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QUANTUM ELECTRONICS

Wigner, Moyal, and Precursors to Canonical Coherent States

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Abstract. The phase space formulation of quantum mechanics pioneered by Wigner and developed by Moyal is shown to contain relevant concepts useful in the study of canonical coherent states.

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The 1932 paper of E. Wigner [1] in which the so-called Wigner quantum mechanical phase space distribution function was introduced started a recognizable cottage industry. It was Moyal [2], however, who expanded the ideas of Wigner and demonstrated how they could be used to provide a reformulation of quantum mechanics that exclusively uses phase space functions. The concept of the Moyal bracket and the implicit star product that are the quantum replacements for the Poisson bracket and the usual product of classical mechanics have been precursors to the recent program of deformation quantization [3]. Even more contemporary is an effort to revive and extend the Moyal program as a tool to analyze situations involving noncommuting geometry [4].

The goal of this short note is far more modest. In particular, our goal is to show how a calculation of Moyal already implies the validity of the resolution of unity formula for canonical coherent states. In Moyal's paper he was naturally led to consider functions of the form ($\hbar = 1$)

$$\Phi_{kl}(x,k) \equiv \int \phi_k(x-q/2)^* \, e^{-ikq} \, \phi_l(x+q/2) \, dq \qquad (1)$$

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for a set of orthonormal functions $\{\phi_k\}_{k=1}^{\infty}$ such that

$$\int \phi_k(q)^* \, \phi_l(q) \, dq = \delta_{kl} \, . \tag{2}$$

In Appendix 2 of his 1949 paper, Moyal showed that

$$\int \Phi_{kl}(x,k)^* \, \Phi_{rs}(x,k) \, dx \, dk = 2\pi \, \delta_{ls} \, \delta_{rk} \, . \tag{3}$$

In addition, Moyal was also led to consider the Fourier transform of basic functions such as those defined in (1). In particular, it follows that

$$\begin{aligned} \phi_{rs}(p,q) &\equiv \frac{1}{2\pi} \int e^{i(kq-xp)} \, \Phi_{rs}(x,k) \, dx \, dk \\ &= \int \phi_r(x+q/2)^* \, e^{ipx} \, \phi_s(x-q/2) \, dx \, . \end{aligned} \tag{4}$$

Since Fourier transformation is an isometry, it follows from (3) that Moyal had implicitly also showed that

$$\int \phi_{kl}(p,q)^* \, \phi_{rs}(p,q) \, dp \, dq = 2\pi \, \delta_{ls} \, \delta_{rk} \, . \tag{5}$$

As we shall see, this formula has important implications.

In 1957 while studying for my PhD at Princeton, I had run across the paper of Moyal and — like many others before and since — really appreciated what a fine paper it was. I was also taking a group theory course taught by Valentine Bargmann that same year. During that course we eventually came to the section dealing with finite dimensional, unitary, irreducible representations of compact groups and the associated group orthogonality relations that such representations admit. In standard notation such group orthogonality relations take the form given by

$$\int (\phi_k, U_a[g] \, \phi_l)^* \, (\phi_r, U_b[g] \, \phi_s) \, d\mu(g) = \frac{\int d\mu(g)}{D_a} \, \delta_{ab} \delta_{ls} \, \delta_{rk} \, . \tag{6}$$

Here, in this relation, U_a and U_b represent two, possibly inequivalent, irreducible group representations, D_a denotes the dimension of the representation a, $d\mu(g)$ denotes the invariant group measure, and $\int d\mu(g)$ the finite group volume. Group orthogonality relations such as these require that the left and right invariant group invariant measures coincide, which is always the case for compact groups.

If one recalls that for the Weyl group there is only one unitary irreducible representation up to unitary equivalence, then it becomes clear that the relation that Moyal had implicitly shown is the exact analog of a group orthogonality relation for the Weyl group. For the Weyl group, it appears that the ratio of the group volume to the representation dimension may be taken as 2π even though both elements of that quotient are divergent. After one of the lectures by Bargmann on group orthogonality, I informed him that such a relation also held true for the Weyl group.

His immediate response was "No, that cannot be true. The Weyl group is noncompact". My reply was "That may be so, but it holds nonetheless". At that point, I took the liberty of bringing Moyal's paper to his attention (or perhaps reminding him of its existence). Of course, Bargmann's instincts were correct since a finite, nonzero quotient for group volume and representation dimension for noncompact groups is far more the exception than the rule.

We can readily extend relation (5) to

$$\int (\psi, U[p,q]\,\lambda)^*\,(\chi, U[p,q]\,\phi)\,dp\,dq/(2\pi) = (\chi,\psi)(\lambda,\phi)\,, \tag{7}$$

where ψ, λ, χ and ϕ denote arbitrary vectors, and we have introduced the shorthand

$$U[p,q] \equiv e^{i(pQ-qP)} = e^{-iqP/2}\,e^{ipQ}\,e^{-iqP/2} \tag{8}$$

for the unitary, irreducible Weyl operators. Indeed, relation (7) is often regarded as just another version of the group orthogonality relations.

One way to "read" relation (7) is to strip off the vectors χ and ϕ. Passing to Dirac notation, the result of this operation is the equation

$$|\psi\rangle\langle\lambda| = \int \langle\lambda|U[p,q]^\dagger\,|\psi\rangle\,U[p,q]\,dp\,dq/(2\pi)\,. \tag{9}$$

This expression may be interpreted as a representation of a certain operator, $|\psi\rangle\langle\lambda|$, as a superposition of Weyl operators. Linear combinations of (9) lead to

$$\mathcal{B} = \int \mathrm{Tr}\big(U[p,q]^\dagger\,\mathcal{B}\big)\,U[p,q]\,dp\,dq/(2\pi)\,, \tag{10}$$

which extends this relation to a general operator $\mathcal{B}$.

A different way to "read" the extended group orthogonality relation (7) is to strip off the vectors χ and ψ. The result, again in Dirac notation, is

$$\langle\lambda|\phi\rangle I = \int U[p,q]\,|\phi\rangle\langle\lambda|\,U[p,q]^\dagger\,dp\,dq/(2\pi)\,. \tag{11}$$

Here I denotes the unit operator. Let us choose $|\lambda\rangle = |\phi\rangle$ and set $\langle\phi|\phi\rangle = 1$. Then if we introduce the notation

$$|p,q\rangle \equiv U[p,q]\,|\phi\rangle\,, \tag{12}$$

it follows that we have established the resolution of unity

$$I = \int |p,q\rangle\langle p,q|\,dp\,dq/(2\pi)\,, \tag{13}$$

which is one of the key relations for canonical coherent states.

Of course, the resolution of unity is but one of the relations that coherent states are required to exhibit. The other important property is continuity, but, in the present context, continuity is clearly satisfied as a direct consequence of the fact that we are dealing with unitary representations of one parameter groups, and one of the defining properties of such groups is weak continuity with regard to the group parameter. Another important aspect of a family of coherent states is their use in a Hilbert space representation by (bounded) continuous functions $\psi(p,q) \equiv \langle p,q|\psi\rangle$ for all $|\psi\rangle \in \mathfrak{H}$. (The word bounded here is in parenthesis depending on whether or not the coherent states in question are themselves uniformly bounded in norm or not — the examples we have in mind in the present context are indeed uniformly bounded in norm.) The result is a functional representation for which the inner product of two elements is given by

$$(\psi, \phi) \equiv \int \psi(p,q)^* \, \phi(p,q) \, dp\,dq/(2\pi) = \langle\psi|\phi\rangle \,. \tag{14}$$

Such Hilbert spaces are particularly useful examples of reproducing kernel Hilbert spaces as well.

Final remarks

Thus, although it was not recognized at the time, one may say that Moyal implicitly established the essence of the resolution of unity appropriate to the family of canonical coherent states for an arbitrary normalized fiducial vector. It is certainly the case that some classic papers of the past contain far more than was recognized at the time they were written. It is safe to say that Moyal's classic paper on the Wigner function and its application to a completely phase space description of quantum mechanics, is just such a paper!

References

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