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QUANTUM ELECTRONICS

Gauss Filtered Back Projection for the Reconstruction of the Wigner Function

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Abstract. Optical homodyne tomography is a quantum state determination process, where measured quadrature distributions are averaged with appropriate sampling functions. We propose Gauss filtered back projection when numerically reconstructing the Wigner function. In this way the result is an s-parametrized quasi-probability distribution where the parameter is determined by the filter.

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1. Introduction

The function Eugene Wigner introduced[b] in 1932 [1] plays an increasingly important role in quantum mechanics and especially in quantum optics. The Wigner function provides a complete picture of a quantum state in phase space. Its measurement, or more precisely its reconstruction from a series of measurements has become a standard practice in laboratories for a variety of physical systems.

Optical homodyne tomography [2] is a method to measure quadrature observables, i.e. the marginal distributions of the Wigner function for light. It was the first method for experimental reconstruction of the Wigner function [3]. Later it was used to first measure the oscillations in the photon statistics of nonclassical light [4].

The set of marginal distributions are related to the Wigner function as its Radon transform. The inversion of the Radon transformation is a well-studied problem in the context of image reconstruction in all tomographic methods [5]. When reconstructing the Wigner functions experimentalist can rely on numerical

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algorithms, developed for the inverse Radon transformation in a general context. In the usual numerical inversion, called filtered back projection, a window function must be introduced in order to filter noisy measured data and ensure convergence by cutting off high frequency contributions. The filtering leads to a distortion of the reconstructed image.

Although there is a substantial literature on the mathematics of homodyne tomography [6–11], in experiments [12] the reconstruction of the Wigner function is achieved through generic numerical algorithms with arbitrary filter function, thereby introducing an unphysical distortion. Sampling functions for quasi-probabilities has been derived in [10] without relating them to filtered back projection.

In this paper we argue that for filtered back projection in optical homodyne tomography the naturally preferred choice for the window function is a Gaussian. With this choice the reconstructed image has a natural physical interpretation as a generalized Wigner function: an s-parametrized quasi-probability distribution in phase space corresponding to general ordering of operators.

2. Gauss Filtered Back Projection

The Radon transformation in 2 dimensions for a function $F(x, y)$ is defined by

$$m(q,\theta) \equiv \int_{-\infty}^{\infty} F(q\cos\theta - p\sin\theta, q\sin\theta + p\cos\theta)dp\,. \tag{1}$$

In Fourier space we have

$$\mathcal{F}_q[m](\xi,\theta) = \mathcal{F}_{x,y}[F](\xi\cos\theta, \xi\sin\theta)\,, \tag{2}$$

note that on the left hand side the Fourier transformation refers only to the first variable, while on the right hand side to both variables. The inversion formula, called back projection, can be cast in the form

$$F(q,p) = \frac{1}{\pi}\int_0^{\pi} m_f(q\cos\theta + p\cos\theta, \theta)d\theta\,, \tag{3}$$

where

$$m_f(x,\theta) = \mathcal{F}_\xi^{-1}\left[\, f\mathcal{F}_q[m]\,\right](x,\theta) \tag{4}$$

with the filter function $f(\xi) = |\xi|$.

Introducing a cutoff, in the form of a filter function, one regularizes the inverse transformation. With a Gaussian filter function $f_G(\xi,\epsilon) = |\xi|\exp\{-\epsilon\xi^2/4\}$ the back projection reads

$$F_\epsilon(q,p) = \frac{1}{2\pi^2}\int_{-\infty}^{\infty} dq' \int_0^{\pi} d\theta\, m(q',\theta)K_\epsilon(q\cos\theta + p\sin\theta - q')\,. \tag{5}$$

The inversion means averaging with respect to a kernel

$$K_\epsilon(x) \equiv \frac{1}{2\pi^2}\,\mathcal{F}_\xi^{-1}[\,f_G\,]\,. \tag{6}$$

The explicit form of the kernel can easily be found

$$K_\epsilon(x) = \frac{\partial}{\partial x}\sqrt{\frac{\pi}{\epsilon}}\,\exp(-x^2/\epsilon)\,\mathrm{erfi}(x/\sqrt{\epsilon})\,. \tag{7}$$

This kernel leads to the reconstruction of quasi-probability functions in optical homodyne tomography, as we will show in the next section.

3. Wigner Function and s-Parametrized Quasi-Probabilities

The measured quantities in homodyne tomography are the rotated quadrature distributions [2]

$$\mathrm{pr}(q,\theta) \equiv \langle q_\theta|\hat{\rho}|q_\theta\rangle\,. \tag{8}$$

The set of quadrature distributions constitutes the Radon transformed Wigner function

$$\mathrm{pr}(q,\theta) = \int_{-\infty}^{\infty} W(q\cos\theta - p\sin\theta, q\sin\theta + p\cos\theta)dp\,. \tag{9}$$

In Fourier space, analogously to Eq. (2), we find the relation to the characteristic function

$$\mathcal{F}_q[\mathrm{pr}](\xi,\theta) = \chi(\xi\cos\theta, \xi\sin\theta; 0)\,. \tag{10}$$

The generalized Wigner functions or quasi-probability densities are defined through the s-parametrized characteristic function

$$\chi(u,v;s) = \mathrm{Tr}\left[\hat{\rho}e^{-iu\hat{q}-iv\hat{p}}\right]\, e^{s(u^2+v^2)/4}\,, \tag{11}$$

through inverse Fourier transformation

$$W(q,p;s) = \mathcal{F}_{u,v}^{-1}[\chi(u,v;s)](q,p) = \mathcal{F}_{u,v}^{-1}[\chi(u,v;0)\,e^{s(u^2+v^2)/4}](q,p)\,. \tag{12}$$

The Wigner function corresponds to $s=0$. Inserting Eq. (10) and following similar lines as in the previous section we arrive at the result

$$W(q,p;s) = \frac{2}{(2\pi)^2}\int_{-\infty}^{\infty} dq' \int_0^\pi d\theta\,\mathrm{pr}(q',\theta) K_{-s}(q\cos\theta + p\sin\theta - q')\,. \tag{13}$$

Here K_{-s} is the Gauss filtered back-projecting kernel (7) (the minus sign in front of s is due to different conventions in the notation).

We note that in cascaded homodyne detection [13] the reconstructing kernel has the same functional form, with different arguments. The cascaded scheme allows local detection of the Wigner function by first displacing the state with a physical process.

4. Conclusions

We derived the kernel of Gauss filtered back projection in a simple, analytical form. In this way we relate the reconstruction of generalized Wigner functions by sampling functions to the standard filtered back projection algorithm with a specific filter. In homodyne tomography experiments this choice of the filter is natural when reconstructing phase space distributions.

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Notes

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b. According to a footnote in the original paper "This expression was found by L. Szilárd and the present author some years ago for a different purpose."

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