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Laser Driven Ion Traps: Beyond the Standard Perturbative Limit

P. Aniello,[1] V.I. Man'ko,[2] G. Marmo,[1] A. Porzio,[3] F. Zaccaria[1] and S. Solimeno[3]

[1] INFN – Naples, Compl. Univ. Monte Sant'Angelo
via Cintia 80126 Napoli, Italy

[2] Lebedev Physics Institute, Moscow, Russia

[3] INFM – Coherentia, Compl. Univ. Monte Sant'Angelo
via Cintia 80126 Napoli, Italy

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Abstract. It is shown that a suitable unitary transformation applied to the N-ion Jaynes–Cummings Hamiltonian allows to go beyond the standard perturbative limit (associated with the laser intensity) which affects the rotating wave approximation (RWA).

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1. Introduction

The classical Jaynes–Cummings model (JCM) [1], with the Hamiltonian

$$H_{\mathrm{JC}} = \nu\, a^{\dagger} a + \frac{1}{2}\omega_{ge}\,\sigma_z + \Omega_R\left(a\,\sigma_+ + a^{\dagger}\,\sigma_-\right),$$

gives a simple description of the main properties of atoms in interaction with electromagnetic fields [2]. This fact makes the JCM of central interest in the emerging field of quantum computing. Indeed, a couple of electronic states of an ion can be regarded as a concrete realization of the concept of qubit [3]. In particular, in a linear chain of trapped ions each ion can be regarded as a two-level system oscillating between a ground and an excited state ($|g\rangle$ and $|e\rangle$). Then, the control of the quantum degrees of freedom is achieved by addressing the ions with controlled laser beams [4]. The mutual Coulomb interactions allow a communication among the trapped ions. The theoretical description of this physical system is given by a Hamiltonian which can be reduced, via a suitable approximation, to the Hamil-

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tonian of the (N-dimensional) JCM. This kind of approximation is known as the *rotating wave approximation* (RWA) [5]. The standard RWA is essentially a perturbative method since its validity depends on the smallness of the Rabi frequency which is proportional to the intensity of the laser field. This is a severe limit since an intense laser field implies a fast coupling between the two internal energy levels of the trapped ions and so a fast QC.

The present paper aims to give a novel approach that allows going beyond the standard perturbative limit of the RWA. The basic idea is that of performing a suitable transformation of the Hamiltonian, so that the transformed one is formally similar to the initial one, but the new perturbative parameter is not proportional any more to the intensity of the laser field but a simple bounded function of it. Thus the RWA, or any other approximation, once applied to the transformed Hamiltonian, will be scarcely sensitive of the laser intensity.

2. Transforming the Basic Hamiltonian

A system of N equal ions in a potential trap, with strong confinement along the y and z axes, and weak harmonic binding of frequency ν_1 along the x axis (the 'trap axis') can be described by a Hamiltonian of the following type ($\hbar = 1$):

$$\sum_{p=1}^{N} \nu_p \, a_p^\dagger a_p + W\left(\{a_p + a_p^\dagger\}\right),$$

where $\{a_p\}$ are the annihilation operators associated with the *normal coordinates* of the ion chain, and $W\left(\{a_p + a_p^\dagger\}\right)$ is the anharmonic component of the ion–ion interaction, which will be neglected from this point onwards. The quantity $x_{jp} \equiv M_{jp}\left(a_p + a_p^\dagger\right)$, with $[M_{jp}]$ a nonsingular real symmetric matrix [6], represents the displacement of the jth ion due to the excitation of the pth mode.

Let us suppose that the jth ion is addressed by a laser beam of frequency ω_L in a traveling wave configuration. Being the jth ion a two-level system oscillating between a ground $|g\rangle$ and an excited state $|e\rangle$, the array will be described by the Hamiltonian of the JC-type $H(t) = H_0 + H_\updownarrow(t)$, where

$$H_0 := \sum_{p=1}^{N} \nu_p \, \hat{n}_p + \frac{1}{2}\omega_{ge}\, \sigma_z^j\,, \quad H_\updownarrow(t) := \Omega_R \left(e^{i\omega_L t}\, \sigma_-^j \, \mathcal{D}^{\dagger\, 2} + e^{-i\omega_L t}\, \sigma_+^j \, \mathcal{D}^2 \right) \tag{1}$$

with $\hat{n}_p = a_p^\dagger \, a_p$ the number operator and $\Omega_R = \wp\, \mathcal{E}$ the Rabi frequency, $\mathcal{E}$ denoting the intensity of the laser field. Moreover, we have set

$$\mathcal{D} := \exp\left(\frac{i}{2} \sum_{p=1}^{N} \eta_{jp}\left(a_p + a_p^\dagger\right) \right) = \prod_{p=1}^{N} D_p\left(i\frac{\eta_{jp}}{2} \right), \quad \eta_{jp} := \frac{k_L \cos\phi}{\sqrt{2\mu\nu_1}}\, M_{jp}\,, \tag{2}$$

where η_{jp} plays the role of a Lamb–Dicke factor and $D_p(\alpha)$ ($\alpha \in \mathbb{C}$) is a displacement operator associated with the pth mode.

The Hamiltonian $H(t)$ can be reduced to a time-independent Hamiltonian switching to the interaction picture with the reference Hamiltonian $\frac{1}{2}\omega_L t\,\sigma_z^j$:

$$\widetilde{H}=R_t\left(H(t)-\frac{1}{2}\omega_L\,\sigma_z^j\right)R_t^\dagger=\sum_{p=1}^{N}\nu_p\,\hat{n}_p+\frac{1}{2}\delta\,\sigma_z^j+\Omega_R\left(\sigma_-^j\,\mathcal{D}^{\dagger\,2}+\sigma_+^j\,\mathcal{D}^2\right), \qquad (3)$$

where $R_t:=\exp\left(i\frac{1}{2}\omega_L t\,\sigma_z^j\right)$ and $\delta:=\omega_{ge}-\omega_L$ is the ion-laser detuning.

We will show that is convenient to apply the standard RWA *after* having suitably transformed the Hamiltonian $\widetilde{H}$. The total transformation can be decomposed into the product of three unitary operators: T_1, T_2 and T_3 where

$$T_1=\frac{1}{\sqrt{2}}\begin{bmatrix}\mathcal{D}^\dagger & \mathcal{D}\\ -\mathcal{D}^\dagger & \mathcal{D}\end{bmatrix},\quad T_2=\begin{bmatrix}\cos\theta/2 & -\sin\theta/2\\ \sin\theta/2 & \cos\theta/2\end{bmatrix},\quad T_3=\begin{bmatrix}\breve{\mathcal{D}} & 0\\ 0 & \breve{\mathcal{D}}^\dagger\end{bmatrix}. \qquad (4)$$

Here we have set

$$\tan\theta=\frac{\omega_{ge}-\omega_L}{2\Omega_R}=\frac{\delta}{2\Omega_R},\quad \breve{\mathcal{D}}:=\prod_{p=1}^{N}D_p\left(i\frac{\breve{\eta}_{jp}}{2}\right)\quad\text{with}\quad \breve{\eta}_{jp}=\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}\,\eta_{jp}\,.$$

The final Hamiltonian reads $\breve{H}=\breve{H}_0+\breve{H}_\updownarrow$, where we have dropped an unessential constant term, and we have set

$$\begin{aligned}\breve{H}_0&=\sum_{p=1}^{N}\nu_p\,\hat{n}_p+\frac{1}{2}\breve{\delta}\,\sigma_z^j,\qquad \breve{\delta}=\sqrt{4\Omega_R^2+\delta^2}\,,\\ \breve{H}_\updownarrow&=\frac{1}{2\tan\theta}\sum_{p=1}^{N}\nu_p\left[i\breve{\eta}_{jp}\,\left(a_p-a_p^\dagger\right)\left(\sigma_-^j\,\breve{\mathcal{D}}^{\dagger\,2}+\text{h.c.}\right)-\breve{\eta}_{jp}^2\,\left(\sigma_-^j\,\breve{\mathcal{D}}^{\dagger\,2}-\text{h.c.}\right)\right].\end{aligned} \qquad (5)$$

The new Hamiltonian $\breve{H}$ enjoys very nice properties. The first point is that the main component $\breve{H}_0$ has a simple diagonal form. The second — but not less important — point is that the spin-flip component has a good behavior in both the weak and strong field regime. Indeed, if the Lamb–Dicke parameters $\{\eta_{jp}\}$ are assumed to be small, $\eta_{jp}\ll 1$, one finds that, *for any intensity of the laser field*,

$$\breve{H}_\updownarrow\simeq\frac{i}{\Delta}\sum_{p=1}^{N}\breve{\eta}_{jp}\,\nu_p\left(a_p-a_p^\dagger\right)\left(\sigma_-^j+\sigma_+^j\right). \qquad (6)$$

3. Discussion

A frequent problem in QM is to find an approximate expression of the evolution operator associated with a Hamiltonian of the type: $H(t)=H_0+H_\updownarrow(t;\{\varepsilon_k\})$, where H_0 is an explicitly diagonalizable, in the sense that it admits a complete set of known eigenvectors, and $\{\varepsilon_k\}$ is a set of parameters. A perturbative treatment of the problem is possible only if $H_{\updownarrow}(t;\{\varepsilon_k\})$ is 'comparatively small' with respect to H_0. Usually, this condition holds only for a restricted range of the parameters $\{\varepsilon_k\}$. A problem of this sort affects the treatment of a Hamiltonian of the Jaynes–Cummings-type describing a set of linearly trapped ions in interaction with laser beams, if the laser intensity is not small. In the present paper, we have shown that such a problem can be bypassed. This is accomplished by performing a suitable transformation of the Hamiltonian. Indeed, recalling formula (5), one observes that the dependence on the Rabi frequency is contained in the bounded factors

$$\frac{1}{2\tan\theta}\,\breve{\eta}_{jp} = \frac{\Omega_R}{\sqrt{4\Omega_R^2+\delta^2}}\,\eta_{jp} \qquad \left(0 \le \left|\frac{1}{2\tan\theta}\,\breve{\eta}_{jp}\right| \le \frac{1}{2}\,|\eta_{jp}|\right)$$

and

$$\frac{1}{2\tan\theta}\,\breve{\eta}_{jp}^2 = \frac{\Omega_R\,\delta}{4\Omega_R^2+\delta^2}\,\eta_{jp}^2 \qquad \left(0 \le \left|\frac{1}{\Delta}\right|\breve{\eta}_{jp}^2 \le \frac{1}{4}\,\eta_{jp}^2\right).$$

Hence the validity of any approximate expression of the evolution operator will be scarcely influenced by the intensity of the laser field. There are two further points that should be highlighted. First, a simple field-intensity corrected condition for resonances, i.e. the degeneracies of the unperturbed component of the Hamiltonian, arises in a natural way from our procedure

$$\nu_p - \breve{\delta} = 0\,, \qquad \breve{\delta} = \sqrt{4\Omega_R^2 + (\omega_{ge}-\omega_L)^2} = \sqrt{4\Omega_R^2+\delta^2}\,,$$

instead of $\nu_p - \delta = 0$. Observe that the corrected detuning $\breve{\delta}$ is always nonnegative, unlike the standard ion-laser detuning δ. Thus, both positive-δ and negative-δ resonances will be associated with terms of the type $a_p\,\sigma_+^j,\ a_p^\dagger\,\sigma_-^j$, in the interaction picture Hamiltonian $\breve{H}$, hence formally treated in the same way, differently from the standard RWA applied to $\widetilde{H}$. Second, one can check that the action on the standard basis of the unitary operator $T = T_1T_2T_3$ is easily computable. This means simple explicit expressions for the time evolution of state vectors.

A more detailed and complete treatment of this approach can be found in a longer paper [7].

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