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QUANTUM
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Entanglement and Bell Theorem, 32 Years Later

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Abstract. We shall discuss the role played in the quantum theory by entanglement. A derivation of the Wigner–Bell inequality is presented.

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1. Introduction

The quantum phenomena can be harnessed to perform tasks impossible in classical physics. Quantum cryptography is already a branch of applied physics, quantum communication is *in statu nascedi* and finally quantum computation has plenty of theoretical results, and an intensive search for operational realizations continues.

Since the paper [1] of Einstein, Podolsky and Rosen (EPR) it was clear that the entangled states play a central role in our attempt to understand the roots of quantumness. EPR hoped to show that such states can be used to define "elements of reality", variables missing in the quantum theory, and in that way to show that quantum mechanics is an incomplete theory. However, they tested their concept only with the use of a single entangled state. Twenty nine years later Bell [2] showed that the introduction of elements of reality cannot lead to a completion of quantum theory, but rather to its revision. Wigner was perhaps the only one of the great minds that took part in the early development of quantum mechanics, who noticed the importance of Bell's theorem. He derived his own "Bell" inequality [3]. We present its derivation, different from the one given by Wigner. It reveals the central role of the Wigner inequality in our attempts to compare the classical probability calculus with the one used in quantum theory. Some direct applications of the Bell theorem will also be presented.

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2. The Bell Inequality of Wigner

The classical theory of probability is essentially a theory of normalized measures on sets. The following relation must be satisfied by such measures:

$$P(B \cap A) + P(B \cap C) \leq P(B) + P(A \cap C)\,, \tag{1}$$

which, together with some other assumptions, is equivalent to the Wigner inequality. The proof can also be given in the following way. Let $\chi_A(x)$ be the characteristic function of the set A. Such a function has only two values: $\chi_A(x) = 1$, when the element x belongs to A, and $\chi_A(x) = 0$ otherwise, and $P(A) = \int \chi(x) d\mu(x)$ with $\int d\mu(x) = 1$. In terms of the characteristic functions of the three sets involved in this relation is equivalent to

$$\chi_B(x)(\chi_A(x) + \chi_C(x)) \leq \chi_B(x) + \chi_A(x)\chi_C(x)\,, \tag{2}$$

which is an obvious relation for numbers that can be equal to 0 an 1.

Let us introduce a symmetric function, which can be called, as we shall see with good justification, the separation between two sets. Seemingly this idea is due to Kolmogorov, and was presented by him in 1948, however, he never applied this notion to quantum probabilities. The author learned about the idea from the paper by Santos [4]. One defines the separation as

$$S(A, B) = P(A) + P(B) - 2P(A \cap B)\,. \tag{3}$$

The first important property of $S(A, B)$ is that it is nonnegative. The second one is, that the inequality can now be reinterpreted, is a triangle inequality. Namely, using the definition (3) one can rewrite it as

$$S(A, B) + S(B, C) \geq S(A, C)\,. \tag{4}$$

This is the Wigner inequality (in disguise). It plays the central role in the discussion of quantum probabilities, because its violation implies the violation of the most elementary property of the separation.

$P(B \cap A)$ can be thought of as the joint probability of yes–no tests A and B upon a quantum system to give a positive result. However, the quantum mechanical rules would require that the two projection operators associated with such tests have to commute (to warrant commeasurability). This can be warranted by requiring that the two yes–no observables are associated with measurements on two different particles very far away from each other (relativity theory guarantees that operators defined in two spatially separated regions of space-time commute). However, in the case of the triangle inequality, there is no way to follow this procedure without additional assumptions. The assumptions made by Wigner (1970), namely those of perfect correlations between two spatially separated subsystems, of the kind present in maximally entangled states, enabled him to interpret (1) as a Bell inequality.[a]

Simply, if a A is an observable for particle 1, then C must be a property of particle 2 (since both appear in $P(C \cap A)$), therefore B pertains to particle 1 (recall we have $P(C \cap B)$), and finally we arrive at the following: the probability $P(A \cap B)$ describes two properties of a single particle, number 1, measured together (i.e. the relativistic principle does not warrant commeasurability). For pairs of noncommuting yes–no observables, there is no quantum mechanical formula that would give their joint probability. The inequalities (1)–(5) cannot be compared with quantum mechanical predictions. To avoid this, Wigner assumed a perfect correlation between the property A of system one, with a property A' of the system two, which is possible when the systems are described by an entangled (i.e. correlated) state. In this way one obtains

$$S(B, A') + S(B, C) \geq S(A, C)\,. \tag{5}$$

Here the properties of the system one appear to the left of those for system two.

A different solution is to consider a quadrangle inequality which is a consequence of two triangle inequalities:

$$S(A', B) + S(A, B) + S(A, B') \geq S(A', B')\,. \tag{6}$$

The tests A and A' are performed upon particle 1, whereas the tests B and B' upon particle 2. There is no problem with the commeasurability of the yes–no operators. The quadrangle inequality can be rewritten with the use of (3) as

$$P(A' \cap B) + P(B \cap A) + P(A \cap B') - P(A' \cap B') - P(A) - P(B) \leq 0\,. \tag{7}$$

this is the Clauser–Horne inequality [5].

3. Direct Applications of Bell's Theorem

Entanglement is the essential feature, which distinguishes the quantum from the classical. It also makes some possible communication tasks, which are absolutely impossible in classical physics. In a recent breakthrough paper [6] Scarani and Gisin have shown that the violation of Bell's inequality is a sufficient condition for the security of quantum key distribution protocols. Extending this result, one may put a general conjecture that only if an entangled state violates a Bell inequality, the efficiency of certain communication protocols, which make use of that state, is higher than of any protocol which does not use such states. Otherwise the protocol is explainable by a local realistic model, and thus achievable for classical physics.

A recent result supports this conjecture. It turns out that one can show that the violation of Bell's inequalities is a necessary and sufficient criterion for the quantum solutions of some communication complexity problems to be more efficient than any classical solution.

Communication complexity problems were introduced in Ref. [7]. Some input data are distributed over n separated parties. Each party knows the local data, but does not the data of the others. The party i obtains an input string z_i. The goal is for each of them to determine the value of some function $f(z_1, \ldots, z_n)$, while exchanging a *restricted* amount of information. This restriction enables the parties to compute the function only with an error. The goal for all parties is to compute the function correctly with a probability as high as possible. An execution is considered successful, if the values determined by *all* parties are correct. Before they start the protocol, the parties are allowed to share any *other* data, which might improve the success of the protocols. They are allowed to process their data locally in whatever way.

It was shown that entanglement can improve the probability of success in communication complexity protocols beyond the limits which are classically possible (see [8] and the list of references therein). For any Bell's inequality for n qubits there exists at least one problem such that the success of its solution in the quantum protocol is higher than in any classical one [8]. The *necessary* and *sufficient* condition for the quantum advantage is that the N qubits violate the Bell inequality. As an exemplary application the complete set of 2^{2^N} of N-qubit Bell's inequalities for correlation functions [9] was used. Extensions of these results also exist. For instance, two entangled qutrits, which violate a specific Bell inequality, can also be utilized to such tasks [10].

Note

a. However, precisely due this assumption Wigner's inequality cannot be tested in the laboratory. Simply, because perfect correlations cannot be observed, due to the inevitable experimental imprecision.

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