Acta Phys. Hung. B 20/1–2 (2004) 7–10

QUANTUM ELECTRONICS

Quantum Fluctuations in Nonlinear Systems

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Received 30 July 2003

Abstract. The Husimi Q function is used to describe the quantum evolution of the field state in the nonlinear process of second-harmonic generation. The Q function for the initial quantum state is used as a given distribution for generating classical random numbers that play a role of the initial values for the classical equations of motion. The evolution of classical and quantum fluctuations is compared for a particular initial state with only two photons in the fundamental mode. It is shown that some features of the quantum state that has no classical counterpart can be recognized from the evolution of classical fluctuations.

Keywords: quantum fluctuations, quantum noise, nonlinear systems
PACS: 42.50.-p, 42.50.Lc

1. Introduction

Nonlinear optical processes are known to produce optical fields with nonclassical properties [1]. One of the best known nonlinear optical phenomena is the second-harmonic generation observed in 1961 [2]. The classical treatment of the problem allows for closed-form solutions, while for quantum fields the closed-form analytical solution has not been found. It has been shown, however, that the quantum states produced in the second-harmonic generation exhibit a number of unique quantum features such as photon antibunching [3] and squeezing [4, 5] for both fundamental and second-harmonic modes (for review see [6]). A nice tool to study properties of the quantum fields are the quasiprobability distributions such as the Wigner function or the Husimi Q function. The latter is always positive and can play the role of the classical probability distribution. Nikitin and Masalov [7] have calculated numerically the Q function, and suggested a possibility of obtaining a Schrödinger cat type of state in the process. The Q function, however, has disadvantage of not leading to correct marginal distributions, and in this respect the Wigner function

1589-9535/04/ $ 20.00

is more appropriate, but it has another disadvantage of taking negative values for nonclassical states, which precludes its use as classical probability distribution. For coherent states or vacuum both the Wigner and the Q functions are Gaussian distributions with different dispersions (the Wigner distribution is narrower). It is easy to simulate random variables with Gaussian distributions, and quantum fluctuations of the field can be simulated by classical random variables that define initial conditions for the classical trajectories. This method has been used by Bajer et al. [8, 9] to explain the sub-Poissonian photon statistics in the second and third-harmonic generation in the no-energy exchange regime. If the initial field has nonzero coherent component (nonzero mean value) the classical trajectory approach reproduces quite nicely the shape of the Q function [6]. How does it work for highly nonclassical initial states? In this paper we give an example of the second-harmonic generation with the initial Fock state with just two photons in the fundamental mode which evolves into the state with one photon of the second-harmonic mode. In this particular case the exact analytical solution is known, and the analytical expression for the Q function can be compared with the classical trajectories approach. We show that even for such a drastically nonclassical state, the classical trajectories reproduce some features of the quantum evolution.

2. Classical Simulation of Quantum Noise

Classical equations of motion for the fields in the second-harmonic generation, with the amplitude of the fundamental mode given by $\alpha = |\alpha| \mathrm{e}^{i\phi_\alpha}$ and the amplitude of the second-harmonic mode $\beta = |\beta| \mathrm{e}^{i\phi_\beta}$, have the following form [6]

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tau}|\alpha| &= -\sqrt{2}\,|\alpha||\beta|\sin\vartheta\,, & \frac{\mathrm{d}}{\mathrm{d}\tau}|\beta| &= \frac{1}{\sqrt{2}}\,|\alpha|^2\sin\vartheta\,, \\ \frac{\mathrm{d}}{\mathrm{d}\tau}\phi_a &= -\sqrt{2}\,|\beta|\cos\vartheta\,, & \frac{\mathrm{d}}{\mathrm{d}\tau}\phi_b &= -\frac{1}{\sqrt{2}}\,\frac{|\alpha|^2}{|\beta|}\cos\vartheta\,, \end{aligned} \tag{1}$$

where $\vartheta = 2\phi_\alpha - \phi_\beta$ and $\tau = \sqrt{2}\,\kappa t$ is the scaled time (κ is the coupling constant). The system (1) has two integrals of motion

$$C_0 = |\alpha|^2 + 2|\beta|^2\,, \qquad C_I = |\alpha|^2|\beta|\cos\vartheta\,. \tag{2}$$

Equations (1) are solved numerically to find the classical trajectories.

The exact quantum solution for the initial state $|2,0\rangle$, with 2 photons in the fundamental mode and no photons in the second-harmonic mode, has the form

$$|\psi(\tau)\rangle \;=\; \cos\tau\,|2,0\rangle - i\sin\tau\,|0,1\rangle\,, \tag{3}$$

and it leads to the exact analytical expressions for the Q functions for the fundamental and second-harmonic modes

$$Q(\alpha,\tau) \;=\; \frac{1}{\pi}e^{-|\alpha|^2}\left(\frac{|\alpha|^4}{2}\cos^2\tau + \sin^2\tau\right) \tag{4}$$

$$Q(\beta,\tau) \quad = \quad \frac{1}{\pi} e^{-|\beta|^2} \left(\cos^2 \tau + |\beta|^2 \sin^2 \tau \right) \ . \tag{5}$$

For $\tau = 0$, $Q(\alpha, 0)$ represents the Q function for the Fock state with 2 photons while $Q(\beta, 0)$ is the Q function for the vacuum. For $\tau = \pi/2$, the fundamental mode is in the vacuum and the second-harmonic mode is in the one-photon state. The Q function for the vacuum has Gaussian shape with the maximum at zero, contrary to the Fock states $|0\rangle$ and $|1\rangle$ for which there is a minimum at zero. These are quite distinct shapes.

In Fig. 1 we compare the classical simulations represented by a cloud of 1000 points to the quantum evolution represented by the contour plots of the Q functions given by (4) and (5). The left-hand side figures show the initial states for the fundamental and second-harmonic modes, and the right-hand side figures represent the states at $\tau = \pi/2$. The initial clouds of points have been obtained by generating random numbers with Gaussian distribution by using the standard computer function for the second-harmonic mode, and using the rejection method for the two-photon state distribution represented by the function $Q(\alpha, 0)$. The initial random points have been used as the initial values for α and β in the classical equations of motion (1) and the evolution was found by numerically solving (1) up to the time $\tau = \pi/2$. The solutions are represented by the clouds of points in the right-hand side figures.

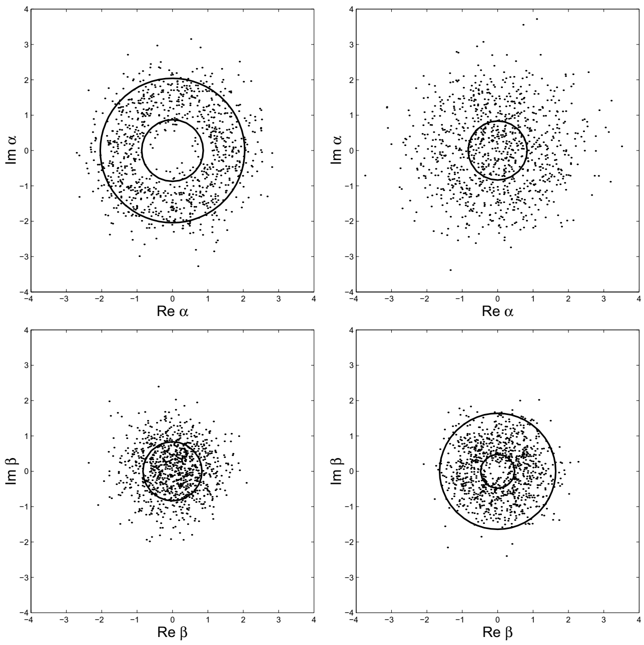

Fig. 1. Contour plots of the Q function and the results of classical simulations for the second-harmonic generation with the initial state $|2, 0\rangle$ (see text)

It is clearly seen from Fig. 1 that, at time $\tau = \pi/2$, the initial minimum at the center for the fundamental mode disappeared, and the resulting distribution has a maximum at the center, but this distribution is much broader than the distribution for the vacuum, as it is seen from the contour plot of the corresponding Q function (compare also to the lower left figure). For the second-harmonic mode we see the appearance of a minimum at the center as it should be for the one-photon state, but the distribution seems to be too narrow in this case. To be more precise let us evaluate the variances

$$\overline{|\alpha|^2}(\tau) = \int |\alpha|^2 Q(\alpha,\tau)\,\mathrm{d}^2\alpha = 2\cos^2\tau + 1 = \langle \hat{a}^+\hat{a}\rangle(\tau) + 1\,, \tag{6}$$

$$\overline{|\beta|^2}(\tau) = \int |\beta|^2 Q(\beta,\tau)\,\mathrm{d}^2\beta = \sin^2\tau + 1 = \langle \hat{b}^+\hat{b}\rangle(\tau) + 1\,, \tag{7}$$

where $\langle \hat{a}^+\hat{a}\rangle(\tau)$ and $\langle \hat{b}^+\hat{b}\rangle(\tau)$ are the mean numbers of photons in the two modes. Thus the quantum results are: $\overline{|\alpha|^2}(0) = 3$, $\overline{|\alpha|^2}(\pi/2) = 1$, $\overline{|\beta|^2}(0) = 1$, and $\overline{|\beta|^2}(\pi/2) = 2$ with the constant of motion $C_0 = 5$. The classical averages obtained from 1000 trajectories give us the results $\overline{|\alpha|^2}(\pi/2) = 2.6$ and $\overline{|\beta|^2}(\pi/2) = 1.2$. The results based on 1000 trajectories are not very accurate but they are repeatable. They clearly illustrate the difference between the quantum and classical evolutions of fluctuations in the nonlinear process of second-harmonic generation with this special choice of the initial state, for which the exact analytical solution exists. The quantum evolution in this case is perfectly periodic with quite deep changes of noise while the classical evolution leads to rather shallow changes of noise. However, even for such highly nonclassical states the classical evolution of noise reproduces some features of quantum evolution.

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