Acta Phys. Hung. B 20/1–2 (2004) 15–23

QUANTUM ELECTRONICS

Gauge-Invariant Relativistic Wigner Functions in External Electromagnetic Fields

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Received 30 July 2003

Abstract. On the basis of the Hamiltonian form of the Klein–Gordon equation of a charged scalar particle field introduced by Feshbach and Villars, the gauge-invariant 2×2 Wigner matrix has been constructed whose diagonal elements describe positive and negative charge densities and the off-diagonal elements correspond to cross-densities in phase-space. The system of coupled transport equations has been derived in case of interaction with an arbitrary external electromagnetic field. A gauge-independent generalization of the free particle representation due to Feshbach and Villars is given, and on the basis of it both the nonrelativistic and the classical limits of the general relativistic quantum Boltzmann–Vlasov equation (RQBVE) is discussed. In the non-relativistic limit ($p/mc \to 0$) the set of equations of motion decouple to two independent quantum transport equations describing the dynamics of oppositely charged positon and negaton densities. In the classical limit ($\hbar \to 0$) two relativistic Boltzmann–Vlasov equations result for the diagonal positon and negaton densities. Even though the Planck constant $\hbar$ is absent in the latter equations, the real part of the positon–negaton cross density does not vanish.

Keywords: Wigner functions, gauge invariance
PACS: 03.65.-w, 03.65.Pm, 05.60.Gg, 52.60.+h

1. Introduction

For some time now, the gauge-invariant relativistic Wigner functions have been the subject of a growing research activity [1–6]. The quantum corrections to the classical non-relativistic and relativistic transport equations can be conveniently studied by using the equations of motion in phase space for the corresponding

1589-9535/04/ $ 20.00

Wigner functions. This way the classical intuition may be taken over to quantum mechanics even in the relativistic domain.

It is known [3] that for a nonrelativistic charged particle the gauge-invariant Wigner function has to be defined as

$$\begin{aligned} W(\mathbf{r},\mathbf{k};t) \;&\equiv\; (2\pi\hbar)^{-3}\int d^3u\,\psi^*\left(\mathbf{r}+\frac{1}{2}\mathbf{u},t\right)\psi\left(\mathbf{r}-\frac{1}{2}\mathbf{u},t\right) \\ &\times\exp\left\{\frac{i}{\hbar}\mathbf{u}\cdot\left[\mathbf{k}+\frac{e}{c}\int_{-1/2}^{+1/2} ds\mathbf{A}(\mathbf{r}+s\mathbf{u},t)\right]\right\}, \end{aligned} \tag{1}$$

where $\mathbf{k}$ here denotes the kinetic momentum variable. Thanks to the line integral in the exponent, this function preserves its form under the simultaneous gauge transformation of first kind $\psi\to\psi'=\exp\left(\frac{ie}{\hbar c}\chi\right)\psi$ and of second kind $\mathbf{A}\to\mathbf{A}'=\mathbf{A}+\nabla\chi$ where χ is the generating function of these transformations. The manifestly covariant relativistic gauge-invariant Wigner function of a charged scalar particle has already long been constructed by Stratonovich [7]

$$\begin{aligned} W(x,p) \;&\equiv\; (2\pi\hbar)^{-4}\int d^4y\,\Phi^*\left(x+\frac{1}{2}y,t\right)\Phi\left(x-\frac{1}{2}y,t\right) \\ &\times\exp\left\{-\frac{i}{\hbar}y\cdot\left[p+\frac{e}{c}\int_{-1/2}^{+1/2} dsA(x+sy,t)\right]\right\}. \end{aligned} \tag{2}$$

For Dirac particles studied by Vasak, Gyulassi and Elze [2] a similar construction leads to a 4×4 Wigner matrix

$$\begin{aligned} W(x,p) \;&\equiv\; (2\pi\hbar)^{-4}\int d^4y\,\Psi\left(x-\frac{1}{2}y,t\right)\bar{\Psi}\left(x+\frac{1}{2}y,t\right) \\ &\times\exp\left\{-\frac{i}{\hbar}y\cdot\left[p+\frac{e}{c}\int_{-1/2}^{+1/2} dsA(x+sy,t)\right]\right\}, \end{aligned} \tag{3}$$

whose components can be expressed by 16 real phase-space densities W_0, W_μ, W_5, $W_{\mu 5}$ and $W_{\mu\nu}$ where μ and ν are spinor indices. The common in these latter two expressions is that the integration with respect to y goes over through the whole space-time. Accordingly, the Wigner functions depend on the four position x and on the four kinetic momentum p. As has been shown [2,7], in deriving the covariant equations of motions one encounters in addition with certain constraint equations which are generalizations of the mass-shell relation on phase-space. In the following we use the Hamiltonian description rather than the covariant formulation.

The motivation of this work is to develop the relativistic phase-space description of the dynamics of charged Klein–Gordon (KG) particles analogously to the

Hamiltonian description used earlier by Bialynicki-Birula, Górnicki and Rafelski for the Dirac particles [1]. The advantage of this method compared with the manifestly covariant descriptions [2, 7] is that a single time parameter is used to describe the dynamics. In Section 2, on the basis of the Fesbach–Villars Hamiltonian formulation of the KG equation [8], we give the gauge-invariant definition of the 2×2 Wigner matrix of charged scalar particles, and summarize the most important physical properties of it. In Section 3 we present the gauge-independent relativistic quantum Boltzmann–Vlasov equation (RQBVE) for the Wigner matrix and derive the coupled set of equations of motion for the four real phase-space distributions related to diagonal and cross densities of KG particles and their antiparticles (positons and negatons). In Section 4 we study the nonrelativistic limit ($p/mc \to 0$) and the relativistic classical limit ($\hbar \to 0$) of the RQBVE. In the latter case we derive an explicit solution for the Wigner matrix in terms of the classical distribution functions of positons and negatons.

2. Feshbach–Villars Formulation of the Klein–Gordon Equation and the Gauge-Invariant Wigner Matrix of Charge Scalar Particles

The Klein–Gordon equation of a charged scalar field of mass m and of charge e,

$$\left[\left(i\hbar\partial - \frac{e}{c}A\right)^2 - (mc)^2\right]\Phi = 0 \tag{4}$$

is a second order equation with respect to the time derivative . We use the convention for the metric $g = \mathrm{diag}(+,-,-,-)$ and the notations $\partial = \{\partial_\mu\} = \partial/\partial x^\mu$, $\{x^\mu\} = (ct, \mathbf{r})$ and $A = \{A^\mu\} = (A_0, \mathbf{A})$. The four product is denoted by $a \cdot b = a_\mu \cdot b^\mu$ and $a^2 = a \cdot a$. In order to get the Hamiltonian form of equation (4) we follow Feshbach and Villars [8] and introduce the two-component wave function:

$$\Psi \equiv \begin{pmatrix} \phi \\ \chi \end{pmatrix}, \quad \phi, \chi \equiv \frac{1}{\sqrt{2}}\left(\Phi \pm \frac{1}{mc}\Pi_0\Phi\right), \quad \Pi_0 \equiv i\hbar\partial_0 - \frac{e}{c}A_0\,, \tag{5}$$

in terms of which (4) can be written as

$$i\hbar\frac{\partial\Psi}{\partial t} = \left[(\tau_3 + i\tau_2)\frac{1}{2m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2 + eA_0 + \tau_3 mc^2\right]\Psi\,. \tag{6}$$

In Eq. (6) and henceforth the notations

$$\tau_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{7}$$

for the Pauli matrices are used. In the frame of the Hamiltonian description the charge density (in the units of e) and the normalization conditions can be expressed as

$$\rho = \phi^*\phi - \chi^*\chi = \Psi^+\tau_3\Psi \equiv \bar{\Psi}\cdot\Psi, \qquad \int d^3r\,\bar{\Psi}\cdot\Psi = \pm 1\,, \tag{8}$$

where $\bar{\Psi} = \Psi^{+}\tau_3$ denotes the Feshbach–Villars adjoint of the wave function Ψ. We see that the charge density is the difference of two positive definite densities. The $\pm$ signs on the right hand side of the normalization condition correspond to positive and negative energy solutions, in other words, they refer to quantum states of positons and negatons, respectively.

Now we define the gauge-invariant Wigner matrix in the Feshbach–Villars representation

$$W(\mathbf{r},\mathbf{p};t) \equiv (2\pi\hbar)^{-3}\int d^3y\,\Psi\left(\mathbf{r}-\frac{1}{2}\mathbf{y},t\right)\bar{\Psi}\left(\mathbf{r}+\frac{1}{2}\mathbf{y},t\right) \times \exp\left\{\frac{i}{\hbar}\mathbf{y}\cdot\left[\mathbf{p}+\frac{e}{c}\int_{-1/2}^{+1/2} ds\mathbf{A}(\mathbf{r}+s\mathbf{y},t)\right]\right\}. \tag{9}$$

The vector parameter $\mathbf{p}$ denotes the (gauge-invariant) kinetic momentum, and the line integral on the right-hand side of (10) secures the gauge invariance of W, as has been shown e.g. in [3]. By construction W is a self-adjoint matrix in the Feshbach–Villars sense, namely $\bar{W} \equiv \tau_3 W^{+}\tau_3 = W$ and its explicit form can be expressed in terms of the components ϕ and χ, yielding

$$2W = 2\begin{pmatrix} W_{\phi\phi} & -W^{*}_{\phi\chi} \\ W_{\phi\chi} & -W_{\chi\chi}\end{pmatrix} = \tau_0 W_0 + i\tau_1 W_1 - i\tau_2 W_2 + \tau_3 W_3. \tag{10}$$

In Eq. (10) W_0, W_1, W_2, W_3 are real and can be expressed as:

$$\begin{pmatrix} W_0 \\ W_1 \\ W_2 \\ W_3 \end{pmatrix} = \mathrm{Tr}\left[W\begin{pmatrix}\tau_0 \\ -i\tau_1 \\ i\tau_2 \\ \tau_3\end{pmatrix}\right]. \tag{11}$$

It can be easily checked that $W_0 = W_{\phi\phi} - W_{\chi\chi}$, $W_1 = 2\,\mathrm{Im}[W_{\phi\chi}]$, $W_2 = 2\,\mathrm{Re}[W_{\phi\chi}]$ and $W_3 = W_{\phi\phi} + W_{\chi\chi}$. W_0 and W_3 are proportional to the charge density and mass density in phase space, respectively, showing that the center of charge and of mass do not necessarily coincide. The sum of W_2 and W_3 is related to the current density:

$$\mathbf{j}(\mathbf{r},t) = \int d^3p\,\mathbf{p}[W_2(\mathbf{r},\mathbf{p},t) + W_3(\mathbf{r},\mathbf{p},t)]\,. \tag{12}$$

The imaginary part W_1 of the cross density $W_{\phi\chi}$ can be directly related to the Zitterbewegung. As simple illustrations we give the Wigner matrix of a free positon and a free negaton plane wave of momentum $\mathbf{q}$

$$W_q^{+} = \frac{1}{V}\delta_3(\mathbf{p}-\mathbf{q})\begin{pmatrix}\cosh^2\xi & \frac{1}{2}\sinh 2\xi \\ -\frac{1}{2}\sinh 2\xi & -\sinh^2\xi\end{pmatrix}, \tag{13}$$

$$W_q^{-} = \frac{1}{V}\delta_3(\mathbf{p}-\mathbf{q})\begin{pmatrix}\sinh^2\xi & \frac{1}{2}\sinh 2\xi \\ -\frac{1}{2}\sinh 2\xi & -\cosh^2\xi\end{pmatrix}, \tag{14}$$

where V is the normalization volume and $\xi = \frac{1}{2}\log\sqrt{1+(q/mc)^2}\,$.

3. Gauge-Independent Relativistic Quantum Boltzmann–Vlasov Equation of Charge Scalar Particles

The equation of motion for the Wigner matrix (10) defined in the previous section can be derived from the Schrödinger-type equation (6) for $\Psi(\mathbf{r} \pm \frac{1}{2}\mathbf{y}, t)$ by using the Gauss theorem (partial integrations with vanishing surface terms) in calculating the $\mathbf{y}$ integral. We obtain

$$\left(\partial_t + \hat{D}_E\right) W + \left(\hat{D}_c + \hat{D}_B\right) \frac{1}{2} \left\{\tau_3 + i\tau_2, W\right\} + \frac{i}{\hbar} \left[H_0(\hat{P}), W\right] = 0\,, \tag{15}$$

where

$$\hat{D}_E \equiv e\hat{\mathbf{E}} \cdot \nabla_p, \quad \hat{\mathbf{E}} = j_0 \left(\frac{\hbar}{2}\nabla_r \cdot \nabla_p\right) \mathbf{E}\left(\mathbf{r}, t\right)\,, \tag{16}$$

$$\hat{D}_c \equiv \frac{1}{m} \left(\mathbf{p} + \Delta\hat{\mathbf{p}}\right) \cdot \nabla_r, \qquad \Delta\hat{\mathbf{p}} \equiv -\frac{e\hbar}{4c}\nabla_p \times \tilde{\mathbf{B}}\,, \tag{17}$$

$$\tilde{\mathbf{B}} = j_1 \left(\frac{\hbar}{2}\nabla_r \cdot \nabla_p\right) \mathbf{B}\left(\mathbf{r}, t\right)\,, \tag{18}$$

$$\hat{D}_B \equiv \frac{e}{mc}[(\mathbf{p} + \Delta\hat{\mathbf{p}}) \times \hat{\mathbf{B}}] \cdot \nabla_p, \quad \hat{\mathbf{B}} = j_0 \left(\frac{\hbar}{2}\nabla_r \cdot \nabla_p\right) \mathbf{B}\left(\mathbf{r}, t\right), \tag{19}$$

$$\hat{P}^2 \equiv \left(\mathbf{p} + \Delta\hat{\mathbf{p}}\right)^2 - \frac{\hbar^2}{4} \left(\nabla_r - \frac{e}{c}\nabla_p \times \hat{\mathbf{B}}\right)^2, \tag{20}$$

$$H_0(\hat{P}) \equiv \left(\tau_3 + i\tau_2\right) \frac{\hat{P}^2}{2m} + \tau_3 mc^2. \tag{21}$$

In equations above the functions $j_0\left(x\right) = \sin x/x = 1 - x^2/3! + x^4/5! \mp \ldots$ and $j_1\left(x\right) = -dj_0\left(x\right)/dx = x/3 - x^3/(3! \cdot 5) + x^5/(5! \cdot 7) \mp \ldots$ are ordinary spherical Bessel functions of first kind of order zero and one, respectively. In the argument of the Bessel functions the gradient ∇_r acts on the electric field strength $\mathbf{E}$ and on the magnetic induction $\mathbf{B}$, and the gradient ∇_p acts on the Wigner functions. Notice that $\hat{\mathbf{E}}$, $\hat{\mathbf{B}}$ and the momentum correction operator $\Delta\hat{\mathbf{p}}$ can be expanded into a power series containing only even powers of $\hbar$. The matrix $H_0(\hat{P})$ in the limit of $\hbar \to 0$ becomes the free particle Hamiltonian $H_0(p)$ in momentum representation. By using the decomposition $W = \mathrm{Re}W + i\,\mathrm{Im}W$ in Eq. (15) we can derive the set of two real equations:

$$\left(\partial_t + \hat{D}_E\right) 2\mathrm{Re}W + \left(\hat{D}_c + \hat{D}_B\right) \frac{1}{2} \left\{\tau_3 + i\tau_2, 2\,\mathrm{Re}W\right\} - \frac{1}{\hbar} \left[H_0(\hat{P}), \tau_1\right] W_1 = 0\,, \tag{22}$$

$$\left(\partial_t + \hat{D}_E\right) \tau_1 W_1 + \frac{1}{\hbar} \left[H_0(\hat{P}), 2\,\mathrm{Re}W\right] = 0\,. \tag{23}$$

In obtaining Eq. (23) we have taken into account that $2\,\mathrm{Im}W = \tau_1 W_1$, and the anticommutator $\{\tau_3 + i\tau_2, \tau_1\}$ vanishes.

The real part of $2\,W$ can be expressed as $2\,\mathrm{Re}W = \tau_0 W_0 - i\tau_2 W_2 + \tau_3 W_3$. With the help of this decomposition the following set of coupled equations can be derived for the real phase-space densities W_0, W_1, W_2, W_3

$$\left(\partial_t + \hat{D}_E\right) W_0 + \left(\hat{D}_c + \hat{D}_B\right)(W_2 + W_3) = 0\,, \tag{24}$$

$$\left(\partial_t + \hat{D}_E\right) W_2 - \left(\hat{D}_c + \hat{D}_B\right) W_0 + 2\omega_c \left(1 + \frac{\hat{P}^2}{2m^2c^2}\right) W_1 = 0\,, \tag{25}$$

$$\left(\partial_t + \hat{D}_E\right) W_3 + \left(\hat{D}_c + \hat{D}_B\right) W_0 - 2\omega_c \frac{\hat{P}^2}{2m^2c^2} W_1 = 0\,, \tag{26}$$

$$\left(\partial_t + \hat{D}_E\right) W_1 - 2\omega_c \frac{\hat{P}^2}{2m^2c^2}(W_2 + W_3) - 2\omega_c W_2 = 0\,, \tag{27}$$

where we introduced the Compton frequency $\omega_c = mc^2/\hbar$. We term the set of equations (24)–(27) or their matrix equivalent (15), relativistic quantum Boltzmann–Vlasov equation (RQBVE). As a boundary condition for the RQBVE in free space it is natural to assume that the functions W_μ vanish sufficiently fast as $\mathbf{r} \to \infty$ and $\mathbf{p} \to \infty$, such that the integrals of W_μ over the whole phase space are finite.

4. The Nonrelativistic Limit and the Classical Limit of the RQBVE

In the present section we derive both the nonrelativistic limit ($p/mc \to 0$) and the relativistic classical limit ($\hbar \to 0$) of the RQBVE with the help of the generalization of the free particle representation introduced by Feshbach and Villars [8] in their classic paper on the relativistic wave functions. In order to do that let us introduce the matrix operator

$$U(\hat{P}) = \exp[-\tau_1 \hat{\xi}(\hat{P})], \qquad \hat{\xi}(\hat{P}) = \frac{1}{2}\log\sqrt{1 + \hat{P}^2/m^2c^2}, \tag{28}$$

where $\hat{P}^2$ is given by Eq. (20). By performing the similarity transformation generated by U on the equation of motion (15) we have

$$\left(\partial_t' + \hat{D}_E'\right) W' + \left(\hat{D}_c' + \hat{D}_B'\right) \frac{mc^2}{E_{\hat{P}}} \frac{1}{2}\{\tau_3 + i\tau_2, W'\} + \frac{i}{\hbar} E_{\hat{P}}\,[\tau_3, W'] = 0\,, \tag{29}$$

where the transformed quantities are $W' = U^{-1}WU$, $\hat{D}_E' = U^{-1}\hat{D}_E U$, etc., and

$$E_{\hat{P}} = \sqrt{m^2c^4 + c^2\hat{P}^2}\,. \tag{30}$$

Now let us assume that the W_μ-s are confined to regions in phase space in which $p/mc \ll 1$ and the characteristic frequencies and wave numbers are much smaller than the Compton frequency and the Compton wavelength, respectively. The latter two conditions can be symbolically written as

$$\partial_t \ll \omega_c = mc^2/\hbar, \qquad \bigtriangledown_r \ll \kappa_c = mc/\hbar\,. \tag{31}$$

Moreover, we assume in addition that

$$Bmc\bigtriangledown_p \ll B_{cr}, \qquad Emc\bigtriangledown_p \ll E_{cr}\,, \tag{32}$$

where $B_{cr} = E_{cr} = m^2c^3/e\hbar$ are the critical field strengths of quantum electrodynamics. Under these conditions $E(\hat{P}) \to mc^2$, $\xi(\hat{P}) \to 0$, hence $U(\hat{P}) \to 1$, and $W' \to W^{NR}$ becomes diagonal. The remaining equations can be combined to yield

$$\left\{\partial_t + (\mathbf{v} + \Delta\hat{\mathbf{v}}) \cdot \nabla_r + e\left[\hat{\mathbf{E}} + \frac{1}{c}(\mathbf{v} + \Delta\hat{\mathbf{v}}) \times \hat{\mathbf{B}}\right] \cdot \nabla_p\right\} F = 0, \tag{33}$$

$$\left\{\partial_t + (\mathbf{v} + \Delta\hat{\mathbf{v}}) \cdot \nabla_r - e\left[\hat{\mathbf{E}} + \frac{1}{c}(\mathbf{v} + \Delta\hat{\mathbf{v}}) \times \hat{\mathbf{B}}\right] \cdot \nabla_p\right\} G = 0\,, \tag{34}$$

where $F(\mathbf{r}, \mathbf{p}, t) = W^{NR}_{\phi\phi}(\mathbf{r}, \mathbf{p}, t)$ and $G(\mathbf{r}, \mathbf{p}, t) = W^{NR}_{\chi\chi}(\mathbf{r}, -\mathbf{p}, t)$ and $\mathbf{v} = \mathbf{p}/m$, $\Delta\hat{\mathbf{v}} = \Delta\hat{\mathbf{p}}/m$. Equation (33) coincides with the nonrelativistic quantum Boltzmann–Vlasov equation derived earlier [3] by the present authors. Equation (34) refers to an oppositely charged Schrödinger particle, demonstrating that the RQBVE describes simultaneously the dynamics of both positons and negatons.

In order to consider the classical limit of the relativistic dynamics in phase space we first observe that

$$\hat{D}_E \to e\mathbf{E} \cdot \nabla_p, \qquad \hat{D}_B \to \frac{e}{mc}(\mathbf{p} \times \mathbf{B})\nabla_p, \qquad U(\hat{P}) \to U(p)\,, \tag{35}$$

$$\hat{D}_c \to \frac{1}{m}\mathbf{p} \cdot \nabla_r, \qquad \hat{P}^2 \to p^2, \qquad H_0(\hat{P}) \to H_0(p) \tag{36}$$

in the $\hbar \to 0$ limit. Equation (22) goes over to a meaningful limit equation only if $W_1 \to 0$ stronger than $\hbar \to 0$. Hence $W^{cl} = \mathrm{Re}W^{cl}$. We note that a more rigorous treatment of the $\hbar \to 0$ limit can be performed by the coarse graining technique introduced by Shin and Rafelski [5] in the case of Dirac electrons. By performing the similarity transformation $W' = U^{-1}(p)W^{cl}U(p)$ generated by the matrix $U(p)$ we have

$$\begin{aligned}(\partial_t + e\mathbf{E} \cdot \nabla_p)\,W' - e\mathbf{E} \cdot \frac{\mathbf{p}c^2}{2E_p^2}[\tau_1, W'] \\ + \quad \frac{mc^2}{E_p}\left[\frac{1}{m}\mathbf{p} \cdot \nabla_r + \frac{e}{mc}(\mathbf{p} \times \mathbf{B}) \cdot \nabla_p\right]\frac{1}{2}\{\tau_3 + i\tau_2, W'\} = 0\,,\end{aligned} \tag{37}$$

$$[\tau_3, W'] = 0\,. \tag{38}$$

According to Eq. (38) W' must be diagonal, hence, by denoting $W'_{\phi\phi}(\mathbf{r},\mathbf{p},t)=f$ and $W'_{\chi\chi}(\mathbf{r},-\mathbf{p},t)=g$ we have:

$$\begin{aligned} W^{cl} &= U(p)\begin{pmatrix} f & 0 \\ 0 & -g \end{pmatrix} U^{-1}(p) \\ &= f\begin{pmatrix} \cosh^2\xi & \frac{1}{2}\sinh 2\xi \\ -\frac{1}{2}\sinh 2\xi & -\sinh^2\xi \end{pmatrix} + g\begin{pmatrix} \sinh^2\xi & \frac{1}{2}\sinh 2\xi \\ -\frac{1}{2}\sinh 2\xi & -\cosh^2\xi \end{pmatrix} \end{aligned} \tag{39}$$

with $\xi = \frac{1}{2}\log\sqrt{1+(p/mc)^2}$. From the off-diagonal part of Eq. (37) the following balance equation can be derived:

$$\int_V d^3r \int d^3p\; e\mathbf{E}\cdot\mathbf{v}(\mathbf{p})(f-g) = -\int_S d\mathbf{s}\cdot\int d^3p\;\gamma mc^2\mathbf{v}(\mathbf{p})(f+g)\,. \tag{40}$$

This means that the work done by the electric field on the positon–negaton field per unit time in a volume V equals to the inward flux of the energy current density over the bounding surface S. The equation of motion for the classical phase-space distribution can be derived from Eq. (37)

$$\partial_t f + \mathbf{v}(\mathbf{p})\cdot\nabla_r f + e\left[\mathbf{E}+\frac{1}{c}\left(\mathbf{v}(\mathbf{p})\times\mathbf{B}\right)\right]\cdot\nabla_p f = 0\,, \tag{41}$$

$$\partial_t g + \mathbf{v}(\mathbf{p})\cdot\nabla_r g - e\left[\mathbf{E}+\frac{1}{c}\left(\mathbf{v}(\mathbf{p})\times\mathbf{B}\right)\right]\cdot\nabla_p g = 0\,, \tag{42}$$

$$\mathbf{v}(\mathbf{p}) = \frac{\mathbf{p}/m}{\sqrt{1+(p/mc)^2}} = \mathbf{p}/m\gamma \tag{43}$$

is the velocity function, and $\gamma = 1/\sqrt{1-v^2/c^2}$ denotes the usual relativistic factor.

As is seen from Eqs. (41) and (42), in the classical limit the RQBVE can be reduced to two uncoupled relativistic Boltzmann–Vlasov equations describing the phase-space dynamics of positons and negatons. According to Eq. (39), in case of positons we take $g=0$ and we have:

$$W_+^{cl} = f\begin{pmatrix} \cosh^2\xi & \frac{1}{2}\sinh 2\xi \\ -\frac{1}{2}\sinh 2\xi & -\sinh^2\xi \end{pmatrix}, \qquad \int d^3r\, d^3p\,\mathrm{Tr}[W_+^{cl}] = +1\,. \tag{44}$$

Similarly for a negaton solution we have

$$W_-^{cl} = g\begin{pmatrix} \sinh^2\xi & \frac{1}{2}\sinh 2\xi \\ -\frac{1}{2}\sinh 2\xi & -\cosh^2\xi \end{pmatrix}, \qquad \int d^3r\, d^3p\,\mathrm{Tr}[W_-^{cl}] = -1\,. \tag{45}$$

Finally, it is interesting to note that the real part of the cross density W_2 does not vanish in the relativistic classical limit. The physical significance of this behavior will be discussed elsewhere.

5. Summary

In order to consider the Hamiltonian description of charged scaler particles we have introduced in Section 2 the Feshbach–Villars formulation of the Klein–Gordon equation and defined the gauge-invariant 2×2 Wigner matrix. In Section 3 we derived the gauge-independent relativistic quantum Boltzmann–Vlasov (RQBVE) equation for this Wigner matrix which leads to a set of four equations for four real phase-space densities whose physical meanings have been given. In Section 4 we have shown that the RQBVE reduces to two uncoupled quantum transport equations for positon and negaton in the nonrelativistic limit. In the classical limit $\hbar \to 0$ one arrives at two uncoupled relativistic Vlasov equations for positon and negaton phase-space densities, however there remains a classical constraint equation (which does not contain $\hbar$). This means that a connection between the positon and negaton densities survives the classical limit. Finally, we have written down the general solution (39) for the Wigner matrix in the classical limit.

Acknowledgements

This work was started during a stay of one of the authors (S.V.) at the Max-Planck-Institut für Quantenoptik, Garching, Germany in the frame of a DAAD Fellowship No. A/01/19250, and was completed in the frame of a NATO Fellowship No. 2082/NATO/02. S.V. is grateful to NSF and NASA for support for a short visit to University of Connecticut. The partial support by the Hungarian National Science Foundation (OTKA) project number T032375 and the European Centre of Excellence Program (KFKI, Condensed Matter Research Centre, contract No.: ICAI-CT-3000-70039) are also acknowledged.

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