Acta Phys. Hung. A 25/1 (2006) 47–56
DOI: 10.1556/APH.25.2006.1.4

HEAVY ION PHYSICS

Description of the Smooth *rms* Charge Radius Surface $R(N, Z)$ and Prediction of Neutron Skin by a Two-Liquid Drop Model

I. Angeli

Institute of Experimental Physics, University of Debrecen
H-4010 Debrecen, Pf. 105, Hungary; E-mail: angeli@tigris.unideb.hu

Received 11 March 2005

Abstract. The smooth N and Z dependence of nuclear *rms* charge radii is interpreted by a two-liquid drop model with $\rho_{\rm p} + \rho_{\rm n} = \rho_0 = \text{const}$. Proton and neutron radii $R_{\rm p}$ and $R_{\rm n}$ are given in closed form. In addition to *rms* charge radii, the model yields the nucleon number dependence of the *skin thickness*, s, e.g. $ds = 0.00124 \times dA$ for Sn isotopes, in agreement with experimental results and theoretical calculations. A strong correlation between the two global parameters of the model — including s_{st} the skin thickness of the stable isotope — is established. If s_{st} is taken from experiment, the other parameter is also fixed; this parameter (m) characterizes the restoring force responsible for the skin thickness. Its value $m = 2.2$ suggests that — in addition to the number of nucleons in the skin — the skin thickness also influences the distribution of "surplus" nucleons between volume and skin.

Keywords: nuclear *rms* charge radii, nucleon skins and densities, nuclear models
PACS: 21.10.Ft, 21.10.Gv, 21.60.n

1. Introduction

Present-day nuclear *rms* charge radius data render it possible to investigate the properties of the smooth *rms charge radius surface* $R(N, Z)$ by fitting simple empirical functions to the experimental data along isotopic, isotonic and isobaric series [1]. One result of the study is that the relative isotopic and isotonic dependence can be described by

$$R = R_0 \left(\frac{A}{A_0} \right)^k , \tag{1}$$

1219-7580/ $ 20.00

where $R_0 \equiv R(A_0)$ is the *rms* charge radius of the reference isotope, $k_Z = 0.156$ for isotopic series, $k_N = 0.478$ for isotonic series, see Table 3 of [1]. For isobaric series we have:

$$R = R_0 \left[1 + f\,(I - I_0)\right] , \qquad I \equiv \frac{N-Z}{A} , \tag{2}$$

with $f \approx -0.161$. These simple formulae average out the shell and deformation effects. The smooth nucleon number dependence makes possible the application of simple models for interpretation. In this paper a two-liquid drop model is presented, which reproduces the empirically found smooth structure of the radius surface $R(N, Z)$, and — in addition — gives quantitative prediction on the nucleon number dependence of the *skin thickness*. This may be of importance in view of the growing number of present-day experimental results, see e.g. [2–4] and references therein. In Section 2 the model is described and its parameters determined. Simple closed formulae are given for the radii of proton and neutron distribution. In Section 3 results are presented: *rms* charge radii and skin thickness for $Z = 50$. The empirical exponent k_Z in (1) is expressed in terms of model quantities. The density ratio $\rho_\mathrm{p}/\rho_\mathrm{n}$ is given as a function of Z and N. The slope ds/dN characterizing the N dependence of the skin is given in closed form. In Section 4 a summary is given and limitations of the model are discussed.

2. The Two-Liquid Drop Model

2.1. Description of the model

Along the valley of stability, experimental radii can be approximated by the radius formula of the traditional liquid drop model:

$$R_{st} = r_0 A_{st}^{1/3} = \left(\frac{3}{4\pi\rho_0}\right) A_{st}^{1/3}, \tag{3}$$

where the radius parameter $r_0 = 1.231$ is expressed by the constant density ρ_0. In this paper an *average density* $\rho_0 = 0.128$ nucleon/fm^3 is used as obtained from fitting the liquid drop formula to stable nuclei, see Table 2 of [1]. (The higher value $\rho_\mathrm{max} \approx 0.16$ nucleon/fm^3, as often found in the literature, refers to the *maximum density* of a Fermi distribution fitted to measurements on the heaviest nuclei [5]). Formula (3) will be generalized for use in the two-liquid drop model, which also works with uniform density distributions, but separately for protons (ρ_p) and neutrons (ρ_n), with radii

$$R_\mathrm{p} = \left(\frac{3}{4\pi\rho_\mathrm{p}}\right)^{1/3} Z^{1/3} \qquad \text{and} \qquad R_\mathrm{n} = \left(\frac{3}{4\pi\rho_\mathrm{n}}\right)^{1/3} N^{1/3} , \tag{4}$$

respectively. The sum of densities is assumed to be constant:

$$\rho_\mathrm{p} + \rho_\mathrm{n} = \rho_0 = \text{const.} , \tag{5}$$

except for the skin region, where the volume density $\rho_{\rm n}$ (or $\rho_{\rm p}$) continues. Alternatively, the radii $R_{\rm p}$ and $R_{\rm n}$ can be expressed also by ρ_0 and $\rho_{\rm p}/\rho_{\rm n}$ using Eqs. (4) and (5):

$$R_{\rm p} = \left[\frac{3Z}{4\pi\rho_0}\left(1+\frac{1}{\rho_{\rm p}/\rho_{\rm n}}\right)\right]^{1/3}, \qquad R_{\rm n} = \left[\frac{3Z}{4\pi\rho_0}\left(1+\frac{\rho_{\rm p}}{\rho_{\rm n}}\right)\right]^{1/3}. \tag{6}$$

These latter forms prove to be more useful than (4).

The difference $S \equiv R_{\rm n} - R_{\rm p}$ is the *skin parameter*. The experimentally measurable *skin thickness* is the difference of the *rms radii*; for uniform distributions the simple relation holds:

$$s = \left\langle r^2\right\rangle_{\rm n}^{1/2} - \left\langle r^2\right\rangle_{\rm p}^{1/2} = \sqrt{\frac{3}{5}}\,(R_{\rm n} - R_{\rm p}) = \sqrt{\frac{3}{5}}S.$$

For nuclei *along the stability line*, the ratio of proton and neutron radii will be approximated by

$$\frac{R_{{\rm p},st}}{R_{{\rm n},st}} \approx \frac{R_{st} - \frac{1}{2}S_{st}}{R_{st} + \frac{1}{2}S_{st}}, \tag{7}$$

and the ratio of densities

$$\frac{\rho_{{\rm p},st}}{\rho_{{\rm n},st}} = \frac{Z_{st}V_{{\rm n},st}}{N_{st}V_{{\rm p},st}} \approx \frac{Z_{st}}{N_{st}}\left(\frac{R_{st} - \frac{1}{2}S_{st}}{R_{st} + \frac{1}{2}S_{st}}\right)^3, \tag{8}$$

where

$$Z_{st} = \frac{A_{st}}{\left(1.98 + 0.016A_{st}^{2/3}\right)}, \qquad N_{st} = A_{st} - Z_{st}\,. \tag{9}$$

Away from stability, some of the "surplus" nucleons, e.g. neutrons are pushed into the volume and shift the proton/neutron density ratio from $\rho_{{\rm p},st}/\rho_{{\rm n},st}$ to a new value $\rho_{\rm p}/\rho_{\rm n}$, while others increase the skin from S_{st} to a new value $S > S_{st}$. Owing to the mutual proton–neutron attraction, there is a tendency to minimize these changes. Therefore, the requirement is raised that the deviation of the actual density ratio $\rho_{\rm p}/\rho_{\rm n}$ from $\rho_{{\rm p},st}/\rho_{{\rm n},st}$ and — simultaneously — the deviation of the mth power of the ratio $R_{\rm p}/R_{\rm n}$ from its stability value be a minimum:

$$\left(\frac{\rho_{\rm p}}{\rho_{\rm n}} - \frac{\rho_{{\rm p},st}}{\rho_{{\rm n},st}}\right)^2 + \left[\left(\frac{R_{\rm p}}{R_{\rm n}}\right)^m - \left(\frac{R_{{\rm p},st}}{R_{{\rm n},st}}\right)^m\right]^2 = \text{min.} \tag{10}$$

The value $m = 1$ corresponds to minimization against change in skin thickness, while $m = 3$ to that against change in skin volume, i.e. in nucleon number in the skin. Note that $R_{\rm p}/R_{\rm n}$ is determined by $\rho_{\rm p}/\rho_{\rm n}$ through Eqs. (4):

$$\frac{R_{\rm p}}{R_{\rm n}} = \left(\frac{Z/N}{\rho_{\rm p}/\rho_{\rm n}}\right)^{1/3}. \tag{11}$$

Performing the minimization according to (10), and having now the value $\rho_{\rm p}/\rho_{\rm n}$ for the given nuclide, $R_{\rm p}$ and $R_{\rm n}$ can be expressed by $\rho_{\rm p}/\rho_{\rm n}$ using Eqs. (6).

2.2. Determination of the model parameters m and S_{st}

The density ratio $\rho_{\rm p}/\rho_{\rm n}$ and — consequently — the radii $R_{\rm p}$ and $R_{\rm n}$ obtained in Eq. (6) depend on the parameters m and S_{st} through (10). To determine these parameters, *model rms charge radii* R_m were calculated using

$$R_m \equiv \left\langle r^2 \right\rangle_{ch,m}^{1/2} = \sqrt{\frac{3}{5}} R_{\rm p} \tag{12}$$

and compared to the empirical radii from Eq. (1) along each of the isotopic series $Z = 10$, 20, 38, 50, 66 and 82. The deviation was measured by the simple, non-weighted standard deviation, which is also a function of the two parameters: $SD(m, S_{st})$. Varying the values of these parameters, the procedure was repeated from (10) to find the minimum SD. The main results of the search are:

- The parameters m and S_{st} are strongly correlated: the left side of Fig. 1 shows the loci of the bottom of the $SD(m, s_{st})$ valleys for the different elements (from here on $s_{st} = \sqrt{3/5}\, S_{st}$ is used). Along these lines the SD does not change significantly; on the other hand, along intersections with constant m or s_{st}, the minimum is well determined.

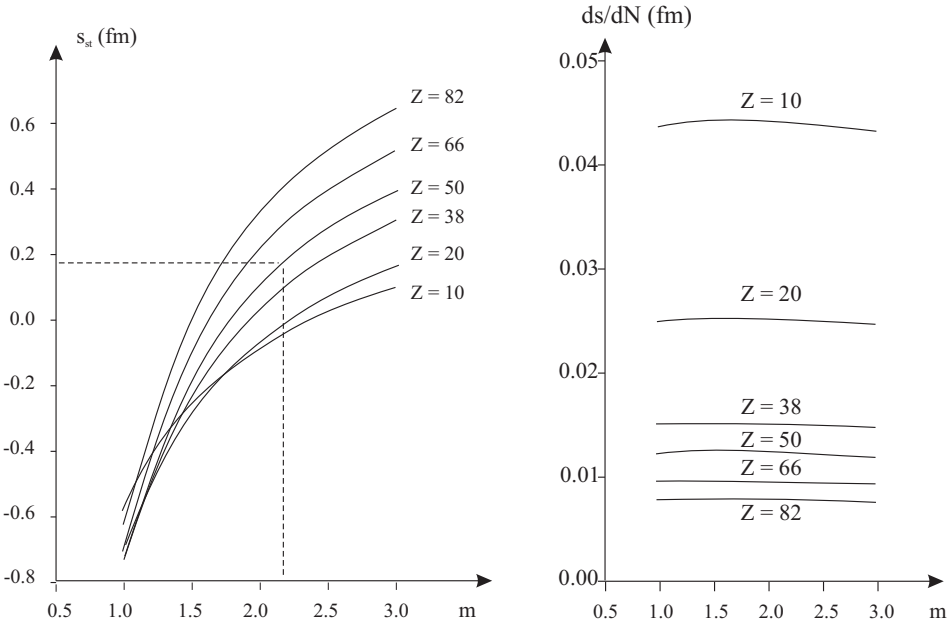

Fig. 1. Left side: correlation of the parameters m and s_{st} for different isotopic series. The horizontal dashed line shows the experimental s_{st} value for $Z = 50$ [4]. Right side: the slope ds/dN of the neutron number dependence of the skin thickness, s, as a function of the model parameter m

- Taking one of the parameters from experiment, e.g. $s_{st,\mathrm{exp}} = 0.181(8)$ fm for $Z = 50$ [4] (horizontal dashed line), the remaining parameter m is also fixed within quite narrow limits: $m = 2.20(2)$ (vertical dashed line). Assuming this m value, s_{st} for other elements can be read from the figure. The value $m = 2.2$ suggests that it is not only the number of nucleons in the skin region but also the skin thickness that determine the restoring force responsible for the distribution of surplus nucleons between volume and skin.

Using these parameter values m and s_{st} in (10), the set of the resulting $\rho_\mathrm{p}/\rho_\mathrm{n}$ ratios for $Z = 50$ isotopes shows an almost linear dependence on the neutron number, Fig. 2:

$$\frac{\rho_\mathrm{p}}{\rho_\mathrm{n}} \approx a_0 + a_1 \times dN\,, \qquad\qquad dN \equiv N - N_{st}\,. \tag{13}$$

The Z dependence of a_0 and a_1 will be determined in the next section.

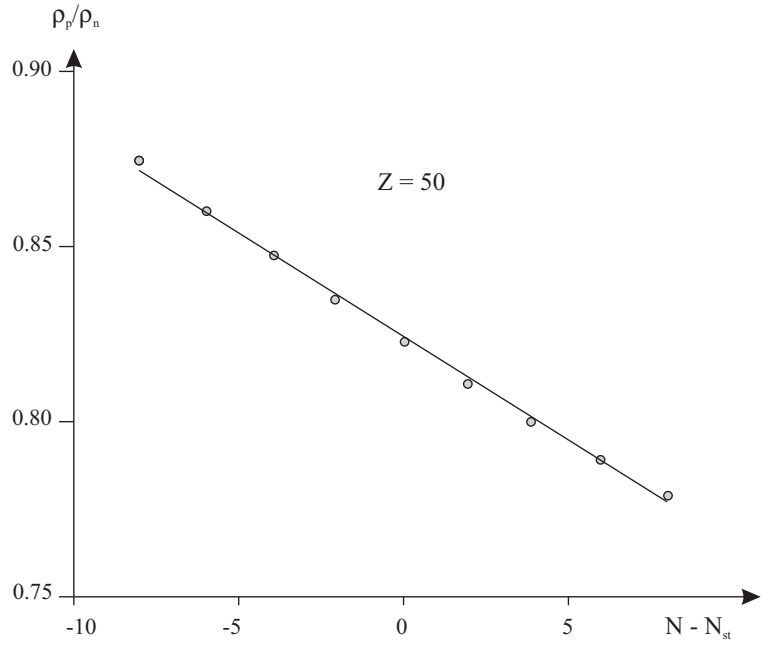

Fig. 2. The ratio of proton and neutron densities $\rho_\mathrm{p}/\rho_\mathrm{n}$ as a function of $N - N_{st}$ for $Z = 50$ derived by the two-liquid drop model

3. Results

This model was formulated with the intention of interpreting the smooth nucleon number dependence of *rms* charge radii. However, there are also some interesting by-products, which also deserve attention. These are: the expression of the empirical exponent k_Z by model quantities, the dependence of the density ratio $\rho_\mathrm{p}/\rho_\mathrm{n}$

on Z and dN, and finally, the neutron number dependence of the skin thickness s within an isotopic series.

3.1. Rms charge radii

Figure 3 shows *rms* charge radii calculated by the model (12), and by the empirical formula (1) applied to the isotopic series $Z = 50$ with reference mass number $A_0 = 118$. The two data sets are practically indistinguishable, the standard deviation being $SD = 0.0002$ fm. This means that the two-liquid drop model describes the neutron number dependence of the smooth charge radius surface $R(N, Z)$ with very good approximation. For comparison, the trend calculated by the $\propto A^{1/3}$ rule of the traditional liquid drop model is also shown.

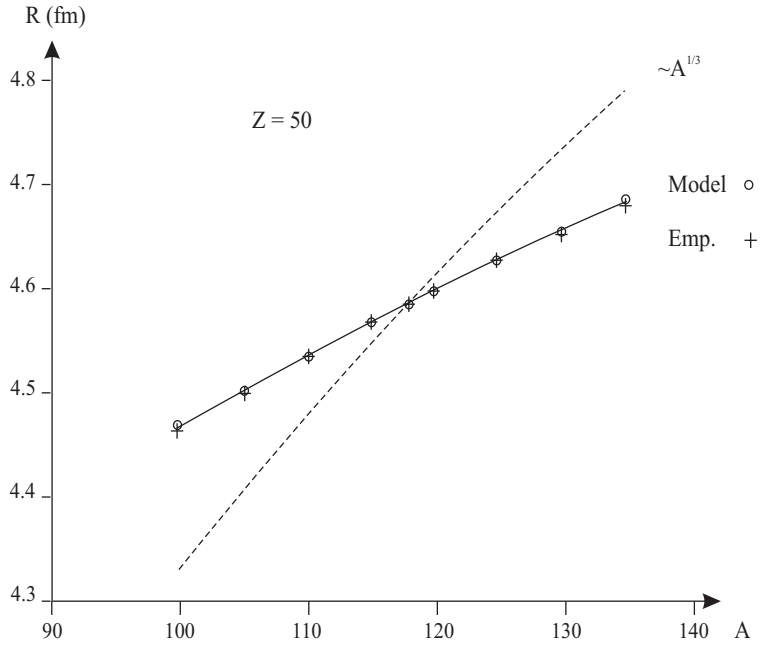

Fig. 3. *Rms* charge radii for $Z = 50$ calculated by the two-liquid drop model (circles), and by the empirical formula (1) relative to $A_0 = 118$ (crosses); the continuous line guides the eye. The dashed line shows an $\propto A^{1/3}$ dependence

3.2. Interpretation of the empirical exponent k_Z by model quantities

Recalling Eq. (13), the proton and neutron radii (6) can be written in the form:

$$R_{\mathrm{p}} = r_0 Z^{1/3} \left(1 + \frac{1}{a_0 + a_1 dN}\right)^{1/3}, \qquad R_{\mathrm{n}} = r_0 N^{1/3} \left(1 + a_0 + a_1 dN\right)^{1/3}. \quad (14)$$

Note that $a_0 = \rho_{\mathrm{p},st}/\rho_{\mathrm{n},st}$. Relating the radius R_p to that for the stable isotope ($dN = 0$) of the series:

$$\frac{R_\mathrm{p}}{R_{\mathrm{p},st}} = \left(\frac{1 + \dfrac{1}{a_0 + a_1 dN}}{1 + \dfrac{1}{a_0}} \right)^{1/3} = 1 - \frac{1}{3} \frac{a_1}{a_0(1+a_0)} dN + \ldots \tag{15}$$

and doing the same with the empirical Eq. (1), with $A_0 = A_{st}$:

$$\frac{R}{R_{st}} = \left(1 + \frac{dN}{A_{st}} \right)^{k_Z} = 1 + \frac{k_Z}{A_{st}} dN + \ldots \tag{16}$$

from the comparison of these equations we have.

$$k_Z = -\frac{A_{st}}{3} \frac{a_1}{a_0(1+a_0)}\,, \tag{17}$$

i.e. the exponent k_Z obtained from the fit to experimental data in [1] can be expressed by the quantities a_0 and a_1 that determine the nucleon density ratio $\rho_\mathrm{p}/\rho_\mathrm{n}$.

3.3. Dependence of the nucleon density ratio $\rho_\mathrm{p}/\rho_\mathrm{n}$ on Z and dN

The nucleon density ratio $\rho_\mathrm{p}/\rho_\mathrm{n}$ is determined by the quantities a_0 and a_1, which depend on Z. The leading term $a_0(Z)$ can be calculated by (8), where s_{st} may be taken from Fig. 1 (left) assuming $m = 2.2$. However, it is more convenient to use the formula

$$a_0(Z) \approx \frac{0.93}{Z^{0.032}}, \tag{18}$$

which is a fairly good approximation. Rearranging (17), the smaller term $a_1(Z)$ is:

$$a_1(Z) = -3 \frac{k_Z}{A_{st}} a_0(1+a_0)\,. \tag{19}$$

Figure 4 shows the Z dependence of a_0 and a_1. The empirical exponent k_Z has been determined in [1] by taking into account all isotopic series longer than three isotopes. It may be regarded as an average on all series, therefore, its value does not depend on Z. Using the Z-dependent coefficients $a_0(Z)$ and $a_1(Z)$ in (13) and (14), the density ratio $\rho_\mathrm{p}/\rho_\mathrm{n}$ — and R_p and R_n — can be calculated for the isotopes of various elements.

3.4. Skin thickness

An important by-product of the model is the skin thickness parameter: $S \equiv R_\mathrm{n} - R_\mathrm{p}$, and consequently, the experimentally measurable $s = \sqrt{3/5}\, S$. Figure 5 shows the mass number dependence of s for $Z = 50$ calculated by the model with three

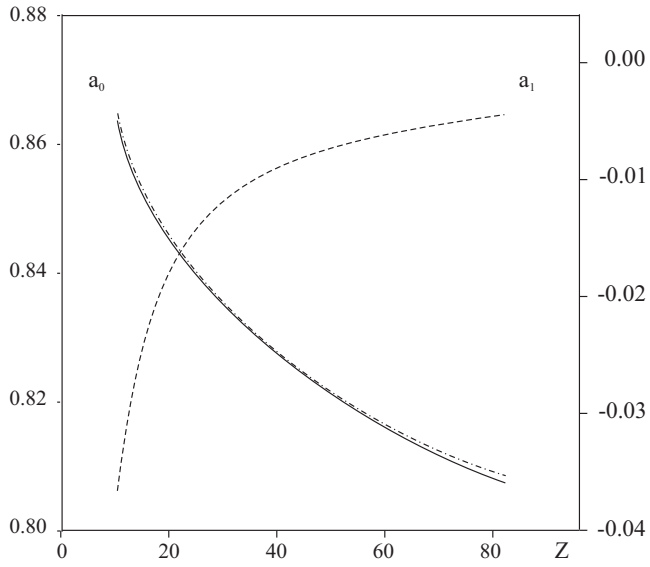

Fig. 4. Z dependence of the density parameters a_0 (scale left) and a_1 (scale right), see Eq. (13). Continuous line: $a_0(Z)$ from (8), dashed-dotted line: $a_0(Z)$ from (18), dashed line: $a_1(Z)$ from (19)

different m values (continuous lines). For comparison, several experimental and theoretical results on the skin for tin isotopes are also plotted. The lines are well approximated by a linear equation:

$$s = \text{const.} + 0.0124 \times (A - 118)\,. \tag{20}$$

It is important to note that for a given Z the slope ds/dN is practically the same in the wide range of the model parameter value m (see Fig. 1, right side), i.e. this slope is well determined by the mass number dependence of the radius surface $R(N, Z)$, independently of the model parameters. Starting from Eqs. (14), the slope ds/dN can be expressed in a closed form:

$$\left(\frac{ds}{dN}\right)_{st} = \frac{R_{\mathrm{n},st}}{\sqrt{15}}\left[\frac{1}{N_{st}} + \frac{a_1}{1+a_0}\left(1 + \frac{1}{a_0}\frac{R_{\mathrm{p},st}}{R_{\mathrm{n},st}}\right)\right], \tag{21}$$

where the subscript st refers to the stablest isotope of the series.

On the other hand, the *constant* in Eq. (20) is not determined, its value should be taken from a separate source of information, e.g. from experimental skin measurements. It can be seen from Fig. 5 that the best fit between model and experiment is achieved with $m = 2.2$; even small changes in m give rise to significant changes in s. Assuming this m value, Fig. 1, left side, suggests that for light nuclei a negative neutron skin, i.e. a *proton* skin is expected, while for medium and heavy nuclei an

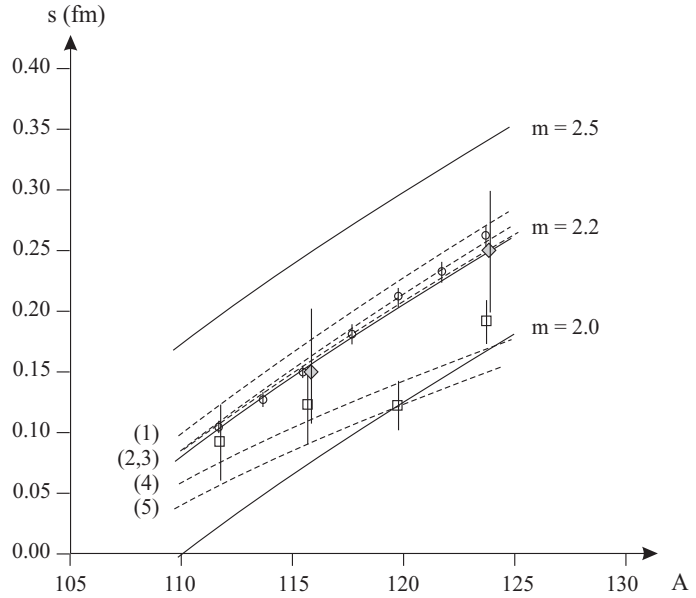

Fig. 5. Neutron number dependence of the skin for $Z = 50$ resulting from the two-liquid drop model calculated with $m = 2.0, 2.2$ and 2.5, respectively (continuous lines). Note that the slopes ds/dN do not depend on the model parameter m. Symbols are for experimental data: from GDR and SDR methods (circles [4]), from (p,p) scattering and from antiprotons (rhombi and rectangles [6]). Dashed lines represent the results of theoretical calculations: (1) RHB/NL3, (3) RHB/NLSH, (4) HFB/SLy4, (5) HFB/SkP by Mizutori et al. [7], and (2) by Lalazissis et al. [8]

increasing positive neutron skin is predicted. This agrees with earlier theoretical results, see Tables 2.1 and 2.2 of [9], pp. 28–29 and 46–47.

4. Summary and Discussion

The smooth nucleon number dependence of *rms* charge radii — as found empirically in [1] and recalled in Eqs. (1) and (2) — has been interpreted by a two-liquid drop model with uniform nucleon densities ρ_p and ρ_n. The ratio of these densities depends linearly on mass number, Eq. (13). Outputs of the model are the proton and neutron radii R_p and R_n of the nucleon distributions, which are given in closed form (6), and can be calculated using Eqs. (14), (18) and (19). From the proton radius parameter *rms* charge radii can be calculated by Eq. (12). These model values are compared to those derived from the empirical formulae (1) and (2), see Fig. 3. The two data sets practically coincide, and differ significantly from the traditional liquid drop rule $\propto A^{1/3}$.

According to the model, the occurrence of the nucleon skin is rather a rule than an exception. For a fixed Z value, the skin thickness s depends approximately linearly on the neutron number. For the isotopic series of tin $ds = 0.0124 \times dA$, Fig. 5. This agrees with the results of experiments and of theoretical calculations. The slope $ds/dA = ds/dN$ is fairly insensitive to the values of model parameters m and s_{st} of Eq. (10), see Fig. 1, right side. These parameters are strongly correlated, as can be seen in Fig. 1, left side. One of these parameters m determines the distribution of "surplus" nucleons between volume and skin. Its value $m = 2.2$ suggests that it is not exclusively the number of nucleons in the skin but also the skin thickness that determines the distribution of surplus nucleons between volume and skin.

The limitations of the model arise from two sources. Firstly, the *model assumption* of uniform density distribution is not justified for light nuclei. Secondly, the *approximations* in Eqs. (7) and (8) may cause systematic errors for nuclei, where $|S_{st}|$ is not small compared to R_{st}. Accepting $m = 2.2$, this occurs for the lightest and heaviest nuclei, see Fig. 1, left side.

References

1. I. Angeli, *Atomic Data and Nuclear Data Tables* **87** (2004) 185.
2. A. Krasznahorkay et al., *Phys. Rev. Lett.* **82** (1999) 3216.
3. R. Schmidt et al., *Phys. Rev.* **C67** (2003) 44308.
4. A. Krasznahorkay et al., *Nucl. Phys.* **A731** (2004) 224, and private communication.
5. L.R.B. Elton, *Nuclear Sizes*, Oxford University Press, 1961.
6. C. Batty et al., *Adv. Nucl. Phys.* **19** (1989) 1.
7. S. Mizutori et al., *Phys. Rev.* **C61** (2000) 44326.
8. G.A. Lalazissis et al., *Phys. Rev.* **C57** (1998) 2294.
9. R.C. Barrett and D.F. Jackson, *Nuclear Sizes and Structure*, Clarendon Press, Oxford, 1977.