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Nonclassicality Made Observable

W. Vogel and Th. Richter

Arbeitsgruppe Quantenoptik, Fachbereich Physik, Universität Rostock
D-18051 Rostock, Germany

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Abstract. A necessary and sufficient hierarchy of observable conditions is derived that is completely equivalent to the failure of the Glauber–Sudarshan P-function to be a probability density.

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1. Introduction

The question of how to characterize the nonclassicality of a quantum system is as old as the field of quantum optics. It became of renewed interest when well-known nonclassical effects such as photon antibunching, sub-Poissonian photon statistics and quadrature squeezing could be experimentally realized. A widely accepted definition of nonclassicality rests on the failure of the Glauber–Sudarshan function $P(\alpha)$ to be a probability density [1–3],

$$P(\alpha) \neq P_{\mathrm{cl}}(\alpha)\,. \tag{1}$$

A shortage of this criterion is that $P(\alpha)$ can be highly singular and thus, in general it cannot be determined from measured data. Therefore we intend to reformulate the condition (1) on the basis of a set of observables that completely characterizes the quantum state under study. Such a set of observables are the phase-sensitive quadrature operators

$$\hat{x}(\varphi) = \hat{a}e^{i\varphi} + \hat{a}^{\dagger}e^{-i\varphi}\,. \tag{2}$$

In a recent letter [4] one of us made the attempt to formulate an observable criterion for nonclassicality in terms of the characteristic function $G(k,\varphi)$,

$$G(k,\varphi) = \langle e^{ik\hat{x}(\varphi)}\rangle\,, \tag{3}$$

of the quadratures. It has been found that the P-function cannot be interpreted as

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a probability density if there exist values of k and φ for which

$$|G(k,\varphi)| > G_{\rm gr}(k)\,, \tag{4}$$

where $G_{\rm gr}(k) = e^{-k^2/2}$ is the characteristic function of the ground (or vacuum) state. The relevance of this criterion for the interpretation of measured data has been demonstrated very recently [5]. In particular, a statistical mixture of a single-photon state and the vacuum state behaves nonclassically according to this condition, even if the Wigner function is non-negative. However, the condition (4) is not equivalent to the condition (1). There exist quantum states that violate the former condition but that are nonclassical according to the latter [6]. Thus the problem of completely characterizing the failure of the P-function to be a probability density in terms of measurable quantities has not been solved yet.

2. Hierarchy of Nonclassicality Conditions

In the following we systematically extend the above criterion (4) to obtain a complete characterization of the nonclassicality of quantum states in terms of measurable quantities. We start from the Bochner criterion [7], which yields a necessary and sufficient condition for the existence of a classical characteristic function, i.e. the Fourier transform of a probability density. To be more specific, the violation of Bochner's criterion for the existence of a classical characteristic function of the P-function serves as our general condition for the nonclassicality of quantum states. Our main result consists in a hierarchy of conditions for nonclassicality, formulated in terms of observable characteristic functions $G(k,\varphi)$ of quadratures. The hierarchy is completely equivalent to the condition (1), in lowest order it reproduces the criterion (4) introduced in [4]. We illustrate the power of our approach for the only example of a nonclassical state published yet [6] that violates the criterion (4).

Let us introduce the characteristic function of the P-function,

$$\Phi(u,v) = \int_{-\infty}^{\infty} P(\alpha_{\rm r},\alpha_{\rm i}) \exp[2i(v\alpha_{\rm r} - u\alpha_{\rm i})]\, d\alpha_{\rm r}\, d\alpha_{\rm i}\,, \tag{5}$$

i.e. its two-fold Fourier transform. The nonclassicality condition (1) can be reformulated in terms of $\Phi(u,v)$ with the help of the theorem by Bochner [7]. Associating with $\Phi(u,v)$ an $n\times n$ matrix with elements $\Phi_{ij} = \Phi(u_i - u_j, v_i - v_j)$, we conclude from the Bochner theorem: A quantum state is classical, i.e. its P-function is a probability density, if and only if for arbitrary real numbers u_k and v_k and for any $k = 2,\cdots,\infty$ the conditions

$$D_k = \begin{vmatrix} 1 & \Phi_{12} & \cdots & \Phi_{1k} \\ \Phi_{12}^* & 1 & \cdots & \Phi_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ \Phi_{1k}^* & \Phi_{2k}^* & \cdots & 1 \end{vmatrix} \geq 0 \tag{6}$$

are valid.

Based on this result we arrive at the following necessary and sufficient criterion for nonclassicality. A quantum state is nonclassical if and only if there exist values u_i, v_i $(i = 1, \cdots, k)$ for which at least one of the determinants D_k $(k = 2, \cdots, \infty)$ becomes negative:

$$D_k < 0\,. \tag{7}$$

We may define nonclassicality of order $k-1$ just by the condition (7). The higher the order of nonclassicality (and thus of the determinant) the more points of the characteristic function are included in the corresponding condition and the finer details of the characteristic function are relevant. Thus, in fact we arrive at a hierarchy of conditions (7) for nonclassicality.

Eventually we reformulate these conditions in terms of the measurable quantities $G(k,\varphi)$ defined in Eq. (3). To this end we make use of the well-known relation

$$G(k,\varphi) = \Phi(k\sin\varphi, k\cos\varphi)G_{\rm gr}(k)\,, \tag{8}$$

which allows us to express Φ_{ij} in the conditions (7) solely in terms of characteristic functions accessible to measurements. Next we discuss the first- and second-order conditions expressed in terms of $G(k,\varphi)$ in more detail.

The first-order nonclassicality condition, $D_2 < 0$, becomes exactly the condition (4), introduced in [4] and applied in experiments [5]. The condition (4) is sufficient to reveal the nonclassicality of important classes of quantum states, such as Fock states, quadrature squeezed states, coherent superpositions of coherent states, Gaussian mixed states and others.

To simplify the discussion we restrict the reformulation of the second-order condition, $D_3 < 0$, to quantum states with phase-insensitive characteristic function, $G(k,\varphi) \equiv G(k)$. The only example of a nonclassical state published yet that violates the first-order condition belongs to this class. Moreover, we assume that the three points in the (k,φ) plane entering the second-order condition lie equidistant on the k axis. Under these conditions and in cases when the state under study does not display first-order nonclassicality the second-order condition becomes

$$G^2(k/2)\,G_{\rm gr}(k/\sqrt{2}) - G(k) > G_{\rm gr}(k)\,. \tag{9}$$

Similar inequalities can be derived for any higher-order condition.

3. An Illustrative Example

Let us consider the mixed state proposed by Diosi [6],

$$\hat{\varrho} = \sum_{n=1}^{\infty} 2^{-n} |n\rangle\langle n|\,, \tag{10}$$

representing a thermal state of mean excitation equal to one, whose ground (or vacuum) state has been suppressed. The P-function of this rotationally symmetric state reads as

$$P(\alpha) = \frac{2}{\pi} e^{-|\alpha|^2} - \delta(\alpha)\,. \tag{11}$$

Obviously it describes a nonclassical state, since it fulfills the condition (1). The corresponding characteristic function $G(k)$ is given by

$$G(k) = \left(2e^{-k^2} - 1\right) e^{-k^2/2}\,. \tag{12}$$

Clearly, $|G(k)| \leq G_{\mathrm{gr}}(k)$, so that the state (10) fails to obey the nonclassicality condition of first order. From the second-order condition (9) and by using Eq. (12) we find that the state (10) is nonclassical of second order if the inequality

$$2\left(2e^{-k^2/4} - 1\right)^2 - \left(2e^{-k^2} - 1\right) > 1\,. \tag{13}$$

holds true for any interval of k-values. For $|k| \to \infty$ the left hand side of Eq. (13) approaches the value of 3, so that the state (10) is indeed nonclassical of second order.

4. Conclusions

We have derived a hierarchy of observable conditions for a quantum state to be nonclassical. These conditions allow one to verify whether or not the Glauber–Sudarshan P-function is a probability density. The hierarchy of conditions naturally yields a classification of nonclassicality with respect to first, second and higher orders. The method is illustrated for the example of a mixed state, that is classical in the first but nonclassical in the second order.

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