APH N.S., Heavy Ion Physics 15/1–2 (2002) 173–179

HEAVY ION PHYSICS
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The Remaining Uncertainty in Quantum Measurement of Noncommuting Discrete Observables

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Received 29 May 2001

Abstract. The quantum mechanical measuring process is analyzed from the standpoint of information theory. We determined the remaining uncertainty in the successive measurements of two discrete noncommuting observables and found its lower bound. Using this lower bound, a new simple form of uncertainty relation for two discrete noncommuting observables is proposed.

Keywords: quantum measurement, Shannon's entropy, uncertainty relations, discrete observables
PACS: 02.10.Lh, 03.67.-a, 06.20.Dk

1. Introduction

Recently, the quantum mechanical measuring process has been the subject of increasing research interest [17]. Many interesting results were achieved by the application of information theory on the quantum measurement [18]. The ultimate aim of a quantum measurement is to decrease the uncertainty of the measured quantum system as much as possible. A measurement is the more effective the more uncertainty of the measured system it removes [11]. By the uncertainty of a measured observable we understand that uncertainty which has the random variable attached to this observable. As it is well-known, according to quantum physics, to any observable A there can be assigned a random variable $\tilde{x}_A$ (see, e.g. [5]) with the probabilistic scheme

S	S_1	S_2	$\dots$	S_n
P	$P(x_1)$	$P(x_2)$	$\dots$	$P(x_n)$
X	x_1	x_2	$\dots$	x_n

1219-7580/02/ $ 5.00

In this scheme the first, second and third row give the eigenstates of A, their probabilities and the corresponding eigenvalues of A, respectively.

The uncertainty measures of $\tilde{x}_A$ can be divided into two classes, namely [10]:
i) The *moment* measures, which give the uncertainty of $\tilde{x}_A$ by formulas that contain both its values and the components of its probability distribution. These measures are mainly given by the *central statistical* moments.
ii) The *probabilistic* or *entropic* measures of uncertainty that contain in their formulas *only* the components of the probability distribution of $\tilde{x}_A$. They determine the spreading out of its probabilistic distribution. The classical measure of uncertainty $S_s(\tilde{x})$ is the *information-theoretical or Shannon entropy* defined for a discrete random variable $\tilde{x}$ as [4]

$$S_s(\tilde{x}) = -\sum_{i=1}^{n} P(x_i) \log P(x_i)\,. \tag{1}$$

In what follows, we adopt the Shannon (information-theoretical) entropy as the uncertainty measure. A general quantum system is characterized by a set of quantum observables $A, B, C, \ldots$ some of which share common eigenvalues (commuting observables), the others do not (noncommuting observables). Due to statistical interdependences between the noncommuting observables, i.e. when measuring one observable from a pair of noncommuting complementary observables always remains a rest uncertainty of the complementary observable.

We start this letter with the determination of the lower bound of the remaining uncertainty in the measurement of two discrete noncommuting observables. Using this bound, a new uncertainty relation is proposed which satisfies all requirements for an uncertainty relation.

2. The Lower Bound of the Remaining Uncertainty

Let $|\phi\rangle$ be a normalized state vector of two observables A and B with noncommuting Hermitean operators $\hat{A}$ and $\hat{B}$ in an N-dimensional Hilbert space, whose corresponding complete orthonormal sets of eigenvectors $\{|x_j\rangle\}$ and $\{|y_i\rangle\}$ $(i = 1, 2, \ldots, N)$ are disjointed and have nondegenerate spectra. The state vector $|\phi\rangle$ can be written in the form

$$|\phi\rangle = \sum_{i}^{N} a_i |x_i\rangle = \sum_{j}^{N} b_j |y_j\rangle\,.$$

The coefficients $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are mutually linked by the equation

$$b_j = \sum_{i=1}^{N} a_i \langle y_j | x_i \rangle, \qquad a_j = \sum_{i=1}^{N} a_i \langle x_j | y_i \rangle\,,$$

where $\langle y_j | x_k \rangle$ and $\langle x_j | y_k \rangle$ $(j, k = 1, 2, \ldots, N)$ are the components of the transformation matrix between A and B.

At the measurement of A or B, the wave function of the measured system collapses, passing into one of its eigenstates $|\psi\rangle = |x_j\rangle$, $j = 1, 2, \ldots, N$ or $|\psi\rangle = |y_j\rangle$, $j = 1, 2, \ldots, N$ depending on which of the both noncommuting observables we measure. The probability of finding a system in its eigenstate $|x_j\rangle$ or $|y_j\rangle$ is $P(x_j) = |\langle x_j|\phi\rangle|^2$ or $P(y_j) = |\langle y_j|\phi\rangle|^2$, respectively. The eigenstates of A and B can be written in terms of $|y_1\rangle, |y_2\rangle, \ldots, |y_N\rangle$ or $|x_1\rangle, |x_2\rangle, \ldots, |x_N\rangle$

$$|x_k\rangle = \sum_j \langle y_j|x_k\rangle |y_j\rangle\,, \qquad k = 1, 2, \ldots, N\,,$$

or

$$|y_k\rangle = \sum_j \langle x_j|y_k\rangle |y_j\rangle\,, \qquad k = 1, 2, \ldots, N\,,$$

respectively. After measurement of A or B, there remains a rest uncertainty of the observable B or A, the value of which is

$$S_k(B) = -\sum_j |\langle y_j|x_k\rangle|^2 \log |\langle y_j|x_k\rangle|^2 \tag{2}$$

or

$$S_k(A) = -\sum_j |\langle x_j|k_k\rangle|^2 \log |\langle x_j|y_k\rangle|^2\,,$$

respectively. Let us consider the successive measurements of observables A and B. We suppose that the pre-measurement wave function of the measured system be $|\psi\rangle = \sum_i^N a_i|x_i\rangle$ and the observable A is to be measured. Entropy of A in the post-measurement state is equal to zero because the result of each measurement of A is one of its eigenvalue, but the mean uncertainty of B after measuring of A turns out to be

$$S(A^*, B) = \sum_i^N |a_i|^2 S_k(B)\,,$$

where $S_k(B)$ is given by Eq. (2). The observable denoted by * is that observable which is measured and the remaining uncertainty is related to the other observable.

From the point of view of the theory of quantum measurement it is interesting to find such a wave function for which $S(A, B)$ takes its minimal values, i.e. to find the lower bound of $S(A^*, B)$. $S(A^*, B)$ represents in fact the mean of a random variable, say $\tilde{z}$, which assumes the values $S_1(B), S_2(B), \ldots, S_N(B)$ with the probabilities $|a_1|^2, |a_2|^2, \ldots, |a_N|^2$. If the values of $\tilde{z}$ are different then it evidently holds

$$S(A^*, B) > S_m(B)\,,$$

where $S_m(B)$ is the smallest value from the set of values $S_1(B), S_2(B), \ldots, S_N(B)$. If the result of the measurement of A is the state vector $|x_m\rangle$ which is attached to $S_m(B)$ then the inequality holds

$$S(A^*, B) \geq S_m(B)\,.$$

The relation '=' holds if all values of $\tilde{z}$ are equal. Denoting the smallest value of the set $\{S_1(B), S_2(B), S_3(B), \ldots, S_N(B)\}$ by the symbol $S^{(\min)}(A^*, B)$ then it represents the exact lower bound of $S(A^*, B)$.

As an example for the calculation of the lower bound of $S(A^*, B)$, we consider a quantum system consisting of a particle with spin $\hbar/2$. The spin components J_x, J_z we take as a pair of noncommuting observables. Here, the state vector is a two-component spinor

$$|\Psi\rangle = \hbar/2 \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} .$$

Since the transformation matrix between J_x and J_z has the form

$$\langle x_j | z_i \rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

we get immediately

$$S^{(\min)}(J_x^*, J_z) = \frac{1}{2} \log 2 ,$$

i.e. the minimum of $S(J_x, J_z)$ is reached when the spin component J_x occurs in its eigenstates. Hence,

$$S(J_x, J_z) \geq \frac{1}{2} \log 2 .$$

3. An Alternative Uncertainty Relation

The fact that A and B cannot simultaneously get sharp eigenvalues represents the cornerstone of the *principle* of uncertainty in quantum mechanics and can be quantitatively expressed in different forms, commonly called uncertainty *relations.* An uncertainty relation, as normally understood, provides an estimate of the minimum uncertainty expected in the outcome of a measurement of an observable, given the uncertainty in outcome of a measurement of another observable. We remark that due to existence of two different types of uncertainty measures there are, at the present time, also two types of the uncertainty relations: i) the Heisenberg (or variance) uncertainty relation and ii) the entropic uncertainty relation. While the Heisenberg uncertainty relation is given as the *product* of the moment uncertainty measures (standard deviation) of two noncommuting observables A and B, the entropic uncertainty relation is given as the *sum* of their Shannon entropies. The Heisenberg uncertainty relation (see, e.g. [7, 15]) is usually written in the form [13]

$$\Delta A \Delta B \geq \frac{1}{2} |\langle \Psi | [\hat{A},\ \hat{B}] | \Psi \rangle| ,$$

where ΔA and ΔB represent the square root of the second central moment (standard deviation or variance) of A and B, respectively, and $[\hat{A}, \hat{B}]$ is their commutator.

The vast literature on the Heisenberg uncertainty relation continues to grow and contains criticisms, notable in the following points [6, 16]:

If one of two noncommuting observables A or B in the Heisenberg uncertainty relation is in its eigenstate then $\Delta A = 0$ or $\Delta B = 0$ so its left-hand side is equal to zero and no non-trivial lower bound of the uncertainty relation exists.

In the last decades, many authors have shown that uncertainty relations given by the sum of entropies of two noncommuting observables do not suffer from the above-mentioned shortcoming of the Heisenberg uncertainty relation. According to quantum transformation theory we have

$$|\phi\rangle = (\sum_i^N a_i \langle y_1|x_i\rangle)|y_1\rangle + (\sum_i^N a_i \langle y_2|x_i\rangle)|y_2\rangle + \ldots = \sum_j^N [\sum_i^N a_i \langle y_j|x_i\rangle |y_j\rangle \,,$$

$$P_i(A) = |\langle x_i|\phi\rangle|^2 = |a_i|^2, \quad Q_j(B) = |b_j|^2 = |\langle y_j|\phi\rangle|^2 = |(\sum_i a_i \langle x_i|y_j\rangle)|^2 \,.$$

With the probability distributions from Eq. (1) the sum of the Shannon entropies of A and B (the right-hand side of the entropic uncertainty relation) becomes

$$S(A) + S(B) = -\sum_i |a_i|^2 \log |a_i|^2 - \sum |(\sum a_i \langle x_i|y_j\rangle)|^2 \log |(\sum a_i \langle x_i|y_j\rangle)|^2 \,.$$

The central problem connected with the entropic uncertainty relation is to find its lower bound. It was shown in the literature that there exists a non-trivial lower bound of the entropy sum of A and B [1, 9] and it depends only on components of the transformation matrix $\langle x_i|y_j\rangle$ between A and B. Deutsch [8] proved that

$$\inf_{|\Phi\rangle}(S(A) + S(B)) \geq -2\ln\left\{\frac{1}{2}(1 + \sup_{i,j} |\langle x_i|y_j\rangle|)\right\},$$

while Maassen and Uffink [3] improved this to

$$\inf_{|\Phi\rangle}(S(A) + S(B)) \geq -2\ln(\sup_{i,j} |\langle x_i|y_j\rangle|) \,.$$

Although the entropic uncertainty relation meets all the requirements put on an uncertainty relation, it has the following disadvantage. The sum of information entropies often represents complicated mathematical expressions whose exact lower bound is difficult to find by the analytical methods [2].

Now we propose an alternative form of the uncertainty relation based on the remaining uncertainty at the measurement of two noncommuting observables. A non-trivial lower bound always exists for these observables unless one of $S_j(B)$ would be zero. In that case, it would be possible determine the eigenvectors of A and B simultaneously which contradicts to the uncertainty principle of quantum mechanics. If we measure A or B from the pair A and B, it holds

$$S(A^*, B) \geq \inf_i S_i(B))$$

or

$$S(A, B^*) \geq \inf_i S_i(A)) \,.$$

Since both lower bounds need not be equal we proceed in the following way: We consider the set $\mathbf{Z} = \{S_1(B), S_2(B), \ldots, S_N(B), S_1(A), S_2(A), \ldots, S_N(A)$. The lower bound of $S(A, B)$ independent of the order of measurement represents the smallest element of $\mathbf{Z}$ and can be written as the alternative uncertainty relation in the form

$$S(A, B) \geq \inf\{\mathbf{Z}\} \,. \tag{3}$$

One can prove that this uncertainty relation satisfied all requirements for the quantitative expression of uncertainty principle in quantum mechanics. For two non-commuting observables, the infimum of $\{\mathbf{Z}\}$ represents the non-trivial lower bound. Uncertainty relation (3) does not suffer from the shortcoming of the Heisenberg uncertainty relation [1]. Supposing that the components of the transformation matrix between A and B are given the entropies $S_i(B)$ and $S_i(A)$ are easy to compound, therefore to determine the exact lower bound of $S(A, B)$ is relatively simple. Moreover, the lower bound of $S(A, B)$ gives the experimenter useful information about the effectiveness of an actual measurement. It is remarkable that the lower bound of $S(J_x, J_z)$ does not depend on the order of measurements of these observables and it is equal to $(1/2) \log 2$. It is identical with the lower bound of the corresponding entropic uncertainty relation for the spin variables J_x and J_z [6].

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