

Comparisons between $\langle\alpha|\hat{q}|\alpha\rangle$, $\langle\alpha|\hat{p}|\alpha\rangle$, $\langle\alpha|\hat{q}^2|\alpha\rangle$, $\langle\alpha|\hat{p}^2|\alpha\rangle$, and $\langle\alpha|\hat{Q}|\alpha\rangle$, $\langle\alpha|\hat{P}|\alpha\rangle$, $\langle\alpha|\hat{Q}^2|\alpha\rangle$, $\langle\alpha|\hat{P}^2|\alpha\rangle$ for

$$
\begin{aligned}
11.5\sqrt{\pi} \le \mathrm{Re}\alpha \le 12.5\sqrt{\pi},\\
11.5\sqrt{\pi} \le \mathrm{Im}\alpha \le 12.5\sqrt{\pi},
\end{aligned}
$$

are presented in Figs. 1–4.

As it should be expected these results are the closest to each other at the nodes of the lattice. In other points of the α plane these differences are slightly greater but not substantially.

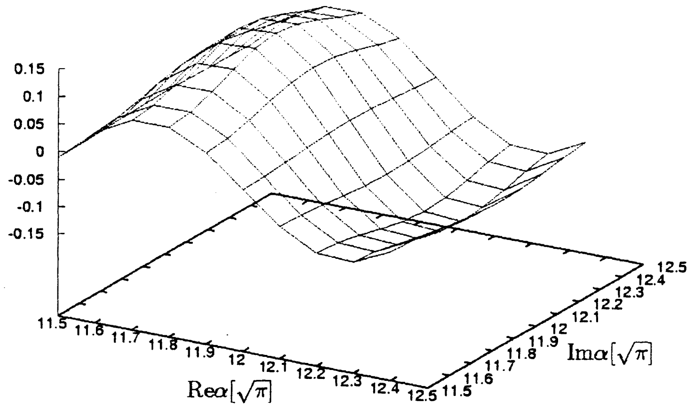

Fig. 1. Variations of $(\langle\alpha_{kl}|\hat{Q}|\alpha_{kl}\rangle - \langle\alpha_{kl}|\hat{q}|\alpha_{kl}\rangle)/\sqrt{2}$ in the cell of the size h

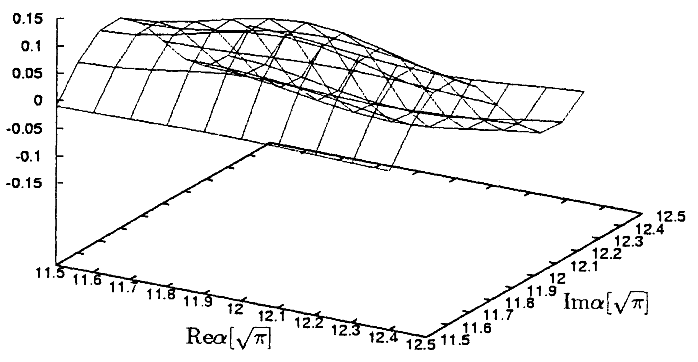

Fig. 2. Variations of $(\langle\alpha_{kl}|\hat{P}|\alpha_{kl}\rangle - \langle\alpha_{kl}|\hat{p}|\alpha_{kl}\rangle)/\sqrt{2}$ in the cell of the size h

4. Conclusions

We can conclude that von Neumann's approach to construction of operators for simultaneous unsharp measurement of coordinate and momentum and his results, which were mainly obtained due to his enormously strong both physical and mathematical intuition, in historical times when general intellectual ambient was not much receptive for such a results, are still not so well known as they deserve. They have been confirmed theoretically, with the greatest mathematical rigor by Perelomov [2]. We believe that our results give further practical support to this approach. Also, the partitioning of phase space in cells of the size h, so often used in all quasi-classical consideration mainly on intuitive grounds, may be now considered as fully founded, both theoretically and practically.

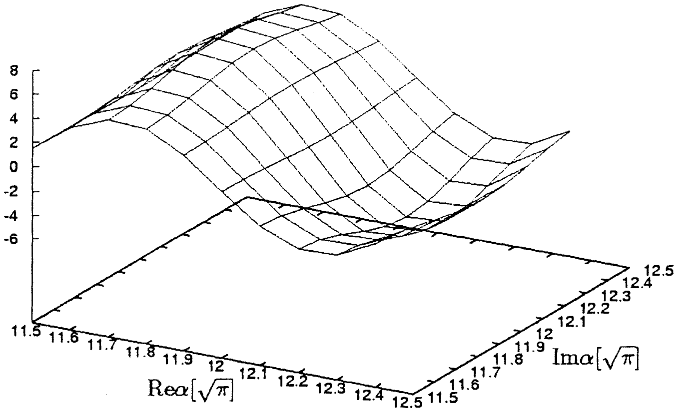

Fig. 3. Variations of $(\langle\alpha_{kl}|\hat{Q}^2|\alpha_{kl}\rangle - \langle\alpha_{kl}|\hat{q}^2|\alpha_{kl}\rangle)/2$ in the cell of the size h

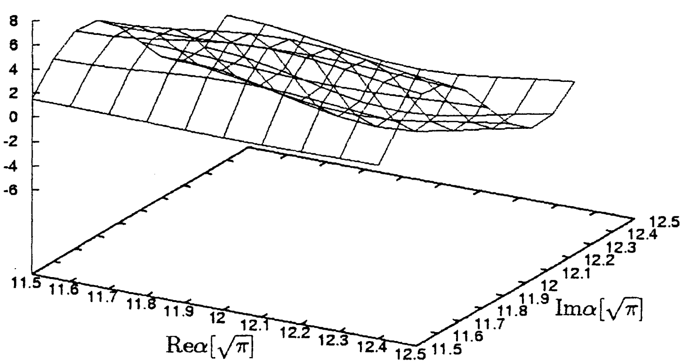

Fig. 4. Variations of $(\langle\alpha_{kl}|\hat{P}^2|\alpha_{kl}\rangle - \langle\alpha_{kl}|\hat{p}^2|\alpha_{kl}\rangle)/2$ in the cell of the size h

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References

1. J. von Neumann, *Mathematische Grundlagen der Quantenmechanik*, Die Grundlagen der mathematischen Wissenschaften, Band XXXVIII, Springer, Berlin, 1932.
2. A.M. Perelomov, *Teor. Matemat. Fizika* **6** (1971) 213.

3. R. Omnès, *The Interpretation of Quantum Mechanics*, Princeton Series in Physics, 1994.
4. D.M. Davidović and D. Arsenović, *Fortschritte der Physik - Progress of Physics* **51**/2-3 (2003) 92.
5. J. Löwdin, *J. Chem. Phys.* **18** (1950) 365.