Acta Phys. Hung. B 23/3–4 (2005) 151–162

QUANTUM ELECTRONICS

Energy–Momentum Tensor for Quasimonochromatic and Quasiplane Waves in Dispersive Media and its Equivalence to a Homogeneous Particle Flow*

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Received 4 March 2005

Abstract. The energy–momentum tensor for quasiplane and quasimonochromatic waves in homogeneous anisotropic dispersive media and an action four-vector in a conservation theorem are calculated in relativistic covariant way. The obtained form of the energy–momentum tensor is similar to that for a homogeneous particle flow in classical mechanics but with a much greater variety of possible dispersion laws for momentum in dependence on group velocity. A relation to quantization is obtained in a very natural way. The nonsymmetry of the obtained energy–momentum tensor and its nonuniqueness create some problems for Einstein's gravitation equations.

Keywords: spatial dispersion, action flow density and action density, conservation law, group velocity, quantization, Einstein's gravitation equations
PACS: 03.50.De, 04.20.Cv, 41.20.Ib, 42.25.Bs

* Based on the talk held at the *11th Central European Workshop on Quantum Optics (CEWQO11)*, 18–20 July 2004, Trieste, Italy.

1. Introduction

The energy–momentum tensor belongs to the few central notions where some of the fundamental concepts of modern physics that are, in particular, particles and fields in the special theory of relativity and conservation theorems, the quantum theory and Einstein's gravitation theory (general theory of relativity) are interwoven in a wonderful and sometimes surprising way, however, also with open problems.

The second paper of Einstein to special theory of relativity [1] deprived the mass from its fundamental character as a conservation quantity and related it to

1589-9535/ $ 20.00

the energy which together with the momentum forms a four-vector and has to be transformed from one to another inertial system according to a Lorentz transformation. In the differential form of conservation of energy and momentum this leads with inevitability to the energy–momentum tensor with relativistic covariant transformation properties.

Aim of the work is the relativistic covariant derivation of the energy–momentum tensor for homogeneous anisotropic media with spatial (wave-vector) and temporal (frequency) dispersion (for spatial dispersion, see, e.g. [2–5]) and to reduce it to a form similar to that for moving mass densities in classical mechanics and hydrodynamics but with uncommon mechanical properties. In connection with action conservation which we formulate this leads to a deep relation to quantum theory. Furthermore, we will show some of its consequences for Einstein's gravitation equations.

2. Basic Equations of Macroscopic Linear Electrodynamics of Dispersive Media

Some four- and three-dimensional (space-time) notations:

four-dimensional: $r = (\boldsymbol{r}, t)$, $k = (\boldsymbol{k}, \omega)$ or $r_\lambda = (r_l, r_4 = \mathrm{i}ct)$, $k_\lambda = (k_l, k_4 = \mathrm{i}\omega/c)$, $\nabla_\lambda = (\nabla_l, \nabla_4 = -(\mathrm{i}/c)\partial/\partial t)$ with no distinction of lower and upper indices; scalar products $kr \equiv k_\mu r_\mu \equiv \boldsymbol{kr} - \omega t$;

three-dimensional: scalar products $\boldsymbol{ab}$ (without point between three-dimensional vectors which are boldface), vector product $\boldsymbol{c} = [\boldsymbol{a}\times\boldsymbol{b}]$ or in vector indices $c_k = \epsilon_{klm} a_l b_m$, ϵ_{klm} Levi-Civita pseudo-tensor; relation between second-rank antisymmetric tensors $c_{lm} = -c_{ml}$ and equivalent axial vectors $c_k = \frac{1}{2}\epsilon_{klm} c_{lm}$, $c_{lm} = \epsilon_{lmn} c_n$.

Our starting equations are Maxwell's equations of macroscopic electrodynamics

$$\begin{aligned} &[\boldsymbol{\nabla}\times\boldsymbol{E}(\boldsymbol{r},t)] + \frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{B}(\boldsymbol{r},t) = \boldsymbol{0}\,, \qquad \boldsymbol{\nabla B}(\boldsymbol{r},t) = 0\,, \\ &[\boldsymbol{\nabla}\times\boldsymbol{B}(\boldsymbol{r},t)] - \frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{D}(\boldsymbol{r},t) = \boldsymbol{0}\,, \qquad \boldsymbol{\nabla D}(\boldsymbol{r},t) = 0\,, \end{aligned} \tag{1}$$

where $\boldsymbol{E}(\boldsymbol{r},t)$ and $\boldsymbol{B}(\boldsymbol{r},t)$ are the averaged microscopic electric and magnetic field (in sense of transition from microscopic to macroscopic electrodynamics). The electric induction $\boldsymbol{D}(\boldsymbol{r},t)$ contains completely the averaged microscopic current density $\overline{\boldsymbol{j}_{\mathrm{micro}}}(\boldsymbol{r},t)$ and charge density $\overline{\varrho_{\mathrm{micro}}}(\boldsymbol{r},t)$ with observation of the continuity equation as follows

$$\overline{\boldsymbol{j}_{\mathrm{micro}}}(\boldsymbol{r},t) \equiv \frac{\partial}{\partial t}\boldsymbol{P}(\boldsymbol{r},t)\,, \qquad \overline{\varrho_{\mathrm{micro}}}(\boldsymbol{r},t) \equiv -\boldsymbol{\nabla P}(\boldsymbol{r},t)\,, \tag{2}$$

where $\boldsymbol{P}(\boldsymbol{r},t)$ is called the polarization and $\boldsymbol{D}(\boldsymbol{r},t) \equiv \boldsymbol{E}(\boldsymbol{r},t) + 4\pi\boldsymbol{P}(\boldsymbol{r},t)$.

The most general linear constitutive equations for homogeneous anisotropic dispersive media are nonlocal with kernel dependence only on the differences of field coordinates

$$D_i(\boldsymbol{r},t) = \int d^3r' \wedge dt' \hat{\varepsilon}_{ij}(\boldsymbol{r}-\boldsymbol{r}', t-t') E_j(\boldsymbol{r}',t') \equiv \varepsilon_{ij}\left(-\mathrm{i}\boldsymbol{\nabla}, \mathrm{i}\frac{\partial}{\partial t}\right) E_j(\boldsymbol{r},t)\,, \quad (3)$$

or after Fourier transformation with respect to space and time variables

$$D_i(\boldsymbol{k},\omega) = \varepsilon_{ij}(\boldsymbol{k},\omega) E_j(\boldsymbol{k},\omega)\,, \qquad \varepsilon_{ij}(\boldsymbol{k},\omega) \equiv \delta_{ij} + 4\pi\chi_{ij}(\boldsymbol{k},\omega)\,, \quad (4)$$

with $\varepsilon_{ij}(\boldsymbol{k},\omega)$ the permittivity tensor and with $\chi_{ij}(\boldsymbol{k},\omega)$ the susceptibility tensor in the analogous relation $P_i(\boldsymbol{k},\omega) = \chi_{ij}(\boldsymbol{k},\omega)E_j(\boldsymbol{k},\omega)$ for the polarization. The dependence of $\varepsilon_{ij}(\boldsymbol{k},\omega)$ on the wave vector $\boldsymbol{k}$ is called spatial dispersion contrary to frequency dispersion [2–5]. Into $\varepsilon_{ij}(\boldsymbol{k},\omega)$ or $\chi_{ij}(\boldsymbol{k},\omega)$, correspondingly, can be included such effects as magnetic susceptibilities, optical gyrotropy, and some others which often are treated in a more specialized way. This concept makes only some difficulties for low frequencies and in limiting transition $\omega \to 0$ which we exclude. On the other side, this concept of spatial and frequency dispersion is almost necessary for our relativistic covariant derivations since the transformation formula from one inertial system $\mathcal{I}$ to another inertial system $\mathcal{I}'$ moving with velocity $\boldsymbol{V}$ in $\mathcal{I}$ is

$$\begin{aligned}\chi'_{ij}\left(\boldsymbol{k}',\omega'\right) = &\left\{\frac{V_iV_k}{\boldsymbol{V}^2} + \gamma\left(\delta_{ik} - \frac{V_iV_k}{\boldsymbol{V}^2} + \frac{\boldsymbol{k}'\boldsymbol{V}\delta_{ik} - V_ik'_k}{\omega'}\right)\right\} \\ &\cdot\left\{\frac{V_jV_l}{\boldsymbol{V}^2} + \gamma\left(\delta_{jl} - \frac{V_jV_l}{\boldsymbol{V}^2} + \frac{\boldsymbol{k}'\boldsymbol{V}\delta_{jl} - V_jk'_l}{\omega'}\right)\right\}\chi_{kl}\left(\boldsymbol{k},\omega\right)\,, \quad (5)\end{aligned}$$

where the transformations of wave-vector and frequency are

$$\boldsymbol{k}' = \boldsymbol{k} + (\gamma-1)\frac{\boldsymbol{k}\boldsymbol{V}}{\boldsymbol{V}^2}\boldsymbol{V} - \gamma\frac{\omega}{c^2}\boldsymbol{V}\,, \quad \omega' = \gamma\left(\omega - \boldsymbol{k}\boldsymbol{V}\right), \quad \gamma \equiv \left(\sqrt{1-\frac{\boldsymbol{V}^2}{c^2}}\right)^{-1}. \quad (6)$$

Frequency-dependent susceptibility tensors $\chi_{kl}(\omega)$ only alone are too narrow since according to (5) this generates in the moving system at once a new susceptibility which depends on frequency and wave-vector. This also explains that spatial dispersion is an important effect in hot gases and hot plasmas where each particle moves with an individual velocity with a statistics described by a partition function.

The magnetic field can be eliminated and in connection with the constitutive equations, one obtains the following linear equation for the electric field

$$\left\{-\nabla_i\nabla_j + \boldsymbol{\nabla}^2\delta_{ij} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\varepsilon_{ij}\left(-\mathrm{i}\boldsymbol{\nabla}, \mathrm{i}\frac{\partial}{\partial t}\right)\right\}E_j(\boldsymbol{r},t) = 0\,, \quad (7)$$

and after Fourier transformation

$$\left\{k_ik_j - \boldsymbol{k}^2\delta_{ij} + \frac{\omega^2}{c^2}\varepsilon_{ij}(\boldsymbol{k},\omega)\right\}E_j(\boldsymbol{k},\omega) = 0\,. \quad (8)$$

We introduce an abbreviation for the operator of this vectorial field equation. However, to obtain relativistic covariance it is necessary to divide the operator in (8) by ω^2/c^2 as the investigation shows and we introduce (heuristically, $c^2\left(k_i k_j - \boldsymbol{k}^2\delta_{ij}\right)/\omega^2 + \delta_{ij}$ multiplied by E_i and E_j is something like the known relativistic invariant $\boldsymbol{E}^2 - \boldsymbol{B}^2$)

$$L_{ij}(\boldsymbol{k},\omega) \equiv \frac{c^2\left(k_i k_j - \boldsymbol{k}^2\delta_{ij}\right)}{\omega^2} + \underbrace{\varepsilon_{ij}(\boldsymbol{k},\omega)}_{=\delta_{ij}+4\pi\chi_{ij}(\boldsymbol{k},\omega)} . \tag{9}$$

Then the vectorial equation for the electric field can be written, alternatively

$$L_{ij}(\boldsymbol{k},\omega)E_j(\boldsymbol{k},\omega) = 0\,, \quad \Leftrightarrow \quad L_{ij}\left(-\mathrm{i}\boldsymbol{\nabla}, \mathrm{i}\frac{\partial}{\partial t}\right)E_j(\boldsymbol{r},t) = 0\,. \tag{10}$$

Now, we make the ansatz of quasiplane and quasimonochromatic waves

$$\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{E}_0(\boldsymbol{r},t)\,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}_0\boldsymbol{r}-\omega_0 t)} + \boldsymbol{E}_0^*(\boldsymbol{r},t)\,\mathrm{e}^{-\mathrm{i}(\boldsymbol{k}_0^*\boldsymbol{r}-\omega_0^* t)}, \tag{11}$$

and from (10) follows for the slowly varying amplitudes $\boldsymbol{E}_0(\boldsymbol{r},t)$

$$\begin{aligned} 0 &= L_{ij}\left(\boldsymbol{k}_0 - \mathrm{i}\boldsymbol{\nabla}, \omega_0 + \mathrm{i}\frac{\partial}{\partial t}\right)E_{0,j}(\boldsymbol{r},t) \\ &= \left\{\left(L_{ij}\right)_0 - \mathrm{i}\left(\frac{\partial L_{ij}}{\partial k_k}\right)_0\nabla_k + \mathrm{i}\left(\frac{\partial L_{ij}}{\partial\omega}\right)_0\frac{\partial}{\partial t} + \ldots\right\}E_{0,j}(\boldsymbol{r},t)\,. \end{aligned} \tag{12}$$

In the following first approximations, we take from the expansion on the right-hand side only the terms up to first-order derivatives of the slowly varying amplitudes.

3. Local Conservation Laws of Action and Energy–Momentum

Local conservation theorems in form of vanishing of four-divergences of vector or tensor fields can only be derived under the assumption of absent dissipation (including amplification) that is the following requirement for the permittivity tensor

$$\varepsilon_{ij}(\boldsymbol{k},\omega) = \left(\varepsilon_{ji}(\boldsymbol{k}^*,\omega^*)\right)^* = \varepsilon_{ji}(-\boldsymbol{k},-\omega)\,, \quad \Leftrightarrow \quad L_{ij}(\boldsymbol{k},\omega) = L_{ji}(-\boldsymbol{k},-\omega)\,. \tag{13}$$

Under this assumption, it is possible to obtain exact conservation theorems contrary to theorems where there remain terms which cannot be written as a four-divergence and which are interpreted as four-forces. Due to these exact conservation theorems, the known problems of the right expressions for the energy–momentum tensor, the Abraham or the Minkowski tensor (e.g. [4]), is not relevant for our considerations.

We give here simplified derivations under the assumption that $\boldsymbol{k}$ and ω are real (no evanescent waves without dissipation, e.g. total reflection) but the more general ones are also possible. The following combination can be represented as the vanishing of the four-divergence of a four-vector $T_\lambda(r)$

$$\begin{aligned} 0 &= \frac{\mathrm{i}}{4\pi}\left\{-E^*_{0,i}(r)L_{ij}(k_0-\mathrm{i}\nabla)E_{0,j}(r)+E_{0,j}(r)L_{ij}(k_0+\mathrm{i}\nabla)E^*_{0,i}(r)\right\} \\ &= \nabla_\lambda T_\lambda(r)\,, \end{aligned} \tag{14}$$

where $T_\lambda(r)$ is given in first approximation by

$$T_\lambda(r) = -\frac{1}{4\pi}\left(\frac{\partial\,(L_{ij})}{\partial k_\lambda}\right)_0 E^*_{0,i}(r)E_{0,j}(r)+\dots\,. \tag{15}$$

With separation in three-dimensional form $T_\lambda(r)\equiv(T_l(\boldsymbol{r},t),\mathrm{i}cs(\boldsymbol{r},t))$ we have

$$\begin{aligned} T_l(\boldsymbol{r},t) &= -\frac{1}{4\pi}\left(\frac{\partial L_{ij}}{\partial k_l}\right)_0 E^*_{0,i}(\boldsymbol{r},t)E_{0,j}(\boldsymbol{r},t)+\dots\,, \\ s(\boldsymbol{r},t) &= \frac{1}{4\pi}\left(\frac{\partial L_{ij}}{\partial \omega}\right)_0 E^*_{0,i}(\boldsymbol{r},t)E_{0,j}(\boldsymbol{r},t)+\dots\,, \end{aligned} \tag{16}$$

and with these terms written more explicitly by means of (9)

$$\begin{aligned} T_l(\boldsymbol{r},t) &= \frac{1}{4\pi}\left\{\frac{c^2}{\omega_0^2}\left([\boldsymbol{E}^*_0(\boldsymbol{r},t)\times[\boldsymbol{k}_0\times\boldsymbol{E}_0(\boldsymbol{r},t)]]_l+[\boldsymbol{E}_0(\boldsymbol{r},t)\times[\boldsymbol{k}_0\times\boldsymbol{E}^*_0(\boldsymbol{r},t)]]_l\right)\right. \\ &\quad \left.-E^*_{0,i}(\boldsymbol{r},t)\left(\frac{\partial\varepsilon_{ij}}{\partial k_l}\right)_0 E_{0,j}(\boldsymbol{r},t)\right\}+\dots\,, \\ s(\boldsymbol{r},t) &= \frac{1}{4\pi}\left\{2\frac{c^2}{\omega_0^3}[\boldsymbol{k}_0\times\boldsymbol{E}^*_0(\boldsymbol{r},t)][\boldsymbol{k}_0\times\boldsymbol{E}_0(\boldsymbol{r},t)]+E^*_{0,i}(\boldsymbol{r},t)\left(\frac{\partial\varepsilon_{ij}}{\partial\omega}\right)_0 E_{0,j}(\boldsymbol{r},t)\right\} \\ &\quad +\dots\,. \end{aligned} \tag{17}$$

The quantity $T_l(\boldsymbol{r},t)$ is the action flow density and $s(\boldsymbol{r},t)$ the action density and the conservation theorem in three-dimensional form

$$\nabla_l T_l(\boldsymbol{r},t)+\frac{\partial}{\partial t}s(\boldsymbol{r},t)=0\,, \tag{18}$$

can also be generalized to inhomogeneous media (including adiabatic invariance). By integration of (18) over the whole three-dimensional space follows the action conservation $\partial S(t)/\partial t=0$ with $S(t)\equiv\int d^3r\,s(\boldsymbol{r},t)$ as the integral action.

The vanishing of a four-divergence of the energy–momentum tensor $T_{\kappa\lambda}(r)$ with right transformation properties is obtained by considering the following combination

$$\begin{aligned} 0 = \frac{\mathrm{i}}{4\pi}&\left\{-E^*_{0,i}(r)\,(k_{0,\kappa}-\mathrm{i}\nabla_\kappa)\,L_{ij}(k_0-\mathrm{i}\nabla)E_{0,j}(r)\right. \\ &\left.+E_{0,j}(r)\,(k_{0,\kappa}+\mathrm{i}\nabla_\kappa)\,L_{ij}(k_0+\mathrm{i}\nabla)E^*_{0,i}(r)\right\} = \nabla_\lambda T_{\kappa\lambda}(r)\,, \end{aligned} \tag{19}$$

with

$$T_{\kappa\lambda}(r) = -\frac{1}{4\pi}\left(\frac{\partial\,(k_\kappa L_{ij})}{\partial k_\lambda}\right)_0 E^*_{0,i}(r)E_{0,j}(r) + \dots\,. \tag{20}$$

In three-dimensional separation

$$T_{\kappa\lambda}(r) \equiv \begin{pmatrix} T_{kl}(\boldsymbol{r},t)\,, & \mathrm{i}cg_k(\boldsymbol{r},t) \\ \frac{\mathrm{i}}{c}S_l(\boldsymbol{r},t)\,, & -w(\boldsymbol{r},t) \end{pmatrix}, \tag{21}$$

where $T_{kl}(\boldsymbol{r},t)$ is the (Maxwell) stress tensor, $g_k(\boldsymbol{r},t)$ the momentum density, $S_l(\boldsymbol{r},t)$ the energy flow density (Poynting–Umov vector) and $w(\boldsymbol{r},t)$ the energy density, the differential form of conservation of momentum and of energy possesses the form

$$\nabla_l T_{kl}(\boldsymbol{r},t) + \frac{\partial}{\partial t}g_k(\boldsymbol{r},t) = 0\,, \qquad \nabla_l S_l(\boldsymbol{r},t) + \frac{\partial}{\partial t}w(\boldsymbol{r},t) = 0\,. \tag{22}$$

One obtains for these quantities written up to the first approximation

$$\begin{aligned} T_{kl}(\boldsymbol{r},t) &= -\frac{1}{4\pi}\left(\frac{\partial(k_k L_{ij})}{\partial k_l}\right)_0 E^*_{0,i}(\boldsymbol{r},t)E_{0,j}(\boldsymbol{r},t) + \dots\,, \\ g_k(\boldsymbol{r},t) &= \frac{1}{4\pi}\left(\frac{\partial(k_k L_{ij})}{\partial \omega}\right)_0 E^*_{0,i}(\boldsymbol{r},t)E_{0,j}(\boldsymbol{r},t) + \dots\,, \\ S_l(\boldsymbol{r},t) &= -\frac{1}{4\pi}\left(\frac{\partial(\omega L_{ij})}{\partial k_l}\right)_0 E^*_{0,i}(\boldsymbol{r},t)E_{0,j}(\boldsymbol{r},t) + \dots\,, \\ w(\boldsymbol{r},t) &= \frac{1}{4\pi}\left(\frac{\partial(\omega L_{ij})}{\partial \omega}\right)_0 E^*_{0,i}(\boldsymbol{r},t)E_{0,j}(\boldsymbol{r},t) + \dots\,. \end{aligned} \tag{23}$$

For length, we do not write down here the more explicit expressions obtained by inserting $L_{ij} \equiv L_{ij}(\boldsymbol{k},\omega)$ according to (9) and by forming the necessary derivatives.

4. Relation between Energy–Momentum and Action Conservation and a Surprising Relation to Quantization

In the limiting transition to plane monochromatic waves, the slowly varying amplitudes $\boldsymbol{E}_0(\boldsymbol{r},t)$ make the transition to constant amplitudes $\boldsymbol{E}_0$ and equation (12) in operator form becomes (we write it now without indices $L_{ij} \to \mathsf{L}$ and $E_j \to \boldsymbol{E}$, indices "$_0$" mean at $\boldsymbol{k} = \boldsymbol{k}_0$, $\omega = \omega_0$)

$$\mathsf{L}_0\boldsymbol{E}_0 = \boldsymbol{0}\,, \qquad \boldsymbol{E}^*_0\mathsf{L}_0 = \boldsymbol{0}\,, \qquad \mathsf{L}_0 \equiv \mathsf{L}\,(\boldsymbol{k}_0,\omega_0)\,. \tag{24}$$

Using this, we find from (20) and (15) in the limiting case (our first magic trick)

$$T_{\kappa\lambda} = k_{0,\kappa}T_\lambda\,, \qquad T_\lambda = -\frac{1}{4\pi}\boldsymbol{E}^*_0\left(\frac{\partial\mathsf{L}}{\partial k_\lambda}\right)_0\boldsymbol{E}_0\,, \tag{25}$$

or in three-dimensional form

$$\begin{aligned} T_{kl} &= k_{0,k} T_l\,, & g_k &= k_{0,k}\, s\,, & & & T_{kl} &= s\, k_{0,k}\, T_l/s = g_k\, T_l/s\,,\\ S_l &= \omega_0 T_l\,, & w &= \omega_0\, s\,, & &\Rightarrow & S_l &= s\, \omega_0\, T_l/s = w\, T_l/s\,. \end{aligned} \tag{26}$$

We now show that T_l/s is exactly equal to the group velocity $v_{0,l}$ in considered point ($\boldsymbol{k} = \boldsymbol{k}_0$, $\omega = \omega_0$) of the dispersion surface given by $|\mathsf{L}\,(\boldsymbol{k}, \omega)| = 0$ ($|\mathsf{L}|$ means determinant of L). We can resolve this dispersion equation with respect to one variable, for example in the form $\omega = \omega(\boldsymbol{k})$. The group velocity is then determined by $\boldsymbol{v} \equiv \partial\omega/\partial\boldsymbol{k}$. Substituting ω in $|\mathsf{L}(\boldsymbol{k}, \omega)|$ by $\omega = \omega(\boldsymbol{k})$, we get the identity $|\mathsf{L}(\boldsymbol{k}, \omega(\boldsymbol{k}))| = 0$ as a function of the wave vector $\boldsymbol{k}$ which we can differentiate and we obtain at the considered point ($\boldsymbol{k} = \boldsymbol{k}_0$, $\omega = \omega_0$) of the dispersion surface (our second magic trick)

$$v_{0,l} = -\frac{\left(\dfrac{\partial|\mathsf{L}|}{\partial k_l}\right)_0}{\left(\dfrac{\partial|\mathsf{L}|}{\partial \omega}\right)_0} = -\frac{\left\langle \left(\dfrac{\partial \mathsf{L}}{\partial k_l}\right)_0 (\overline{\mathsf{L}})_0 \right\rangle}{\left\langle \left(\dfrac{\partial \mathsf{L}}{\partial \omega}\right)_0 (\overline{\mathsf{L}})_0 \right\rangle} = -\frac{\boldsymbol{E}_0^* \left(\dfrac{\partial \mathsf{L}}{\partial k_l}\right)_0 \boldsymbol{E}_0}{\boldsymbol{E}_0^* \left(\dfrac{\partial \mathsf{L}}{\partial \omega}\right)_0 \boldsymbol{E}_0} = \frac{T_l}{s}\,. \tag{27}$$

Herein, $\langle \mathsf{A} \rangle$ means the trace of three-dimensional operators A. In the second step it was used that the differentiation of a determinant with respect to a parameter λ is $\partial|\mathsf{L}|/\partial\lambda = \big\langle (\partial\mathsf{L}/\partial\lambda)\overline{\mathsf{L}} \big\rangle$, where $\overline{\mathsf{L}}$ denotes the complementary operator to L with property $\mathsf{L}\overline{\mathsf{L}} = \overline{\mathsf{L}}\mathsf{L} = |\mathsf{L}|\mathsf{I}$ and where in our case due to $|\mathsf{L}_0| = 0$ the complementary operator is proportional to the dyadic product $\boldsymbol{E}_0 \circ \boldsymbol{E}_0^*$. We see that to obtain (27) there is no reason to make any explicit calculations of the group velocity.

Thus in considered limiting case we find the factorization

$$T_{kl} = s\, k_{0,k}\, v_{0,l}\,, \qquad g_k = s\, k_{0,k}\,, \qquad S_l = s\, \omega_0\, v_{0,l}\,, \qquad w = s\, \omega_0\,. \tag{28}$$

The group velocity $\boldsymbol{v}$ as a regular velocity as is well known is not spatial part of a four-vector and the modified four-vector u of the velocity can be defined by

$$u \equiv \left(\frac{\boldsymbol{v}}{\sqrt{1 - \dfrac{\boldsymbol{v}^2}{c^2}}}, \mathrm{i}\frac{c}{\sqrt{1 - \dfrac{\boldsymbol{v}^2}{c^2}}} \right), \quad \Rightarrow \quad u^2 = -c^2. \tag{29}$$

This leads to the energy–momentum tensor (25) in the relativistic covariant form

$$T_{\kappa\lambda} = s_0 k_{0,\kappa} u_{0,\lambda}\,, \qquad \left(s_0 \equiv s\sqrt{1 - \frac{\boldsymbol{v}_0^2}{c^2}} \right), \tag{30}$$

where s_0 is the action density in the inertial system of resting wave and s the action density in the considered inertial system where the wave possesses the group velocity $\boldsymbol{v}_0$. The tensor $T_{\kappa\lambda}$ for anisotropic dispersive media is, in general, nonsymmetric ($T_{\kappa\lambda} \neq T_{\lambda\kappa}$) since wave vector $\boldsymbol{k}_0$ and corresponding group velocity $\boldsymbol{v}_0$ do not possess, in general, the same direction.

The energy–momentum tensor (30) in its structure is very similar to the energy–momentum tensor for a homogeneous particle flow in classical mechanics and hydrodynamics

$$T_{\kappa\lambda} = n_0 p_{0,\kappa} u_{0,\lambda}\,, \qquad \left(n_0 \equiv n\sqrt{1 - \frac{\boldsymbol{v}_0^2}{c^2}} \right), \tag{31}$$

where $p_{0,\kappa}$ is the four-vector of momentum and n_0 the particle density in the system where the particle rests and n the particle density in the system where it moves with velocity $\boldsymbol{v}_0$. In reality, classical mechanics considers only the dispersion law $p_{0,\kappa} = m_0 u_\kappa$ where $m_0 = m\sqrt{1 - \boldsymbol{v}^2/c^2}$ is the rest mass of one particle and m the mass in the considered system and the energy–momentum tensor becomes $T_{\kappa\lambda} = \mu_0 u_{0,\kappa} u_{0,\lambda}$ with $\mu_0 = n_0 m_0$ the mass density in the system of resting particles.

If we take seriously the analogue of electrodynamic (30) to mechanical energy–momentum tensor (31), we may introduce an abbreviation $\hbar$ for the following quantity (please, forget for a moment that it already exists!)

$$\hbar \equiv \frac{s}{n} = \frac{s_0}{n_0} \equiv \frac{\text{action density}}{\text{density}} = \text{action of 1 particle}\,. \tag{32}$$

It is relativistically invariant (a relativistic scalar) and, moreover, it remains an invariant under adiabatic changes of the system as one may realize. It also cannot change from one to another system because in other case there would appear strange effects at the boundary between two such systems. This suggests that it should have a universal meaning. The energy–momentum tensor (30) can now be written

$$T_{\kappa\lambda} = n_0 p_{0,\kappa} u_{0,\lambda}\,, \qquad p_{0,\kappa} \equiv \frac{s_0}{n_0}\, k_{0,\kappa} \equiv \hbar k_{0,\kappa}\,. \tag{33}$$

Practically, with these relations we arrived at quantum mechanics (of quasiparticles) but such a reasoning as mentioned was possible, in principle, already within the time after the creation of special theory of relativity in 1905 and the development of rigorous quantum theory from 1925 on.

Electrodynamics of dispersive media provides a much greater variety of possible dispersion laws $\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{v})$ (or $\boldsymbol{k} = \boldsymbol{k}(\boldsymbol{v})$) than classical mechanics which only knows $\boldsymbol{p} = m\boldsymbol{v}$ with m the particle mass. By direct calculations in electrodynamics, one usually finds primarily the dispersion law in the form $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{k})$ of group velocity $\boldsymbol{v}$ as function of wave vector $\boldsymbol{k}$ and its inversion $\boldsymbol{k} = \boldsymbol{k}(\boldsymbol{v})$ is difficult and not to make in general form for the whole variety of wave vectors $\boldsymbol{k}$. A full analogy to classical mechanics is only obtained for transverse waves in a cold plasma with the scalar permittivity $\varepsilon(\omega) = 1 - \omega_p^2/\omega^2$ (ω_p is plasma frequency) where the photons possess an equivalent scalar rest mass $m_0 = \hbar\omega_p/c^2$ which tends to zero in the transition $\omega_p \to 0$

to the vacuum. Furthermore, in optics (e.g. crystal optics) of nondispersive media (or in such approximation), we have the relation $\boldsymbol{ns} = (c\boldsymbol{k}/\omega)(\boldsymbol{v}/c) = \boldsymbol{kv}/\omega = 1$ (leading to $T_{kk} = w$) for refraction vectors $\boldsymbol{n} \equiv c\boldsymbol{k}/\omega$ and ray vectors $\boldsymbol{s} \equiv \boldsymbol{v}/c$ which can be continued to a whole list of dualities between description with refraction and ray quantities (see, e.g. [7]). These duality relations are no more true for dispersive media as, for example, the explicit calculation of their group velocities shows.

5. Nonuniqueness of the Energy–Momentum Tensor

It is known (e.g. [6]) that the energy–momentum tensor $T_{\kappa\lambda}(r)$ as quantity in the local conservation theorems (22) is nonunique. The same is with the action four-vector $T_\lambda(r)$ introduced in this paper. From this arise problems in Einstein's gravitation equations where the energy–momentum tensor stands on the right-hand side in its absolute form (see next section).

We begin with the action four-vector $T_\lambda(r)$ which in the local conservation theorem possesses the following nonuniqueness (i.e. $\nabla_\lambda T'_\lambda(r) = \nabla_\lambda T_\lambda(r)$)

$$T'_\lambda(r) = T_\lambda(r) + \nabla_\mu \psi_{\lambda\mu}(r)\,, \qquad \psi_{\lambda\mu}(r) = -\psi_{\mu\lambda}(r)\,, \tag{34}$$

where $\psi_{\lambda\mu}(r)$ is an arbitrary second-rank antisymmetric tensor function. Written in three-dimensional vector form this means $(T_\lambda(r) = (T_l(\boldsymbol{r},t), \mathrm{i}cs(\boldsymbol{r},t)))$

$$\boldsymbol{T}'(\boldsymbol{r},t) = \boldsymbol{T}(\boldsymbol{r},t) + [\boldsymbol{\nabla} \times \boldsymbol{\psi}(\boldsymbol{r},t)] + \frac{\partial}{\partial t}\boldsymbol{\chi}(\boldsymbol{r},t)\,, \quad s'(\boldsymbol{r},t) = s(\boldsymbol{r},t) - \boldsymbol{\nabla}\boldsymbol{\chi}(\boldsymbol{r},t)\,. \tag{35}$$

with a pseudo-vector function $\psi_l(\boldsymbol{r},t) = \frac{1}{2}\epsilon_{lmn}\psi_{mn}(\boldsymbol{r},t)$ and a vector function $\chi_l(\boldsymbol{r},t) = (\mathrm{i}/c)\psi_{4l}(\boldsymbol{r},t) = -(\mathrm{i}/c)\psi_{l4}(\boldsymbol{r},t)$.

The corresponding most general nonuniqueness of the energy momentum tensor $T_{\kappa\lambda}(r)$ for local conservation theorem (i.e. $\nabla_\lambda T'_{\kappa\lambda}(r) = \nabla_\lambda T_{\kappa\lambda}(r)$) is described by

$$T'_{\kappa\lambda}(r) = T_{\kappa\lambda}(r) + \nabla_\mu \psi_{\kappa\lambda\mu}(r)\,, \qquad \psi_{\kappa\lambda\mu}(r) = -\psi_{\kappa\mu\lambda}(r)\,, \tag{36}$$

with a third-rank four-tensor function $\psi_{\kappa\lambda\mu}(r)$ which is antisymmetric in the last two indices [6]. In three-dimensional form this means for the stress tensor $T_{kl}(\boldsymbol{r},t)$ and the momentum density $g_k(\boldsymbol{r},t)$

$$\begin{aligned} T'_{kl}(\boldsymbol{r},t) &= T_{kl}(\boldsymbol{r},t) + \epsilon_{lmn}\nabla_m\psi_{kn}(\boldsymbol{r},t) + \frac{\partial}{\partial t}\chi_{kl}(\boldsymbol{r},t)\,, \\ g'_k(\boldsymbol{r},t) &= g_k(\boldsymbol{r},t) - \nabla_l\chi_{kl}(\boldsymbol{r},t)\,, \end{aligned} \tag{37}$$

and for the energy flow density $S_l(\boldsymbol{r},t)$ and the energy density $w(\boldsymbol{r},t)$

$$\begin{aligned} S'_l(\boldsymbol{r},t) &= S_l(\boldsymbol{r},t) + \epsilon_{lmn}\nabla_m\psi_n(\boldsymbol{r},t) + \frac{\partial}{\partial t}\chi_l(\boldsymbol{r},t)\,, \\ w'(\boldsymbol{r},t) &= w(\boldsymbol{r},t) - \nabla_l\chi_l(\boldsymbol{r},t)\,, \end{aligned} \tag{38}$$

where $\psi_{kn}(\boldsymbol{r},t) \equiv \frac{1}{2}\epsilon_{lmn}\psi_{klm}(\boldsymbol{r},t)$ and $\chi_{kl}(\boldsymbol{r},t) \equiv -(\mathrm{i}/c)\psi_{kl4}(\boldsymbol{r},t)$ are two arbitrary second-rank tensor functions and $\psi_n(\boldsymbol{r},t) \equiv \frac{1}{2}\epsilon_{lmn}\psi_{lm}(\boldsymbol{r},t)$ and $\chi_l(\boldsymbol{r},t) \equiv -(\mathrm{i}/c)\psi_{l4}(\boldsymbol{r},t)$ are two arbitrary vector functions.

As a simple example, we consider the limiting case of plane monochromatic waves in vacuum. The energy flow density and energy density (and, analogously, stress tensor and momentum density) derived from general relations obtained without the approximation of slowly varying amplitudes possess then "high-frequency terms" according to (the terms with only slowly varying amplitudes are announced by points)

$$\begin{aligned} \boldsymbol{S}' &= \frac{c}{4\pi}\left\{\ldots + \frac{c}{\omega_0}\left[\boldsymbol{E}_0\times\left[\boldsymbol{k}_0\times\boldsymbol{E}_0\right]\right]\mathrm{e}^{\mathrm{i}2(\boldsymbol{k}_0\boldsymbol{r}-\omega_0 t)} + \mathrm{c.c.}\right\}, \\ w' &= \frac{1}{8\pi}\left\{\ldots + \left(\boldsymbol{E}_0\boldsymbol{E}_0 + \frac{c^2}{\omega_0^2}\left[\boldsymbol{k}_0\times\boldsymbol{E}_0\right]\left[\boldsymbol{k}_0\times\boldsymbol{E}_0\right]\right)\mathrm{e}^{\mathrm{i}2(\boldsymbol{k}_0\boldsymbol{r}-\omega_0 t)} + \mathrm{c.c.}\right\}. \end{aligned} \tag{39}$$

If we choose

$$\boldsymbol{\chi}(\boldsymbol{r},t) = \frac{c}{4\pi}\left\{\mathrm{i}\frac{c}{2\omega_0^2}\left[\boldsymbol{E}_0\times\left[\boldsymbol{k}_0\times\boldsymbol{E}_0\right]\right]\mathrm{e}^{\mathrm{i}2(\boldsymbol{k}_0\boldsymbol{r}-\omega_0 t)} + \mathrm{c.c.}\right\}, \tag{40}$$

and calculate $\partial\boldsymbol{\chi}(\boldsymbol{r},t)/\partial t$ and $-\boldsymbol{\nabla}\boldsymbol{\chi}(\boldsymbol{r},t)$, we find that the high-frequency terms in (39) can be removed. One has to be a little careful with these relations because one has to suppose in these derivations that the amplitudes $\boldsymbol{E}_0$ are slowly varying quantities and then we have further terms which are derivatives of these amplitudes. However, they may be considered in first approximation as small terms which vanish in the limiting transition to plane monochromatic waves. This means that the nonuniqueness functions among other actions play a role to reduce the energy–momentum tensor to forms without the high-frequency and high-wave-vector terms.

6. Problems with Einstein's Gravitation Equation

The basic equation of Einstein's gravitation theory is [8] (and, e.g. [6])

$$R_{\kappa\lambda} - \frac{1}{2}\, g_{\kappa\lambda}\, R = \frac{8\pi\gamma}{c^4}\, T_{\kappa\lambda}\,, \tag{41}$$

where $R_{\kappa\lambda} \equiv g^{\mu\nu}R_{\mu\kappa\nu\lambda}$ is the Ricci tensor (field) ($R_{\mu\kappa\nu\lambda}$ is the Riemann curvature tensor (field)), $g_{\kappa\lambda}$ the metric tensor (field), $T_{\kappa\lambda}$ the energy–momentum tensor (field) and γ the gravitation constant. The Riemann curvature tensor is defined by the metric tensor and by its not higher than second-order derivatives and this continues to the Ricci tensor (see, e.g. [6] for full definitions). Both tensors, the Ricci tensor and the metric tensor are by definition symmetric (i.e. $R_{\kappa\lambda} = R_{\lambda\kappa}$ and $g_{\kappa\lambda} = g_{\lambda\kappa}$). Therefore, Einstein's equations are finitely second-order differential equations for the 10 independent components of the metric tensor $g_{\kappa\lambda}$ for known energy–momentum tensor $T_{\kappa\lambda}$ as the source of gravitational field.

For unity of physics, one has to suppose that on the right-hand side of (41) stands the full energy–momentum tensor $T_{\kappa\lambda}$ consisting of the tensor for the moving mass distributions plus the tensor of the electromagnetic field. From this arise two problems. The first problem is that the tensor of the electromagnetic field for dispersive media is, in general, nonsymmetric ($T_{\kappa\lambda} \neq T_{\lambda\kappa}$ with 16 independent components), whereas the Ricci tensor $R_{\kappa\lambda}$ and the metric tensor $g_{\kappa\lambda}$ are by definition symmetric (10 independent components) and a general equality in (41) is not possible. The second problem is that the energy–momentum tensor as a quantity in the local conservation theorem is not uniquely defined. This nonuniqueness should not have influence on the metric tensor $g_{\kappa\lambda}$. It is not proved to our knowledge (and apparently wrong) that the arbitrary nonuniqueness functions $\psi_{kn}(\boldsymbol{r},t)$, $\chi_{kl}(\boldsymbol{r},t)$ and $\psi_n(\boldsymbol{r},t)$, $\chi_l(\boldsymbol{r},t)$ (see Section 5) do not have influence via equation (41) on the metric tensor $g_{\kappa\lambda}$. Usually, this problem is not considered by assuming the symmetric tensor $T_{\kappa\lambda}(\boldsymbol{r},t) = \mu(\boldsymbol{r},t)u_\kappa u_\lambda$ as the right form of the energy–momentum tensor for a moving mass distribution ($\mu(\boldsymbol{r},t)$ mass density). Concerning electrodynamics in vacuum, there are made great efforts to show that the energy–momentum tensor is equivalent to a symmetric one, mostly already before the treatment of Einstein's gravitation equations (41) where the problem of nonuniqueness is then usually no more mentioned (see, e.g. [6]). On the other side, one could think that the nonuniqueness of the energy–momentum tensor $T_{\kappa\lambda}$ which for homogeneous anisotropic dispersive media is basically obtained as a nonsymmetric one could be used to make this tensor to a symmetric tensor. However, it seems that the energy–momentum tensor for such media is intrinsically nonsymmetric and we did not find a possibility to make it to a symmetric one using the nonuniqueness functions.

Although one is surely far from measurability of the influence of dispersive media on the gravitational field, the unity of physical laws to which one believes suggests that there are some problems with Einstein's gravitation equations (41) in connection with the energy–momentum tensor $T_{\kappa\lambda}$ of anisotropic dispersive media. We cannot solve here and to this time these problems and could only mention them.

7. Conclusion

We have calculated in relativistic covariant way the action four-vector and the energy–momentum tensor for homogeneous anisotropic dispersive media in macroscopic electrodynamics and have discussed relations to quantization and to problems for Einstein's gravitation equation. For spatially and (or) temporally inhomogeneous and dispersive media, the energy–momentum tensor in electrodynamics is no more a quantity in a conservation theorem and only the action-four-vector remains to be such a conservation quantity. We hope that we can show this in future although there is yet the difficulty that we do not have in this more general case such basic solutions as the plane monochromatic waves for homogeneous media which we can separate from slowly varying amplitudes as made in the derivations. It is intended to write an article with more details and references and also with more results which were not possible to discuss here.

Note

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