Acta Phys. Hung. B 20/1–2 (2004) 95–99

QUANTUM
ELECTRONICS

Quantum Measurement with a Positive Operator-Valued Measure

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Received 30 July 2003

Abstract. In the quantum theory of measurement, the positive operator-valued measure (POVM) is an important concept, and its implementation can be useful. Following a brief review of the mathematics of POVMs in quantum theory, a particular implementation of a POVM for use in the measurement of nonorthogonal photon polarization states is reviewed. The two-dimensional Hilbert space of the POVM implementation can be embedded in the three-dimensional Hilbert space of an ordinary projective-valued measure. Also, analytical expressions can be obtained for the maximum Renyi information loss from the device to a disturbing probe.

Keywords: quantum measurement, positive operator-valued measure,
quantum cryptography, quantum information
PACS: 03.67.-a, 03.67.Dd, 03.67.Hk, 03.65.Ta

1. Introduction

A positive operator valued measure (POVM) is a set of nonnegative Hermitian operators A_m that act in the Hilbert space of a quantum system and sum to the identity operator, namely,

$$\sum_m A_m = 1\,. \tag{1}$$

The index m labels the various possible outcomes of a measurement implementing the POVM. The probability P_m of outcome m, if the system is in a state described by the density matrix ρ, is given by

$$P_m = \mathrm{Tr}\,(A_m \rho)\;. \tag{2}$$

The advantage of a POVM is that it may allow the extraction of more information than can the usual von Neumann-type projective measurement. A brief history of

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the use of POVMs in quantum theory is given in [1]. In the present work, I address an all-optical implementation of a POVM which can distinguish two nonorthogonal photon polarization states some of the time [1–7].

2. Photonic Implementation of a POVM

If one wants to be able to distinguish conclusively between two nonorthogonal photon polarization states at least some of the time, it is useful to consider a POVM used in quantum cryptography. This POVM consists of the following set of three nonnegative Hermitian operators:

$$A_u = (1 + \langle u|v\rangle)^{-1} [1 - |v\rangle \langle v|] \,, \tag{3}$$

$$A_v = (1 + \langle u|v\rangle)^{-1} [1 - |u\rangle \langle u|] \,, \tag{4}$$

$$A_? = 1 - A_u - A_v \,, \tag{5}$$

in which kets $|u\rangle$ and $|v\rangle$ represent the two nonorthogonal single photon polarization states. The states $|u\rangle$ and $|v\rangle$ are here taken to be linear-polarization states with the Dirac bracket $\langle u|v\rangle$ given by

$$\langle u|v\rangle = \cos\theta \,, \tag{6}$$

where θ is the angle between the two polarization vectors.

A general normalized single-photon polarization state is given by

$$|\psi\rangle = \alpha|u\rangle + \beta|v\rangle \,. \tag{7}$$

The probability P_u that an arbitrary photon polarization state $|\psi\rangle$ given by Eq. (7) is measured to be in the u-polarization state $|u\rangle$ can be calculated with Eqs. (2), (3) and (7); the calculation yields

$$P_u = \langle\psi|A_u|\psi\rangle = |\alpha|^2(1 - \cos\theta) \,. \tag{8}$$

Analogously for state $|v\rangle$, one obtains

$$P_v = \langle\psi|A_v|\psi\rangle = |\beta|^2(1 - \cos\theta) \,, \tag{9}$$

and the probability of an inconclusive result is

$$P_? = \langle\psi|A_?|\psi\rangle = |\alpha + \beta|^2 \cos\theta \,. \tag{10}$$

An all-optical implementation of this POVM is shown schematically in Fig. 1 [1–4]. The straight lines with arrows represent possible optical paths for a photon to move through the device. The path labeled $|\psi\rangle$ is the incoming path for a photon in an arbitrary polarization state given by Eq. (7). Also in Fig. 1, D_u, D_v, and $D_?$ designate photodetectors representing the measurement operators A_u, A_v, and $A_?$, respectively. Shown also is a Wollaston prism W, which is aligned so that an

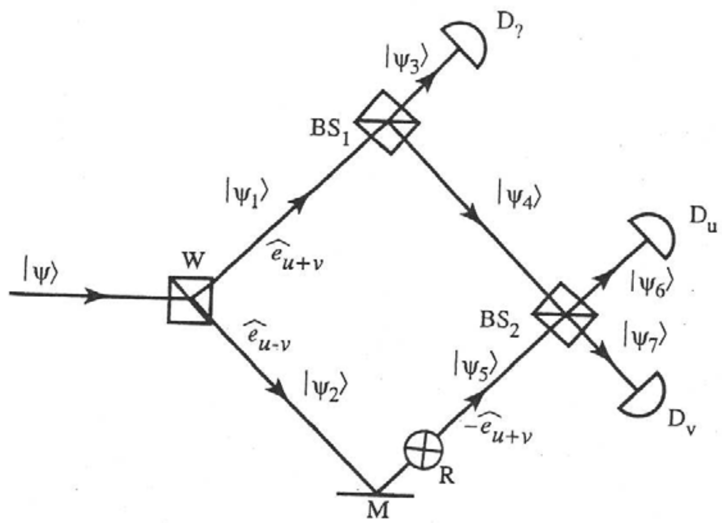

Fig. 1. POVM implementation

incident photon with polarization vector $\hat{e}_{u+v}$ takes the path labeled by $\hat{e}_{u+v}$ and the state $|\psi_1\rangle$, and not the path labeled by the polarization vector $\hat{e}_{u-v}$ and the state $|\psi_2\rangle$. Here $\hat{e}_{u+v}$ denotes a unit polarization vector corresponding to polarization state $|u+v\rangle = |u\rangle + |v\rangle$, and is perpendicular to the unit polarization vector $\hat{e}_{u-v}$ corresponding to the polarization state $|u-v\rangle = |u\rangle - |v\rangle$. Also in Fig. 1, M denotes a mirror, R denotes a 90° polarization rotator, and BS_1 and BS_2 designate beam splitters with reflection coefficients $\tan^2(\theta/2)$ and $1/2$, respectively. The various paths through the device are designated in Fig. 1 by corresponding photon states $|\psi_i\rangle$, $i = 1, 2, \ldots, 7$ [1, 2, 4], which, when calculated for ideal detector D_u, D_v and $D_?$, yield Eqs. (8)–(10).

Based on Neumark's extension theorem, the two dimensional Hilbert space of the POVM implementation can be embedded in a three-dimensional Hilbert space of a projective valued (PV) measure [3]. Figure 2 displays the geometry of the embedding in a three dimensional rectangular Cartesian space with coordinates x, y and z. One has

$$|u\rangle = 2^{1/2}\sin(\theta/2)|x\rangle + \left(1 - \frac{1}{2}\sin^2(\theta/2)\right)^{1/2} |z\rangle\,, \tag{11}$$

$$|v\rangle = 2^{1/2}\sin(\theta/2)|y\rangle + \left(1 - \frac{1}{2}\sin^2(\theta/2)\right)^{1/2} |z\rangle\,, \tag{12}$$

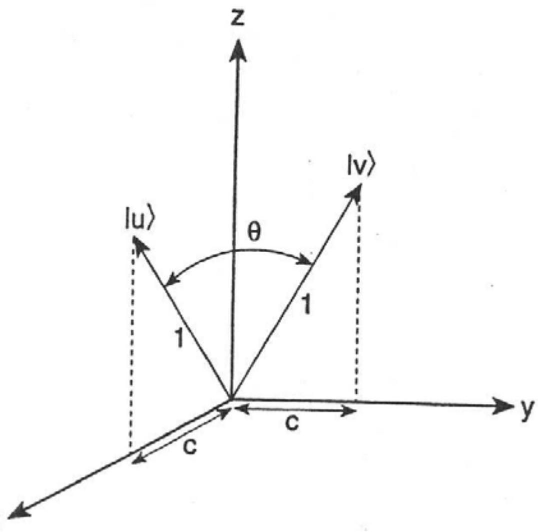

Fig. 2. Geometry of POVM embedding

where $|x\rangle$, $|y\rangle$ and $|z\rangle$ are unit kets along the x, y and z axes, respectively. The vectors $|u\rangle$ and $|v\rangle$ lie in the x–z and y–z planes, respectively, and make an angle θ with each other.

A PV measure corresponding to the POVM Eqs. (3)–(5) is given by [3]

$$M = |x\rangle\langle x| - |y\rangle\langle y| \,. \tag{13}$$

The probability of outcome m $(m = u, v, ?)$ for a general state $|\psi\rangle$, with unit normalization in the three-dimensional Hilbert space, is $\langle\psi|D_m|\psi\rangle$, where the projective detection operators D_m, corresponding to projections on the respective eigenvectors, are given by

$$D_u = |x\rangle\langle x| \,, \tag{14}$$

$$D_v = |y\rangle\langle y| \,, \tag{15}$$

$$D_? = 1 - D_u - D_v = |z\rangle\langle z| \,. \tag{16}$$

The POVM operator A_m is the projection of the detection operator D_m into the plane spanned by $|u\rangle$ and $|v\rangle$. It can be shown, using Eqs. (7), (8)–(12) and (14)–(16), that for any vector $|\psi\rangle$ in the plane defined by $|u\rangle$ and $|v\rangle$, one has [3]

$$\langle\psi|D_m|\psi\rangle = \langle\psi|A_m|\psi\rangle \,, \tag{17}$$

and therefore M is equiv alent to the POVM, Eqs. (3)–(5).

It can be useful for quantum cryptographic applications [5–7] to also consider a general unitary probe which interacts with an individual photon having equiprobable nonorthogonal polarization states $|u\rangle$ or $|v\rangle$ encoding Boolean states $|0\rangle$ or $|1\rangle$, respectively, and the probe state becomes entangled with the photon state. This occurs while the photon is on its way from a transmitter to the POVM device acting as a receiver, and the probe state is measured only after the photon has been measured by the receiver. The probe is characterized by four probe parameters which quantify the general unitary transformation producing the entanglement of the signal with the probe. If the error rate E induced by the probe in the POVM receiver is fixed, or alternatively if both the error and inconclusive rates are fixed, algebraic expressions for the maximum information gain by the probe can be calculated [5–7].

Acknowledgements

This work was supported by the U.S. Army Research Laboratory and the Defense Advanced Research Projects Agency. The author wishes to thank Prof. József Janszky for the invitation to present this paper at the *Centennial Conference in Commemoration of Eugene Paul Wigner* in Pécs, Hungary, 8–12 July, 2002.

Note

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