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QUANTUM ELECTRONICS

Generation of Schrödinger Cats in Trapped Ions*

B. Militello and A. Messina

Dipartimento di Scienze Fisiche ed Astronomiche dell'Università di Palermo
Via Archirafi, 36, I-90123 Palermo, Italy

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Abstract. A quantum system in interaction with a repeatedly measured one is subjected to a non-unitary time evolution provoking the decay of some states in favor of the remaining ones. Under appropriate hypotheses the system may be addressed exactly toward a quantum state or pushed into a pre-selected finite-dimensional subspace. On the basis of such a general strategy, we propose to exploit suitable vibronic couplings in order to 'extract' trapped ion center of mass states of motion characterized by well defined absolute value of an angular momentum projection. In particular, since it implies the simultaneous presence of opposite angular momentum projections, we show the realizability of Schrödinger Cat superpositions involving clockwise and counterclockwise two-dimensional motions. Specific examples are provided. The efficiency and the fidelity of the method are discussed.

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1. Introduction

Over the last twenty years trapped ions have offered a very fruitful platform to test fundamental aspects of quantum mechanics and to realize applications in the context of quantum processing [1, 2]. Very recently important experiments on quantum teleportation have also been concretized [3].

In this physical framework we propose a scheme devoted to the generation of non-classical states, in particular angular momentum Schrödinger Cat states of a single trapped ion center of mass motion. Such superpositions have already been studied and relevant generation schemes proposed [9]. Nevertheless we propose a more efficient strategy requiring a not so critical time control during the experiment

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as in the previously mentioned proposal. Moreover, we generalize the class of states providing a good control of the relative quantum phase between the two terms of the Schrödinger Cat superposition terms.

Our approach is in some sense the generalization of the quantum non-demolition measurement scheme already proposed in the context of trapped ions for the generation of Fock states [4], and at the same time to measure the population of a Fock state into a vibrational state of the trapped ion. The scheme is based on the idea of alternatively subjecting the system to vibronic couplings and measurement of the internal state of the ion, that is its atomic state. Recently an analogous procedure has been used also for the realization of purification of quantum states [5]. Such a purification scheme has been analyzed introducing a new formalism, that is showing that the evolution of a physical system in interaction with a repeatedly measured one may be described through the action of a suitable non-unitary operator, providing an effective way to forecast the result of the entire process.

Following the same approach as in Ref. [5], we introduce a non-unitary operator describing the dynamics of a trapped ion center of mass when its atomic state is repeatedly measured after suitable laser pulses inducing vibronic couplings responsible for entanglement between the bosonic and fermionic degrees of freedom. The analysis of such a non-unitary operator directly leads to the individuation of some *privileged* subspaces which the system is forced to go onto [8].

The paper is organized as follows. In the next section we present the general sketch of the generation scheme, then, in third section, we consider some specific applications related to the generation of eigenstates or coherent superpositions of eigenstates of angular momentum. Finally we give some conclusive remarks.

2. Extraction of Vibrational Subspaces

2.1. The physical system

A trapped ion is an ion subjected to a suitable configuration of inhomogeneous time-dependent e.m. fields able to confine a charged particle into a finite region of the space [6]. It may be described by an hamiltonian which takes into account both the vibrational degrees of freedom related to the center of mass motion confined into an effective parabolic well, and the internal degrees of freedom, that is the electronic configuration of the ion. Since in most of the cases, the atomic levels effectively involved in the dynamics are two, the relevant "free" hamiltonian for a "trapped" ion is expressible as ($\hbar = 1$)

$$\hat{H}_0 = \hat{H}_A + \hat{H}_B \tag{1}$$

with

$$\hat{H}_A = \frac{\omega_0}{2}\hat{\sigma}_3\,; \qquad \hat{H}_B = \sum_{i=x,y,z} \nu_i \hat{a}_i^\dagger \hat{a}_i$$

the fermionic (i.e. atomic) and bosonic (i.e. vibrational) unperturbed hamiltonians,

where ν_i are the center of mass harmonic oscillator frequencies, $\hat{a}_i$ ($\hat{a}_i^\dagger$) the related annihilation (creation) operators, ω_0 is the Bohr frequency between the two atomic levels considered, and $\hat{\sigma}_3$ is the diagonal Pauli operator.

The action of suitable configurations of laser fields on a trapped ion is described by corresponding vibronic couplings whose features depend on the laser frequencies, directions of propagation, wavelengths, polarizations and strengths. Generally speaking, the $\hat{H}_0$ interaction picture hamiltonian model evaluated in the Rotating Wave Approximation (*RWA*) turns out to be time-independent and expressible as [1, 2, 7]

$$\hat{H}_v = \gamma \hat{\Omega} \hat{\sigma}_+ + \text{h.c.}\,, \tag{2}$$

where $\hat{\sigma}_+ = |+\rangle\langle -|$ ($\hat{\sigma}_- = |-\rangle\langle +|$) is the Pauli raising (lowering) operator, $|\pm\rangle$ being the internal ionic states, and γ is a positive coupling constant substantially related to laser intensities and initial phases. The generic time-independent vibrational operator $\hat{\Omega}$, whose form is determined by the specific laser configuration [7], is a function of the annihilation and creation operators $\{\hat{a}_x, \hat{a}_y, \hat{a}_z, \hat{a}_x^\dagger, \hat{a}_y^\dagger, \hat{a}_z^\dagger\}$.

2.2. Repeatedly measuring the electronic state: purification

Assume the ion, initially prepared into a factorized state $\rho = \rho_v |+\rangle\langle +|$ — ρ_v describing a generic vibrational state —, evolves for a time τ under the action of H_v given by Eq. (2), and then measure its internal state assuming to find the ion into the state $|+\rangle$. Then let the system still evolve for a time τ and once again measure the state $|+\rangle$, and so on N times.

The *conditional* (i.e. realized under the condition that the results of the N measurements are the "right" ones, that is $|+\rangle$ N times) evolution of the system may be described through the non-unitary operator

$$\hat{W}_+^{(N)}(\tau) \equiv \aleph_N \left[|+\rangle\langle +| e^{-i\hat{H}_0\tau} e^{-i\hat{H}_v\tau} \right]^N |+\rangle\langle +| \tag{3}$$

which, may be cast in the form

$$\hat{W}_+^{(N)}(\tau) = e^{i\phi} \aleph_N \left[e^{-i\hat{H}_B\tau} \hat{V}(\tau) \right]^N |+\rangle\langle +| \tag{4}$$

with

$$\hat{V}(\tau) \equiv \langle +| e^{-i\hat{H}_v\tau} |+\rangle\,, \tag{5}$$

ϕ a phase coming from the action of $e^{-i\hat{H}_A t}$ on $\langle +|$, and $\aleph_N = [\prod_{k=1}^N \sqrt{\wp_k}]^{-1}$, $\wp_k$ being the probability of finding the ion into the state $|+\rangle$ at the kth measurement step.

Under the hypothesis $[e^{-i\hat{H}_B\tau}, \hat{V}(\tau)] = 0$ the operator $\hat{W}_+^{(N)}(\tau)$ may be rearranged in the convenient form

$$\hat{W}_+^{(N)}(\tau) = e^{i\phi} \aleph_N \; e^{-i\hat{H}_B N\tau} \left[\hat{V}(\tau) \right]^N |+\rangle\langle +|\,. \tag{6}$$

In the specific case considered, it is easy to find [8]

$$\hat{V}(\tau) = \cos\left(\gamma\tau\sqrt{\hat{\Omega}\hat{\Omega}^{\dagger}}\right) , \tag{7}$$

hence showing that such a non-unitary and in general non-hermitian operator admits a spectral decomposition which is traceable back to that of the non-negative hermitian operator $\hat{\Omega}\hat{\Omega}^{\dagger}$. Consider indeed the following expansion

$$\hat{\Omega}\hat{\Omega}^{\dagger} = \sum_{k} \omega_k \sum_{l=1}^{g_k} |\omega_k, l\rangle \langle\omega_k, l| , \tag{8}$$

$\{|\omega_k, l\rangle\}$ being the eigenstates corresponding to the eigenvalues $\{\omega_k\}$, with l an index spanning the generic degenerate subspace of dimension g_k. The Nth power of $\hat{V}(\tau)$ is easily evaluated as

$$\left[\hat{V}(\tau)\right]^N = \sum_{k} \cos^N\left(\gamma\tau\sqrt{\omega_k}\right) \sum_{l=1}^{g_k} |\omega_k, l\rangle \langle\omega_k, l| . \tag{9}$$

Introducing the set $I(\tau) := \left\{k : \exists q_k \epsilon \mathbb{Z} | \gamma\tau\sqrt{\omega_k} = q_k\pi\right\}$, and provided it is not empty (that is setting the pulse area $\gamma\tau$ in such a way that there exists at least one k satisfying the condition) one easily finds the following limit for the Nth power of $\hat{V}(\tau)$:

$$\left[\hat{V}(\tau)\right]^N \rightarrow \sum_{k\epsilon I(\tau)} (-1)^{N q_k} \sum_{l=1}^{g_k} |\omega_k, l\rangle \langle\omega_k, l| . \tag{10}$$

This practically means that, provided the electronic state is found into its upper level at any measurement step, the vibrational part of the system undergoes a non-unitary evolution (substantially governed by $\hat{V}(\tau)^N$) that, for large enough number of measurements (N), forces the ion center of mass into a subspace generated by the eigenstates whose corresponding eigenvalues satisfy the condition $\gamma\tau\sqrt{\omega_k} = q_k\pi$. In other words the procedure amounts at realizing the projection operator

$$\hat{P}_d := \sum_{k\epsilon I(\tau)} \sum_{l=1}^{g_k} |\omega_k, l\rangle \langle\omega_k, l| . \tag{11}$$

up to a unitary transformation acting on this subspace.

This is the essence of our generation scheme.

The efficiency of the procedure, that is the probability of success of the subspace extraction, is equal to the probability that the measured electronic state is $|+\rangle$ at each step. It is straightforward to prove that such a probability, is given by [8]

$$\mathfrak{E}_N = Tr\{\hat{V}(\tau)^N \rho_v\} \tag{12}$$

which in the limit $N \rightarrow \infty$ gives

$$\mathfrak{E} := \lim_{N\rightarrow\infty} \mathfrak{E}_N = Tr\{\hat{P}_d\ \rho_v\} , \tag{13}$$

i.e. the norm of the projection of the initial state ρ_v onto the target subspace $\hat{P}_d$.

3. Generation of Angular Momentum Non-Classical States

The scheme here reported contains as a very specific application the standard quantum non-demolition measurements scheme for the generation of Fock states and/or for the measure of the related populations [4]. Indeed, such a scheme corresponds to the case wherein $\hat{\Omega}$ is a strongly non-linear function of the bosonic excitation number operators, realized through the action of a laser field tuned to the atomic Bohr frequency ω_0.

Moreover, other specific applications are realizable. In this section we will consider the generation of angular momentum eigenstates or eigenstate superpositions.

3.1. Extracting angular momentum subspaces

Consider an isotropic two-dimensional trap, that is a trapped ion with two equal frequencies along two orthogonal directions, for instance $\nu_x = \nu_y =: \nu$. Act on such a system through two laser beams propagating along the directions x and y and both tuned to the second red sideband. The induced vibronic coupling, in the Rotating Wave Approximation and in the so called Lamb–Dicke limit (corresponding to the fact that the oscillations of the ion are very smaller compared with the laser wavelength) [7, 9], in the interaction picture takes the form

$$\hat{H}_v = \gamma \left(\hat{a}_x^2 + \hat{a}_y^2\right) \hat{\sigma}_+ + \text{h.c.}\,, \tag{14}$$

corresponding to $\hat{\Omega} = \hat{a}_x^2 + \hat{a}_y^2$, hence providing

$$\hat{\Omega}\hat{\Omega}^\dagger = \sum_{j,k=x,y} \hat{a}_j^2 \hat{a}_k^{\dagger 2} = \hat{N}_T^2 + 4\hat{N}_T - \hat{L}_z^2 + 4 \tag{15}$$

being $\hat{N}_T = \hat{a}_x^\dagger \hat{a}_x + \hat{a}_y^\dagger \hat{a}_y$ the total excitation number, and $\hat{L}_z = i\left[\hat{a}_x \hat{a}_y^\dagger - \hat{a}_y \hat{a}_x^\dagger\right]$ the angular momentum component corresponding to the two-dimensional motion here considered. Introducing the relevant quantum numbers n_T and m, it is easy to see that in general the square roots $\sqrt{n_T^2 + 4n_T - m^2 + 4}$ are incommensurable for different values such quantum numbers, therefore in general the condition $\gamma\tau\sqrt{n_T^2 + 4n_T - m^2 + 4} = q_{n_T,m}\pi$, with $q_{n_T,m}\epsilon\mathbb{Z}$, turns out to be selective enough to provide the possibility of forcing the system toward a subspace characterized by specific n_T and absolute value of m (due to the fact that just m^2 appears in the argument of the square root). Hence, on the one hand if $m = 0$ is considered the extraction of a single angular momentum eigenstate is possible, on the other hand if $m \neq 0$ is considered the extracted subspaces are twofold degenerate and generated by $|n_T, m\rangle$ and $|n_T, -m\rangle$.

3.2. Angular momentum Schrödinger Cats

The circumstance of degeneracy of the extracted subspaces in the case $m \neq 0$ naturally leads to the possibility of generating coherent superpositions of the form

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left[|n_T, m\rangle + e^{i\xi} |n_T, -m\rangle \right] . \tag{16}$$

It is possible to show that states $|n_T, m = \pm n_T\rangle$ are SU(2) coherent states describing clockwise and counterclockwise circular motions, and therefore their coherent superposition may be considered as a *mesoscopic* Schrödinger Cat-like superposition [9].

On the basis of all the previous considerations we can state that, if we start our experiment from a vibrational state containing a coherent superposition of the states $|n_T, m = \pm n_T\rangle$ with equal weights, the procedure here reported makes it possible to extract the corresponding angular momentum "Schrödinger Cat".

The Fock state $|n_x = n_T, n_y = 0\rangle$ possessing n_T excitations along the direction x and no excitation along y possesses such a kind of superposition, and in particular it turns out $\langle n_x = n_T, n_y = 0|\ n_T, m = \pm n_T\rangle = 1/2^{n_T/2}$ [8]. Therefore the superposition

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left[|n_T, m = n_T\rangle + |n_T, m = -n_T\rangle \right] \tag{17}$$

may be generated starting from the Fock state $|n_x = n_T, n_y = 0\rangle$ with an efficiency $\mathfrak{E} = 1/2^{n_T - 1}$, provided $\gamma\tau\sqrt{4\,(n_T + 1)} = q\pi$.

The scheme may be generalized considering a Fock state directed along directions different from x. Denoting by θ the angle between the initial direction of motion of the Fock state and the x axis, one obtains the state

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left[|n_T, m = n_T\rangle + e^{i2n_T\theta} |n_T, m = -n_T\rangle \right] , \tag{18}$$

again with an efficiency $\mathfrak{E} = 1/2^{n_T - 1}$ [8], such an efficiency being independent of the relative phase between the two terms of the superposition.

4. Conclusive Remarks

Summarizing, in this paper we have shown the possibility of realizing the generation of interesting non-classical states through their *extraction* from the initial condition. The scheme reported, which generalizes quantum non-demolition measurement procedures, is a practical realization of a projection operator onto a subspace of the vibrational part of the Hilbert space associated with a trapped ion.

On the basis of such a strategy we have reported an effective procedure for generating non-classical states, and in particular coherent superpositions of angular momentum eigenstates, which may be interpreted as Schrödinger Cat states involving SU(2) coherent states. The relative quantum phase between the two terms of

the superposition turns out to be well controllable being proportional to the angle between the initial trapped ion center of mass motion and the x axis.

The efficiency of the method is related to some specific conditions, in particular to the amount of overlap between the initial condition of the ion center of mass motion and the state we want to obtain. In the considered examples such an efficiency is reasonable: for $n_T = 4$, the efficiency is $1/8$ and for $n_T = 3$ we have $\mathfrak{E} = 1/4$. Moreover, we mention that the number of steps necessary to obtain a good fidelity is also reasonable. For instance, in order to obtain the Schrödinger Cat in Eq. (18) with just a 5% error, in the case $n_T = 4$, five steps (i.e. interactions followed by measurements) are enough.

In our future work we will analyze the effects of decoherence during the process.

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