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RESEARCH PAPER

LOCAL EXISTENCE AND NON-EXISTENCE FOR A FRACTIONAL REACTION–DIFFUSION EQUATION IN LEBESGUE SPACES

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Abstract

We consider the following fractional reaction-diffusion equation

$$u_t(t) + \partial_t \int_0^t g_\alpha(s)\mathcal{A}u(t-s)ds = t^\gamma f(u),$$

where $g_\alpha(t) = t^{\alpha-1}/\Gamma(\alpha)$ $(0 < \alpha < 1)$, $f \in C([0,\infty))$ is a non-decreasing function, $\gamma > -1$, and $\mathcal{A}$ is an elliptic operator whose fundamental solution of its associated parabolic equation has Gaussian lower and upper bounds. We characterize the behavior of the functions f so that the above fractional reaction-diffusion equation has a bounded local solution in $L^r(\Omega)$, for non-negative initial data $u_0 \in L^r(\Omega)$, when $r > 1$ and $\Omega \subset \mathbb{R}^N$ is either a smooth bounded domain or the whole space $\mathbb{R}^N$. The case $r = 1$ is also studied.

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain or the whole space $\mathbb{R}^N$. Without loss of generality we assume that Ω contains the origin. We study

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the local existence in Lebesgue spaces of the following problem

$$\begin{cases} u_t + \partial_t \int_0^t g_\alpha(s)\mathcal{A}u(t-s)ds & = t^\gamma f(u), \quad \text{in } \Omega \times (0,T), \\ \mathcal{B}u & = 0, \quad \text{on } \partial\Omega \times (0,T), \\ u(0) & = u_0 \geq 0, \quad \text{in } \Omega, \end{cases} \tag{1.1}$$

where $g_\alpha(t) = t^{\alpha-1}/\Gamma(\alpha)$, $0 < \alpha < 1$, $\gamma > -1$, $f : [0,\infty) \to [0,\infty)$ is a continuous and non-decreasing function, $u_0 \in L^r(\Omega)$, $1 \leq r < \infty$, and

(H.1) The pair $(\mathcal{A}, \mathcal{B})$ defines an unbounded operator $A : D(A) \to L^2(\Omega)$ that generates a C_0-semigroup $(e^{-tA})_{t\geq 0}$ in $L^2(\Omega)$, with Green's function (or fundamental solution) K such that

$$(e^{-tA})\psi(x) = \int_\Omega K(x,y;t)\psi(y)dy, \tag{1.2}$$

for $\psi \in C_c^\infty(\Omega)$, that is, $e^{-tA}\psi$ is the solution of the linear problem

$$\begin{cases} u_t = & -\mathcal{A}u, \quad \text{in } \Omega \times (0,T), \\ \mathcal{B}u = & 0, \quad \text{on } \Omega, \\ u(0) = & \psi, \quad \text{in } \Omega. \end{cases} \tag{1.3}$$

Here, the set $C_c^\infty(\Omega)$ denotes the space of infinitely differentiable functions with compact support in Ω.

(H.2) K has a Gaussian upper bound

$$K(x,y;t) \leq C_1 t^{-N/2} \exp\left(-\lambda_1 \frac{|x-y|^2}{t}\right), \text{ for } t > 0, x, y \in \Omega, \tag{1.4}$$

with positive constants C_1, λ_1.

(H.3) K has a Gaussian lower bound: in the case $\Omega = \mathbb{R}^N$

$$K(x,y;t) \geq C_2 t^{-N/2} \exp\left(-\lambda_2 \frac{|x-y|^2}{t}\right), \text{ for } t > 0, x, y \in \mathbb{R}^N, \tag{1.5}$$

with positive constants C_2, λ_2; and in the case that $\Omega \subset \mathbb{R}^N$ is a bounded domain

$$K(x,y;t) \geq C_2 t^{-N/2} \exp\left(-\lambda_2 \frac{|x-y|^2}{t}\right), \text{ for } x, y \in \Omega' \subset \Omega, \tag{1.6}$$

and $0 < t < \min\{1, d^2(y, \partial\Omega)/8\}$, where Ω' is a convex subset of Ω such that $d(\Omega', \partial\Omega) > 0$.

The most simple example of such $\mathcal{A}$ is the Laplacian operator $-\Delta$ with Dirichlet boundary conditions. Nevertheless, we can consider more general operators such as

$$\mathcal{A}u = -\sum_{i,j=1}^N a_{ij}(x)u_{x_i x_j} - \sum_{j=1}^N b_j(x)u_{x_j} - c(x)u$$

with Robin boundary condition, considered also by Fujita and Watanabe [12]. More details in Section 5.

On one hand, the local existence of the semilinear parabolic problem

$$\begin{cases} u_t - \Delta u &= f(u), \qquad \text{in } \Omega \times (0,T), \\ u &= 0, \qquad \text{in } \partial\Omega \times (0,T), \\ u(0) &= u_0 \geq 0, \quad \text{in } \Omega, \end{cases} \tag{1.7}$$

has been studied by Brezis and Cazenave [6], Celik and Zhou [7], and Weissler [24, 25, 26], for $f(u) = u^p$, $p > 1$ and $u_0 \in L^r(\Omega), u_0 \geq 0$. More precisely,

(i) Problem (1.7) has a local solution $C([0,T], L^r(\Omega)), r > 1$ if and only if $r \geq N(p-1)/2$.

(ii) If $1 \leq r < N(p-1)/2$ or $r = N(p-1)/2 = 1$, then there exists an initial data $u_0 \in L^r(\Omega), u_0 \geq 0$ such that problem (1.7) does not admit any non-negative solution $C([0,T], L^r(\Omega))$.

Such results were recently obtained, for a more general f, by Laister et al. [14] in the following way.

Let $f : [0,\infty) \to [0,\infty)$ be a continuous and non-decreasing function.

(i) Assume that Ω is a bounded domain.

  (a) Problem (1.7) has a local solution for every $u_0 \in L^r(\Omega)$ with $r > 1$ if and only if
$$\limsup_{\tau\to\infty} \tau^{-(1+2r/N)} f(\tau) < \infty. \tag{1.8}$$

  (b) Problem (1.7) has a local solution for every $u_0 \in L^1(\Omega)$ if and only if
$$\int_1^\infty \tau^{-(1+2/N)} F(\tau) d\tau < \infty, \text{ where } F(\tau) = \sup_{1\leq\sigma\leq\tau} f(\sigma)/\sigma. \tag{1.9}$$

(ii) When $\Omega = \mathbb{R}^N$, the statements (a) and (b) remain valid if we replaced conditions (1.8) and (1.9) by
$$\limsup_{\tau\to\infty} \tau^{-(1+2r/N)} f(\tau) < \infty \quad \text{and} \quad \limsup_{\tau\to 0} f(\tau)/\tau < \infty,$$
$$\int_1^\infty \tau^{-(1+2/N)} \tilde{F}(\tau) dt < \infty \quad \text{and} \quad \limsup_{\tau\to 0} f(\tau)/\tau < \infty,$$
respectively, where $\tilde{F} = \sup_{0\leq\sigma\leq\tau} f(\sigma)/\sigma$.

It is worth point out that these results were extended by Kexue Li in [16] for problem (1.7) with the fractional Laplacian $(-\Delta)^\alpha$ with either $\Omega = B_R$ is the ball or $\Omega = \mathbb{R}^N$. Similar result for weakly coupled systems were obtained recently by Aparcana et al. in [1].

On the other hand, the integral version of the fractional diffusion equation

$$u_t(t,x) = \partial_t(g_\alpha * \Delta u)(t,x) + f, \quad t > 0,\ x \in \Omega, \tag{1.10}$$

where $\Omega \subset \mathbb{R}^N$, was studied initially by Schneider and Wyss [22], with $f \equiv 0$. In [19] the authors present an extensive list of systems displaying anomalous dynamical behavior of subdiffusive type that can be modeled by (1.10). We emphasize that (1.10) and

$$\partial_t^\alpha u = \Delta u + f,\ t > 0,\ x \in \Omega,\ 0 < \alpha < 1, \tag{1.11}$$

where $\partial_t^\alpha u$ denotes the Caputo fractional derivative of u, are equivalent only for $f \equiv 0$, but not in general. Other mathematical difference between these two problems lies on the fact that, for $f(u) = u^\rho, \rho > 1$, the critical Fujita's exponent for (1.10) $\rho_F = 1 + \frac{2}{\alpha N}$ (see [9, 23]) while it is $\rho_F = 1 + \frac{2}{N}$ (the same as the heat equation, case $\alpha = 1$), for (1.11), according to [27]. It may suggest that the fractional parameter α plays a more influential role in (1.10) than in (1.11). From the physical point of view, we emphasize that considering a source in the diffusion process with memory as (1.10) avoids the reaction to be affected by the memory effect, as observed by Metzler et al. [18, p. 346], unlike in (1.11). Recently, Lophushansky et al. [17] studied the existence and uniqueness of solutions in Bessel potential spaces, for an equation that is equivalent to (1.11), by means of the abstract approach. More recently, de Andrade et al. [10] proved, among other issues, a result on the local well-posedness for (1.10) with $u_0 \in L^q(\Omega)$ and for $f(u) \simeq u^\rho, \rho > 1$. More precisely, their existence results can be read as follows: let $v_0 \in L^q(\Omega)$, if either

(i) $q \geq 1$, $q > \frac{\alpha N}{2}(\rho - 1)$, and $\rho\alpha > 1$; or
(ii) $q \geq \rho$ and $q > \frac{N}{2}(\rho - 1)$; or
(iii) $1 \leq q < \rho$ and $q \geq \frac{N}{2}(\rho - 1)$; or
(iv) $1 < q = \frac{\alpha N}{2}(\rho - 1)$ and $\rho\alpha > 1$;

then, there exist $T > 0$ and $R > 0$ such that (1.10) with initial condition $u_0 \in B_{L^q(\Omega)}(v_0, R/4)$ has a L^q-mild solution $u : [0,T] \to L^q(\Omega)$.

Therefore, the above mentioned results motivate us to study conditions for the existence and non-existence of a L^r-local solution for problem (1.1) with $u_0 \in L^r$ (see Definition 2.1 for the concept of solution we consider).

Our main result is the following.

Theorem 1.1. *Let $f : [0,\infty) \to [0,\infty)$ be a continuous non-decreasing function, $\gamma > -1$ and*

$$p^\star = 1 + \frac{2r(\gamma+1)}{\alpha N}. \tag{1.12}$$

(i) *Existence.* *Assume* $p^\star > (1+\gamma)/\alpha$, $r > 1$ *and*

$$\limsup_{\tau\to\infty} \tau^{-p^\star} f(\tau) < \infty, \text{ if } \Omega \text{ is bounded}$$

or

$$\limsup_{\tau\to\infty} \tau^{-p^\star} f(\tau) < \infty \text{ and } \limsup_{\tau\to 0} f(\tau)/\tau < \infty, \text{ if } \Omega = \mathbb{R}^N.$$

Then, for every non-negative $u_0 \in L^r(\Omega)$, *problem (1.1) has a non-negative local* L^r*-solution. For* $r = 1$, *existence holds with* $N = 1$.

(ii) *Non-existence. Let* $r \geq 1$. *Suppose that*

$$\limsup_{\tau\to\infty} \tau^{-p^\star} f(\tau) = \infty, \text{ if } \Omega \text{ is bounded}$$

or

$$\limsup_{\tau\to\infty} \tau^{-p^\star} f(\tau) = \infty \text{ or } \limsup_{\tau\to 0} f(\tau)/\tau = \infty, \text{ if } \Omega = \mathbb{R}^N.$$

Then there exists $u_0 \in L^r(\Omega), u_0 \geq 0$ *such that problem (1.1) has no non-negative local* L^r*-solution.*

REMARK 1.1. Here are some comments on Theorem 1.1.

(i) When $\gamma = 0$ and $\alpha = 1$, we recover Laister et al. [14] characterization in L^r. Indeed, our proofs work well for $\alpha = 1$, with fewer constraints.

(ii) When $f(\tau) = \tau^p, \tau \geq 0$ and $\gamma = 0$, the condition $p^\star > (1+\gamma)/\alpha = 1/\alpha$ in the existence part coincides with the one required in [9, Theorem 1] to show the global existence.

(iii) As usual in local existence results, we do not require the smallness of $\|u_0\|_{L^r}$ as it was in [9, Theorem 1], where global solutions are sought. We rather use that $\lim_{\tau\to 0^+} t^{\frac{N\alpha}{2}(\frac{1}{r}-\frac{1}{\eta})}\|S_\alpha(t)u_0\|_{L^\eta} = 0$, see Lemma 3.1 for details.

In our second result, we consider a weakened assumption for the non-existence result given in Theorem 1.1 for $r = 1$. Precisely, we have the following.

THEOREM 1.2. *Let* $f : [0,\infty) \to [0,\infty)$ *be a non-decreasing function,* $F(s) = \sup_{1\leq\tau\leq s} f(\tau)/\tau$, *and* $\tilde{F}(s) = \sup_{0<\tau\leq s} f(\tau)/\tau$. *Assume that either*

$$\int_1^\infty s^{-p^*} F(s)ds = \infty, \text{ if } \Omega \text{ is a bounded domain,}$$

or

$$\int_0^\infty s^{-p^*} \tilde{F}(s)ds = \infty, \text{ if } \Omega = \mathbb{R}^N,$$

where p^* *is given by (1.12) with* $r = 1$. *Then there exists* $u_0 \in L^1(\Omega), u_0 \geq 0$ *such that problem (1.1) has no non-negative local* L^r*-solution.*

Concerning our results, we observe that the case $r = 1$ offers some extra difficulties in the application of the technique given in [14], mostly because the Mittag-Leffler family $S_\alpha(t)\phi$ does not enjoy the semigroup property $T(t+s) = T(t)T(s)$, see e.g. [20]. Moreover, $L^1 \to L^\infty$ bounds for $S_\alpha(t)$, with $N \geq 2$, are not available in the literature, and it is a sensitive subject indeed. We overcame the earlier barrier but not the latter one. Therefore, our existence results in L^1 holds for $N = 1$. The precise results are in Subsection 4.2. On the other hand, this work extends Laister et al. [14] in by considering a nonlinearity with temporal weight and the fractional diffusion. Our results are of the same nature as those in [14], recover and extend them (except for $N = 1$). We follow the methodology used in [14], but difficulties inherent to nonlocal-in-time problems had to be overcome as well as the analysis with the time-dependent nonlinearity demanded more effort.

This work is organized as follows. In Section 2, we prove some lower bounds and $L^q - L^r$ bounds for the Mittag-Leffler family $(S_\alpha(t))_{t\geq 0}$. The proofs of existence and non-existence results are split into two cases: $r > 1$ and $r = 1$, which are subjects of Sections 3 and 4, respectively. In Section 5, we gather some results on Gaussian bounds for the heat kernels associated with some more general elliptic operators in order to exhibit existence and non-existence results on fractional reaction–diffusion equations involving such general elliptic operators.

2. **Auxiliary results**

2.1. **The parabolic equation.** We denote by $B_\rho(x) \subset \mathbb{R}^N$ the ball centered at x with radius ρ, by χ_ρ the characteristic function on $B_\rho(0)$, and by ω_N the volume of the unit ball in $\mathbb{R}^N$.

In view of our assumptions (H.1)–(H.3) on the operator $(\mathcal{A}, \mathcal{B})$, we can obtain an estimate from below of the solution of the linear problem (1.3).

Lemma 2.1. *Assume that Ω is bounded and let $\rho > 0$, $\delta \in (0,1)$ such that $B_{\rho+2\delta} \subset \Omega$. There exist positive constants c_* and c_0, which depend only on N, C_1, γ_1, q, r, and N, C_2, γ_2, respectively, such that*

$$\|e^{-At}\varphi\|_{L^r(\Omega)} \leq c_\star t^{-\frac{N}{2}\left(\frac{1}{q}-\frac{1}{r}\right)} \|\varphi\|_{L^q(\Omega)}, \tag{2.1}$$

for $1 \leq q \leq r \leq \infty$, and

$$e^{-tA}\chi_\rho \geq c_0 \left(\frac{\rho}{\max\{\rho, \sqrt{t}\}} \right)^N \chi_{\rho+\sqrt{t}}, \tag{2.2}$$

for all $0 < t \leq \delta^2/8$.

P r o o f. We first note that the identity (1.2) holds, for $u_0 \in L^r(\Omega), 1 \leq r \leq \infty$. Indeed, it follows either from (1.4) and the density of C_0^∞ in $L^r(\Omega)$, with $1 \leq r < \infty$ or from [2], in general. To show (2.1), we use (1.2), (1.4), and the Young inequality in the following standard way: define p by $\frac{1}{p} = 1 + \frac{1}{r} - \frac{1}{q}$, then

$$\begin{aligned}\|e^{-At}\varphi\|_{L^r(\Omega)} &= \left\|\int_\Omega K(x,y;t)\varphi(y)dy\right\|_{L^r(\Omega)} \\ &\leq C_1 \left\|t^{-\frac{N}{2}} \exp\left(-\gamma_1 \frac{|\cdot|^2}{t}\right)\right\|_{L^p(\mathbb{R}^N)} \|\chi_\Omega \varphi\|_{L^q(\mathbb{R}^N)} \\ &\leq c_* t^{-\frac{N}{2}+\frac{N}{2p}} \|\varphi\|_{L^q(\Omega)} \\ &= c_\star t^{-\frac{N}{2}\left(\frac{1}{q}-\frac{1}{r}\right)} \|\varphi\|_{L^q(\Omega)}.\end{aligned}$$

Next, we argue as in the proof of [14, Lemma 2.1]. From (1.2) and (1.6), for $\Omega' = B_{\rho+\delta}(0)$, we have that $d(\Omega', \partial\Omega) > \delta$. Thus, for $0 < t < \min\{1, \delta^2/8\} = \delta^2/8$ and $|x| < \rho + \sqrt{t}$ we have

$$\begin{aligned}e^{-tA}\chi_\rho(x) &= \int_\Omega K(x,y;t)\chi_\rho dy \\ &\geq C_2 t^{-N/2} \int_{|y|\leq\rho} \exp\left(-\frac{\gamma_2|x-y|^2}{t}\right)dy \\ &\geq C_2 \int_{B_{\rho/\sqrt{t}}((1+\frac{\rho}{\sqrt{t}})\mathbf{u})} \exp(-\gamma_2|z|^2)dz.\end{aligned}$$

Since $|z - (1+\frac{\rho}{\sqrt{t}})\mathbf{u}| \leq |z - 2\mathbf{u}| + |(1-\frac{\rho}{\sqrt{t}})\mathbf{u}|$ we conclude that $B_1(2\mathbf{u}) \subset B_{\rho/\sqrt{t}}((1+\frac{\rho}{\sqrt{t}})\mathbf{u})$ if $\sqrt{t} \leq \rho$. Then,

$$S(t)\chi_\rho \geq C_2 \int_{B_1(2\mathbf{u})} \exp(-\gamma_2|z|^2)dz.$$

On the other hand, if $\sqrt{t} \geq \rho$, we have $S(t)\chi_\rho \geq C_2\omega_N \exp(-9\gamma_2)(\rho/\sqrt{t})^N$. These lead to (2.2). □

Remark 2.1. Since the proof of Lemma 2.1 is based on the lower bound of the Green function K, it is possible to conclude, using (1.5), that estimate (2.2) also holds for $\Omega = \mathbb{R}^N$ and for all $t > 0$. The same proof for (2.1) works when $\Omega = \mathbb{R}^N$.

2.2. **The fractional problem.**

We recall some facts on the resolvent family associated to the equation

$$\begin{cases} u_t + \partial_t \displaystyle\int_0^t g_\alpha(s)\mathcal{A}u(t-s)ds = 0, & \text{in } \Omega \times (0,T), \\ \mathcal{B}u = 0, & \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0 \geq 0, & \text{in } \Omega, \end{cases} \tag{2.3}$$

Indeed, after integrating (2.3), we rewrite it as the following abstract Volterra integral equation

$$u(t) = u_0 + \int_0^t g_\alpha(s)Au(t-s)ds. \tag{2.4}$$

Recalling that A is the generator of a C_0-semigroup in $L^2(\Omega)$, the subordination principle in [5] gives that A also generates the resolvent family

$$S_\alpha(t) = \frac{1}{2\pi i}\int_{H_a} e^{\lambda t}\lambda^{\alpha-1}(\lambda^\alpha + A)^{-1}d\lambda,\ t \geq 0, \tag{2.5}$$

for an arbitrary Hankel path H_a, and that satisfies

$$S_\alpha(t)\varphi = \int_0^\infty M_\alpha(\sigma)e^{-\sigma t^\alpha A}\varphi d\sigma, \tag{2.6}$$

for every distribution φ, where M_α is the Wright function

$$M_\alpha(z) = \sum_{n=0}^\infty \frac{(-z)^n}{n!\Gamma(-\alpha n - \alpha + 1)}. \tag{2.7}$$

This family $(S_\alpha)_{t\geq 0}$ of bounded operators in $L^2(\Omega)$, with $S_\alpha(0) = I$, is called the Mittag-Leffler family associated to the operator $(\mathcal{A}, \mathcal{B})$, and is the same given by the formal application of the Laplace transform in (2.3).

Clearly, the above reasoning works well with a Banach space X in place of $L^2(\Omega)$, provided that A generates a C_0-semigroup in X.

The case of $\Omega = \mathbb{R}^N$ can also be approached by means of the Duhamel principle applied to (2.3), and by recalling that the Mittag-Leffler function $E_\alpha(-z)$ is an analytic Laplace transformable function obeying $\widehat{E_\alpha(-zt^\alpha)}(\lambda) = \frac{\lambda^{\alpha-1}}{\lambda^\alpha + z}$ and

$$E_\alpha(-z) = \int_0^\infty M_\alpha(t)e^{-zt}dt, \quad z \in \mathbb{C}.$$

We will use the following properties of the Wright function: $M_\alpha(t) \geq 0$ for all $t \geq 0$,

$$\int_0^\infty M_\alpha(\sigma)d\sigma = 1, \tag{2.8}$$

and

$$\int_0^\infty M_\alpha(\sigma)\sigma^\delta d\sigma = \frac{\Gamma(\delta+1)}{\Gamma(\alpha\delta+1)}, \quad \text{if } \ \delta > -1. \tag{2.9}$$

LEMMA 2.2. *Let $1 \le q \le r \le \infty$ be such that $1/q - 1/r < 2/N$, then*

$$\|S_\alpha(t)\varphi\|_{L^r(\Omega)} \le C_\star(N,\alpha,q,r) t^{-\frac{\alpha N}{2}\left(\frac{1}{q}-\frac{1}{r}\right)} \|\varphi\|_{L^q(\Omega)},$$

where $C_\star > 0$ depends only on $N, \alpha, q,$ and r.

P r o o f. Combining (2.6), (2.1), and (2.9), we have

$$\begin{aligned}\|S_\alpha(t)\varphi\|_{L^r} &\le \int_0^\infty M_\alpha(\sigma)\|e^{-\sigma t^\alpha A}\|_{L^r} d\sigma \\ &\le c_* t^{-\alpha\frac{N}{2}\left(\frac{1}{q}-\frac{1}{r}\right)} \int_0^\infty M_\alpha(\sigma)\sigma^{-\frac{N}{2}\left(\frac{1}{q}-\frac{1}{r}\right)} d\sigma \|\varphi\|_{L^q} \\ &=: C_* t^{-\alpha\frac{N}{2}\left(\frac{1}{q}-\frac{1}{r}\right)}\|\varphi\|_{L^q}.\end{aligned}$$

□

LEMMA 2.3. *Let $\rho > 0$, $\delta \in (0,1)$ such that $B_{\rho+2\delta} \subset \Omega$. There exists a constant $\tilde{c}_0 > 0$, which depend only on N, C_2 and γ_2 and such that*

$$S_\alpha(t)\chi_\rho \ge \tilde{c}_0 \left(\frac{\rho}{\max\{\rho, \sqrt{t^\alpha}\}}\right)^N \chi_{\rho+\sqrt{t^\alpha}}, \tag{2.10}$$

for all $0 < t^\alpha \le \delta^2/16$.

P r o o f. From (2.6) and Lemma 2.1

$$\begin{aligned} S_\alpha(t)\chi_\rho &= \int_0^\infty M_\alpha(\sigma) e^{-\sigma t^\alpha \mathcal{A}} \chi_\rho d\sigma \\ &\ge c_0 \int_0^{t^{-\alpha}\delta^2/8} M_\alpha(\sigma) \left(\frac{\rho}{\max\{\rho,\sqrt{\sigma t^\alpha}\}}\right)^N \chi_{\rho+\sqrt{\sigma t^\alpha}}\, d\sigma \\ &\ge c_0 \left(\frac{\rho}{\max\{\rho,\sqrt{t^\alpha}\}}\right)^N \chi_{\rho+\sqrt{t^\alpha}} \int_1^{t^{-\alpha}\delta^2/8} M_\alpha(\sigma)\sigma^{-\frac{N}{2}} d\sigma \qquad (2.11) \\ &\ge 2^{-N/2} c_0 \int_1^2 M_\alpha(\sigma) d\sigma \left(\frac{\rho}{\max\{\rho,\sqrt{t^\alpha}\}}\right)^N \chi_{\rho+\sqrt{t^\alpha}} \\ &= \tilde{c}_0 \left(\frac{\rho}{\max\{\rho,\sqrt{t^\alpha}\}}\right)^N \chi_{\rho+\sqrt{t^\alpha}}, \end{aligned}$$

where $\tilde{c}_0 = 2^{-N/2} c_0 \int_1^2 M_\alpha(\sigma) d\sigma$. □

REMARK 2.2.

(i) When $\Omega = \mathbb{R}^N$, from Remark 2.1, it is possible to observe that estimate (2.10) holds for all $t > 0$.

(ii) The smoothing effect in Lemma 2.2 holds for $\alpha = 1$, without the constraint $1/q - 1/r < 2/N$, actually, it is Lemma 2.1. Such a constraint makes ranges tighter and restricts the dimension for distant exponents. For instance, Lemma 2.2 holds for $(q, r) = (1, \infty)$, only when $N = 1$.

The local L^r-mild solutions for the problem (1.1) are understood in the following sense.

Definition 2.1. Given $u_0 \in L^r(\Omega)$, $r \geq 1$. We say that $u \in L^\infty((0,T), L^r(\Omega))$ is a local L^r-mild solution, or simply, a local solution of problem (1.1) when there exists $T > 0$ such that

$$u(t) = S_\alpha(t)u_0 + \int_0^t S_\alpha(t-\sigma)\sigma^\gamma f(u(\sigma))d\sigma, \tag{2.12}$$

for $t \in (0,T)$.

We also need a comparison principle for equation (2.12). So, it is convenient to define what we understand by a supersolution for (2.12).

Definition 2.2. Given $u_0 \in L^r(\Omega)$, $r \geq 1$, a non-negative function $\bar{u} \in L^\infty((0,T), L^r(\Omega))$ is a local L^r-mild supersolution, or simply, a *supersolution* of (2.12), if

$$\bar{u}(t) \geq S_\alpha(t)u_0 + \int_0^t S_\alpha(t-\sigma)\sigma^\gamma f(\bar{u}(\sigma))d\sigma.$$

Subsolutions are defined analogously, with reversed inequality.

Lemma 2.4. *Assume that $r \geq 1$ and $f : [0,\infty) \to [0,\infty)$ is a continuous and non-decreasing function. Let $u_0 \in L^r(\Omega)$ be a non-negative function. Then, problem (2.12) admits a local solution in $L^\infty((0,T), L^r(\Omega))$ if and only if it admits an supersolution in $L^\infty((0,T), L^r(\Omega))$.*

Proof. It is clear that every solution is also a supersolution of the problem (1.1). We must prove the converse. In fact, we follow the argument used in [21]. Suppose that there exists a supersolution $\bar{u}$ of the problem (2.12) in $(0,T)$, and define the operator $\mathcal{F}$ by

$$\mathcal{F}(v)(t) = S_\alpha(t)u_0 + \int_0^t S_\alpha(t-\sigma)\sigma^\gamma f(v(\sigma))d\sigma,$$

for $t \in (0,T)$. Note that $\bar{u} \geq \mathcal{F}(\bar{u})$ in $(0,T)$.

Using the monotonicity of f and the positivity preserving of $S_\alpha(t)$ (see [9]), we obtain that $\mathcal{F}$ is a non-increasing operator in the set of the non-negative and measurable functions. Now, consider the following sequence

$\left\{\mathcal{F}^k(\bar{u})\right\}_{k\geq 0}$, where $\mathcal{F}^{k+1}(\bar{u}) = \mathcal{F}(\mathcal{F}^k(\bar{u}))$. By $\bar{u} \geq \mathcal{F}(\bar{u})$ and the monotonicity of the operator $\mathcal{F}$, we conclude that the sequence $\left\{\mathcal{F}^k(\bar{u})\right\}_{k\geq 0}$ is a non-decreasing and non-negative sequence in $(0, T)$. Taking the pointwise limit

$$u(x,t) = \lim_{k\to\infty} \left[\mathcal{F}^k(\bar{u})\right](x,t) \text{ whenever there exist,}$$

we have that u verifies (2.12). Indeed, for continuity of f and by the monotone convergence theorem, it is possible to conclude that $\lim_{k\to\infty} \mathcal{F}(u_k) = \mathcal{F}(u)$ a.e. in $\Omega \times (0,T)$ where $\mathcal{F}^k(\bar{u}) := u_k$ for all $k \in \mathbb{N}$. Thus, due to the construction of the sequences we have $u = \mathcal{F}(u)$ a.e. in $\Omega \times (0,T)$. Moreover, since $\bar{u}(t), u_0 \in L^r(\Omega)$ and f is a non-decreasing function, we have

$$0 \leq \mathcal{F}(\bar{u}(t)) \leq S_\alpha(t)u_0 + \int_0^t S_\alpha(t-\sigma)\sigma^\gamma \bar{u}(\sigma)d\sigma,$$

whence

$$\|\mathcal{F}(\bar{u}(t))\|_{L^r} \leq C_\star \|u_0\|_{L^r} + C_\star \frac{T^{\gamma+1}}{\gamma+1}\|\bar{u}\|_{L^\infty(0,T;L^r)}, \tag{2.13}$$

by Lemma 2.2. Now, the monotonicity of the operator $\mathcal{F}$ implies that the sequence $\left\{\mathcal{F}^k(\bar{u})\right\}_{k\geq 0}$ belongs to $L^r(\Omega)$, by induction, and so $u(t) \in L^r(\Omega)$, a.e $t > 0$. □

3. **Proof of Theorem 1.1**

In this section, we give two preliminary lemmas, and we present the proof of Theorem 1.1, divided into two parts: existence and non-existence in L^r.

LEMMA 3.1. *Let $\alpha \in (0,1)$, $u_0 \in L^r(\Omega)$, and $1 \leq r < \eta \leq \infty$. If $\frac{N}{2}\left(\frac{1}{r} - \frac{1}{\eta}\right) < 1$, then*

$$\lim_{t\to 0^+} t^{\frac{N\alpha}{2}(\frac{1}{r}-\frac{1}{\eta})}\|S_\alpha(t)u_0\|_{L^\eta} = 0.$$

P r o o f. Given $u_0 \in L^r(\Omega)$, there is a sequence (φ_n) test functions converging to u_0. Hence, Lemma 2.2 gives

$$\begin{aligned} t^{\frac{\alpha N}{2}\left(\frac{1}{r}-\frac{1}{\eta}\right)}\|S_\alpha(t)u_0\|_{L^\eta} \leq & t^{\frac{\alpha N}{2}\left(\frac{1}{r}-\frac{1}{\eta}\right)}\|S_\alpha(t)(u_0-\varphi_n)\|_{L^\eta} \\ & + t^{\frac{\alpha N}{2}\left(\frac{1}{r}-\frac{1}{\eta}\right)}\|S_\alpha(t)\varphi_n\|_{L^\eta} \\ \leq & C\|u_0-\varphi_n\|_{L^r} + t^{\frac{\alpha N}{2}\left(\frac{1}{r}-\frac{1}{\eta}\right)}\|\varphi_n\|_{L^\eta}. \end{aligned}$$

The result follows now by passing the limit as $t \to 0$, and then $n \to 0$. □

The next lemma ensures that the constants appearing in the proof of Theorem 1.1 are finite and allows us to use the fixed point method to prove the existence of solutions for the problems 3.2 and 3.6.

Lemma 3.2. *Let* $r > 1$, $\gamma > -1, \alpha > 0$, $N \geq 1$, $p^* = 1 + \frac{2r(1+\gamma)}{N\alpha}$ *and* $\beta = \frac{\alpha N}{2}(\frac{1}{r} - \frac{1}{\eta})$. *Assume that* $p^* > (1+\gamma)/\alpha$. *Then, there exists* $\eta > r, \eta > p^*$ *such that*

(i) $\frac{N}{2}\left(\frac{1}{r} - \frac{1}{\eta}\right) < 1$;
(ii) $\beta p^* < 1 + \gamma$;
(iii) $\frac{N}{2\eta}(p^* - 1) < 1$;
(iv) $1 + \beta - \frac{\alpha N}{2\eta}(p^\star - 1) - p^\star\beta + \gamma = 0$.

3.1. **Existence.** We consider two situations.

Case 1. Ω is a bounded domain. Since $\limsup_{\tau\to\infty} \tau^{-p^*} f(\tau) < \infty$, there exists a positive constant C_1 such that

$$f(\tau) \leq C_1(1 + \tau^{p^\star}) \tag{3.1}$$

for $\tau \geq 0$, where $p^\star = 1 + \frac{2r(\gamma+1)}{\alpha N}$.

Next, we obtain a local L^r-mild solution for the following auxiliary problem

$$\begin{cases} v_t - \partial_t \displaystyle\int_0^t g_\alpha(s)\mathcal{A}v(t-s)ds &= C_1 t^\gamma(1 + v^{p^\star}), \quad \text{in } \Omega \times (0,T), \\ \mathcal{B} &= 0, \quad \text{on } \partial\Omega \times (0,T), \\ u(0) &= u_0 \geq 0, \quad \text{in } \Omega \end{cases} \tag{3.2}$$

with $u_0 \in L^r(\Omega)$. Note that from the existence of a solution for problem (3.2), Lemma 2.4 and (3.1) we have the result.

We use a fixed point argument, as in [6, 9, 10]. We warn that the constant C_* may vary along the proof. Keeping in mind the definition of mild solution given by (2.12) with $f(u(\sigma))$ replaced by $C_1(1 + v^{p^*}(\sigma))$, we define the operator $\Psi_1 : K \to E$ by

$$\Psi_1(v)(t) = S_\alpha(t)u_0 + C_1 \int_0^t S_\alpha(t-\sigma)\sigma^\gamma[1 + v^{p^\star}(\sigma)]d\sigma,$$

where $E = L^\infty_{loc}((0,T), L^\eta(\Omega))$, η is given by Lemma 3.2,

$$K = \left\{v \in E : v \geq 0 \text{ and } t^\beta \|v(t)\|_{L^\eta} \leq \delta\right\},$$

$\beta = \frac{\alpha N}{2}\left(\frac{1}{r} - \frac{1}{\eta}\right)$, and $\delta > 0$ will be chosen later. The set K is a complete metric space endowed with the metric $d(v,w) = \sup_{t\in(0,T)} t^{\beta}\|v(t)-w(t)\|_{L^\eta}$, for all $v, w \in K$.

It is easy to see that $\Psi_1(v) \geq 0$, for any $v \in K$. From Lemmas 2.2 and 3.2 ((i), (ii) and (iv)) we have

$$\begin{aligned}
& t^{\beta}\|\Psi_1(v)(t)\|_{L^\eta} \\
\leq\ & t^{\beta}\|S_\alpha(t)u_0\|_{L^\eta} + C_1 t^{\beta}\int_0^t \sigma^{\gamma}\|S_\alpha(t-\sigma)1\|_{L^\eta}\,d\sigma \\
& + C_1 t^{\beta}\int_0^t \sigma^{\gamma}\|S_\alpha(t-\sigma)v^{p^*}(\sigma)\|_{L^\eta}\,d\sigma \\
\leq\ & t^{\beta}\|S_\alpha(t)u_0\|_{L^\eta} + \frac{C_1 C_\star}{1+\gamma}\|1\|_{L^\eta} t^{1+\gamma+\beta} \\
& + C_1 C_* t^{\beta}\int_0^t \sigma^{\gamma}(t-\sigma)^{-\frac{\alpha N}{2}(\frac{p^*}{\eta}-\frac{1}{\eta})}\|v(\sigma)\|_{L^\eta}^{p^*}\,d\sigma \\
\leq\ & t^{\beta}\|S_\alpha(t)u_0\|_{L^\eta} + \frac{C_1 C_*}{1+\gamma}|\Omega|^{1/\eta} t^{1+\gamma+\beta} \\
& + C_1 C_* \delta^{p^*}\int_0^1 (1-\sigma)^{-\frac{\alpha N}{2}(\frac{p^*}{\eta}-\frac{1}{\eta})}\sigma^{-\beta p^*+\gamma}\,d\sigma. \qquad (3.3)
\end{aligned}$$

Let $\delta > 0$ be such that

$$C_1 C_* \delta^{p^*}\int_0^1 (1-\sigma)^{-\frac{\alpha N}{2}(\frac{p^*}{\eta}-\frac{1}{\eta})}\sigma^{-\beta p^*+\gamma}\,d\sigma < \delta/2. \qquad (3.4)$$

From Lemma 3.1, there exists $T > 0$ such that $t^{\beta}\|\Psi_1(v)(t)\|_{L^\eta} \leq \delta$. So, $\Psi_1 v \in K$.

Arguing similarly, it is possible to show that

$$\begin{aligned}
& t^{\beta}\|\Psi_1(v)(t) - \Psi_1(\tilde v)(t)\|_{L^\eta} \qquad (3.5) \\
\leq\ & C_1 C_\star \delta^{p^*-1}\int_0^1 (1-\sigma)^{-\frac{N}{2}(\frac{p^*}{\eta}-\frac{1}{\eta})}\sigma^{-\beta p^*+\gamma}\,d\sigma \sup_{t\in(0,T)} t^{\beta}\|v(t)-\tilde v(t)\|_{L^\eta}.
\end{aligned}$$

Thus, by (3.4), we have that Ψ_1 is a strict contraction. Therefore, there exists a fixed point v for the map $\Psi_1 : K \to K$, that is, $v = \Psi_1 v$. Since, $v \in E$, $\eta > r$, and Ω is bounded, we have that $v \in L^\infty((0,T); L^r(\Omega))$.

Thus, we obtained the existence of a local L^r-mild solution for (3.2) when Ω is a bounded domain.

Case 2. $\Omega = \mathbb{R}^N$. Since

$$\limsup_{\tau\to 0} f(\tau)/\tau < \infty \text{and} \limsup_{\tau\to\infty} \tau^{-p^*} f(\tau) < \infty,$$

we have that there exists a constant $C_2 > 0$ such that $f(\tau) \leq C_2(\tau + \tau^{p^*})$. With small modifications in the arguments of the Case 1, it is possible to prove the existence of a solution of the following problem

$$\begin{cases} w_t - \partial_t \int_0^t g_\alpha(s)\mathcal{A}w(t-s)ds &= C_2 t^\gamma (w + w^{p^\star}), \quad \text{in } \mathbb{R}^N \times (0,T), \\ u(0) &= u_0, \quad \text{in } \mathbb{R}^N, \end{cases} \tag{3.6}$$

in the space $E = L^\infty_{loc}((0,T), L^\eta(\mathbb{R}^N) \cap L^\infty((0,T), L^r(\mathbb{R}^N))$. Indeed, let $M \geq \|u_0\|_{L^r}$, and $\delta_0 < 1$, which will be chosen later. The set

$$K_0 := \left\{ v \in E : v \geq 0, \|v(t)\|_{L^r} \leq C_* M + 1, t^\beta \|v(t)\|_{L^\eta} \leq \delta_0 \right\},$$

is a non-empty complete metric space endowed with the metric

$$d_0(v,w) := \max\left\{ \sup_{t\in(0,T)} \|v(t) - w(t)\|_{L^r}, \sup_{t\in(0,T)} t^\beta \|v(t) - w(t)\|_{L^\eta} \right\}.$$

The values of η and β are given as in the Case 1. Let $\Psi_2 : K_0 \to E$ be given by

$$\Psi_2(v)(t) = S_\alpha(t)u_0 + C_2 \int_0^t S_\alpha(t-\sigma)\sigma^\gamma v(\sigma)d\sigma + C_2 \int_0^t S_\alpha(t-\sigma)\sigma^\gamma v^{p^\star}(\sigma)d\sigma.$$

Using Lemma 3.2, and arguing as in the derivation of (3.3) and (3.5), we have

$$\begin{aligned} &\|\Psi_2(v)(t)\|_{L^r} \\ \leq\ & C_*\|u_0\|_{L^r} + C_2 C_* \int_0^t \sigma^\gamma \|v(\sigma)\|_{L^r} d\sigma \\ &+ C_2 C_* \int_0^t \sigma^\gamma (t-\sigma)^{-\frac{\alpha N}{2}\left(\frac{p^\star}{\eta} - \frac{1}{r}\right)} \|v(\sigma)\|^{p^*}_{L^\eta} d\sigma \\ \leq\ & C_* M + \frac{C_2 C_* t^{1+\gamma}}{1+\gamma}(C_* M + 1) + C_2 C_* \delta_0^{p^*} \int_0^1 (1-\sigma)^{\frac{\alpha N}{2}(\frac{p^*}{\eta} - \frac{1}{r})} \sigma^{\gamma - \beta p^*} d\sigma, \end{aligned}$$

and

$$\begin{aligned} &t^\beta \|\Psi_2(v)(t)\|_{L^\eta} \\ \leq\ & t^\beta \|S_\alpha(t)u_0\|_{L^\eta} + C_2 t^\beta \int_0^t \sigma^\gamma \|v(\sigma)\|_{L^\eta} d\sigma \\ &+ C_2 C_* t^\beta \int_0^t \sigma^\gamma (t-\sigma)^{-\frac{\alpha N}{2}\left(\frac{p^\star}{\eta} - \frac{1}{\eta}\right)} \|v(\sigma)\|^{p^*}_{L^\eta} d\sigma \\ \leq\ & t^\beta \|S_\alpha(t)u_0\|_{L^\eta} + \frac{C_2 t^{1+\gamma}}{1+\gamma-\beta}\delta + C_2 C_* \delta_0^{p^*} \int_0^1 (1-\sigma)^{-\frac{\alpha N}{2}(\frac{p^*}{\eta} - \frac{1}{r})} \sigma^{\gamma - \beta p^*} d\sigma. \end{aligned}$$

Similarly, it is possible to show that

$$\begin{aligned}&\|\Psi_2(v)(t)-\Psi_2(w)(t)\|_{L^r}\\ \leq\ & \left(\frac{C_2C_*t^{1+\gamma}}{1+\gamma}+C_2C_*\delta_0^{p-1}\int_0^1(1-\sigma)^{-\frac{\alpha N}{2}(\frac{p^*}{\eta}-\frac{1}{r})}\right)d(v,w),\\ &t^{\beta}\|\Psi_2(v)(t)-\Psi_2(w)(t)\|_{L^\eta}\\ \leq\ & \left(\frac{C_2t^{1+\gamma}}{1+\gamma-\beta}+C_2C_*\delta_0^{p-1}\int_0^1(1-\sigma)^{-\frac{\alpha N}{2}(\frac{p^*}{\eta}-\frac{1}{\eta})}\right)d(v,w).\end{aligned}$$

From these estimates and Lemma 3.1 we observe that Ψ_2 is a strict contraction if $\delta_0>0$ and $T>0$ are sufficiently small. Therefore, there exists a fixed point of Ψ_2, which is the desired solution. □

3.2. **Non-existence.** We show the non-existence part of Theorem 1.1. Assume first that $\limsup_{\tau\to\infty}\tau^{-p^*}f(\tau)=\infty$. Then, there exists a sequence $\{z_k\}_{k\in\mathbb{N}}$ such that

$$z_k\geq k \text{ and } f(z_k)\geq z_k^{p^\star}\mathrm{e}^{k/r}. \tag{3.7}$$

Let $r_k=(k^{-2}z_k^{-1})^{r/N}$. Since $\lim_{k\to\infty}r_k=0$, we can suppose that $r_k<1$ and $B_{3r_k}\subset\Omega$, for all $k\in\mathbb{N}$. Now, define $u_k=\tilde{c}_0^{-1}z_k\chi_{r_k}$ and

$$u_0=\sum_{k=1}^{\infty}u_k, \tag{3.8}$$

where $\tilde{c}_0$ is the constant of Lemma 2.3. Then,

$$\|u_0\|_{L^r}\leq\sum_{k=1}^{\infty}\|u_k\|_{L^r}=\omega_N^{1/r}\tilde{c}_0^{-1}\sum_{k=1}^{\infty}k^{-2}<\infty.$$

We argue now by contradiction and assume that problem (1.1) has a solution, in the sense of (2.12), for every initial data in $L^r(\Omega)$. In particular, for u_0 there exists a function $u\in L^\infty((0,T),L^r(\Omega))$ verifying (2.12).

By the non-negativity of $S_\alpha(t)$, we have $u(t)\geq S_\alpha(t)u_0\geq S_\alpha(t)u_k$ for $t\in(0,T)$. Thus, from (2.12) we obtain

$$u(t)\geq\int_0^t\sigma^\gamma S_\alpha(t-\sigma)f(S_\alpha(\sigma)u_k)d\sigma, \tag{3.9}$$

since f is non-decreasing. Lemma 2.3 implies that

$$S_\alpha(\sigma)u_k=\tilde{c}_0^{-1}S_\alpha(\sigma)(z_k\chi_{r_k})\geq z_k\ \chi_{r_k+\sqrt{\sigma^\alpha}}\geq z_k\ \chi_{r_k},$$

for all $0<\sigma^\alpha\leq r_k^2/16$. Hence,

$$f(S_\alpha(\sigma)u_k)\geq f(z_k\chi_{r_k})\geq f(z_k)\chi_{r_k}, \tag{3.10}$$

for $0<\sigma^\alpha\leq r_k^2/16$. From (3.10) and Lemma 2.3 we obtain

$$S_\alpha(t-\sigma)f(S_\alpha(\sigma)u_k)\geq\tilde{c}_0f(z_k)\chi_{r_k} \tag{3.11}$$

for $0 < \sigma^\alpha \le t^\alpha \le r_k^2/16$. Now, define $t_k = \left(r_k^2/16\right)^{1/\alpha}$. For $t \in [t_k/2, t_k]$, we use (3.9) and (3.11) to obtain

$$\begin{aligned} u(t) &\ge \int_0^t \sigma^\gamma S_\alpha(t-\sigma) f(S_\alpha(\sigma)u_k)d\sigma \\ &\ge \tilde{c}_0 f(z_k)\chi_{r_k} \int_0^{t_k/2} \sigma^\gamma d\sigma \\ &= \frac{\tilde{c}_0 2^{-(\gamma+1)(1+4/\alpha)}}{\gamma+1} r_k^{2(\gamma+1)/\alpha} f(z_k)\chi_{r_k}. \end{aligned}$$

This and (3.7) imply

$$\begin{aligned} \|u(t)\|_{L^r} &\ge C r_k^{2(\gamma+1)/\alpha} f(z_k) r_k^{N/r} \\ &= C r_k^{Np^*/r} f(z_k) \\ &\ge C k^{-2p^*} e^{k/r} \to \infty, \end{aligned}$$

as $k \to \infty$. This contradicts the fact that $u \in L^\infty((0,T), L^r(\Omega))$.

For the case $\Omega = \mathbb{R}^N$, assume that $\limsup_{\tau\to 0^+} f(\tau)/\tau = \infty$. Then, there exists a sequence $\{\lambda_k\}_{k\ge 1}$, with $\lambda_k < k^{-2}$ such that $f(\lambda_k) \ge e^k \lambda_k$. Let $\rho_k = (\lambda_k k^2)^{-r/N} > 1$ and $v_0 = \sum_{k=1}^\infty v_k$, with $v_k = \tilde{c}_0^{-1}\lambda_k \chi_{\rho_k}$. Thus,

$$\|v_0\|_{L^r} \le \sum_{k=1}^\infty \|v_k\|_{L^r} = \tilde{c}_0^{-1}\omega_N^{1/r} \sum_{k=1}^\infty k^{-2} < \infty.$$

As in the anterior case, we argue by contradiction and assume that problem (1.1) has a solution, in the sense of (2.12), for every initial data in $L^r(\mathbb{R}^N)$. In particular, for v_0, there exists a function $v \in L^\infty((0,T), L^r(\mathbb{R}^N))$ verifying (2.12). Arguing as in the derivation of (3.9)-(3.11) it is possible to obtain

$$S_\alpha(t-\sigma) f(S_\alpha(\sigma)v_k) \ge \tilde{c}_0 f(\lambda_k)\chi_{\rho_k},$$

for $0 < \sigma^\alpha \le t^\alpha \le 1 < \rho_k^2$. So, for $t \in (0,1)$ we have

$$\begin{aligned} v(t) &\ge \int_0^t \sigma^\gamma S_\alpha(t-\sigma) f(S_\alpha(\sigma)v_k)d\sigma \\ &\ge \frac{\tilde{c}_0 t^{\gamma+1}}{(\gamma+1)} f(\lambda_k)\chi_{\rho_k}. \end{aligned}$$

Therefore,

$$\begin{aligned} \|v(t)\|_{L^r} &\ge C t^{\gamma+1} f(\lambda_k)\rho_k^{N/r} \\ &\ge C e^k t^{\gamma+1} k^{-2} \to \infty, \end{aligned}$$

as $k \to \infty$. This contradicts the fact that $v \in L^\infty((0,T), L^r(\mathbb{R}^N))$. □

4. **Initial data in** $L^1(\Omega)$

In this section, we prove the non-existence result in L^1. To do this, we first give sufficient conditions for the non-existence and recall the Lemma from [14] that identify the assumptions in Theorem 1.2 with such conditions. We also discuss the existence in L^1. The combination of the results gives optimal conditions of existence in L^1, only for $N = 1$.

4.1. **Non-existence.** Note that, in the statement of the following proposition, we do not require f to be continuous.

PROPOSITION 4.1. *Assume* $f : [0, \infty) \to [0, \infty)$ *is non-decreasing and that there exists a sequence* $\{s_k\}_{k\geq 1}$ *such that* $s_{k+1} \geq \theta s_k$ *for some constant* $\theta > 1$, *and*

$$\sum_{k=1}^{\infty} s_k^{-[1+2(\gamma+1)/(\alpha N)]} f(s_k) = \infty. \tag{4.1}$$

Then, there exists a non-negative initial condition $u_0 \in L^1(\Omega), u_0 \geq 0$ *such that the problem (1.1) has no local* L^1 *solution.*

P r o o f. Let $\rho_k = \tilde{c}_0^{-1} s_k$, where $\tilde{c}_0$ is given in Lemma 2.3. Set $u_n = \frac{1}{n^2}\theta_n^N \chi_{1/\theta_n}$, where $\theta_n = (n^2 \rho_{\xi_n})^{1/N}$, and ρ_{ξ_n} to be chosen later. Set $u_0 = \sum_{n=n_0}^{\infty} u_n$, with n_0 chosen such that $1/\theta_{n_0} < \delta_0 := 1/3 \inf_{x\in\partial\Omega} |x|$. Remember that we are assuming that $0 \in \Omega$.

Note that, $1/\theta_n \leq \delta_0$ and so $B_{1/\theta_n+2\delta_0} \subset \Omega$ for all $n \geq n_0$. Also,

$$\|u_0\|_{L^1} \leq \sum_{n=n_0}^{\infty} \|u_n\|_{L^1} \leq \sum_{n=1}^{\infty} \|u_n\|_{L^1} = \omega_N \sum_{n=1}^{\infty} n^{-2} < \infty.$$

First, we will see the action of $S_\alpha(\sigma)$ over every function u_n. Choose $\rho = 1/\theta_n$ and $\delta = \delta_0$ in Lemma 2.3, thus

$$S_\alpha(\sigma)u_n \geq \frac{\tilde{c}_0}{n^2} \cdot \frac{\theta_n^N}{\left(1 + \theta_n^{2/\alpha}\sigma\right)^{\alpha N/2}} \cdot \chi_{1/\theta_n+\sqrt{\sigma^\alpha}} \geq \tilde{c}_0\, \rho_k\, \chi_{1/\theta_n+\sqrt{\sigma^\alpha}}, \tag{4.2}$$

for $\sigma \leq t_k = \min\left\{\delta_0^2/16, \left(\frac{n^{-2}}{\rho_k}\right)^{2/\alpha N} - \frac{1}{\theta_n^{2/\alpha}}\right\}$, $\rho_k \leq n^{-2}\theta_n^N$, and $|x| \leq 1/\theta_n + \sqrt{\sigma^\alpha}$.

Now, suppose by contradiction that there exist $u \in L^\infty((0, T), L^1(\Omega))$ a local solution of problem (1.1) with initial condition u_0. We can assume that with $T < \delta_0^2/16$. Thus, by using the non-negativity, Lemma 2.3, estimate (4.2), and that f is non-decreasing, we have

$$\begin{aligned}
\int_\Omega u(t)dx &\geq \int_\Omega \int_0^t \sigma^\gamma S_\alpha(t-\sigma) f(S_\alpha(\sigma)u_0) d\sigma dx \\
&\geq \sum_k \int_\Omega \int_{t_{k+1}}^{t_k} \sigma^\gamma S_\alpha(t-\sigma) f(S_\alpha(\sigma)u_0) d\sigma dx \\
&= \sum_k \int_{t_{k+1}}^{t_k} \int_\Omega \sigma^\gamma S_\alpha(t-\sigma) f(S_\alpha(\sigma)u_0) dx d\sigma \\
&\geq C \sum_k f(\tilde{c}_0\rho_k) \int_{t_{k+1}}^{t_k} \int_\Omega \sigma^\gamma S_\alpha(t-\sigma)\chi_{1/\theta_n+\sqrt{\sigma^\alpha}}\ dx d\sigma \\
&\geq C \sum_k f(\tilde{c}_0\rho_k) \int_{t_{k+1}}^{t_k} \sigma^\gamma \left(\frac{1}{\theta_n}+\sqrt{\sigma^\alpha}\right)^N d\sigma \\
&\geq C \sum_k f(\tilde{c}_0\rho_k) \int_{t_{k+1}}^{t_k} \sigma^{\gamma+\alpha N/2} d\sigma, \qquad (4.3)
\end{aligned}$$

for $n \geq n_0$, where the sum in k is taken over those values for which $\frac{1}{\theta_n^N} \leq \frac{n^{-2}}{\rho_k} \leq \left(t + \frac{1}{\theta_n^{2/\alpha}}\right)^{\alpha N/2}$.

Consider the following additional constraint $\frac{\rho_{k+1}}{\theta_n^N n^{-2}} \leq \frac{1}{2}$. Since $s_{k+1} \geq \theta s_k$, for each k we have

$$\begin{aligned}
&\int_{t_{k+1}}^{t_k} \sigma^{\gamma+\frac{\alpha N}{2}} d\sigma \\
= \ &\frac{1}{1+\gamma+\alpha N/2}\left(t_k^{1+\gamma+\frac{\alpha N}{2}} - t_{k+1}^{1+\gamma+\frac{\alpha N}{2}}\right) \\
= \ &\frac{1}{1+\gamma+\alpha N/2}\left(\left[\left(\frac{n^{-2}}{\rho_k}\right)^{2/\alpha N} - \frac{1}{\theta_n^{2/\alpha}}\right]^{1+\gamma+\frac{\alpha N}{2}}\right. \\
&\left. - \left[\left(\frac{n^{-2}}{\rho_{k+1}}\right)^{2/\alpha N} - \frac{1}{\theta_n^{2/\alpha}}\right]^{1+\gamma+\frac{\alpha N}{2}}\right) \\
= \ &\frac{1}{1+\gamma+\alpha N/2}\left(\frac{n^{-2}}{\rho_k}\right)^{\frac{2}{\alpha N}(1+\gamma)+1} \times \left(\left[1 - \left(\frac{\rho_k}{n^{-2}\theta_n^N}\right)^{\frac{2}{\alpha N}}\right]^{1+\gamma+\frac{\alpha N}{2}}\right. \\
&\left. - \left(\frac{\rho_k}{\rho_{k+1}}\right)^{\frac{2}{\alpha N}(1+\gamma)+1}\left[1 - \left(\frac{\rho_{k+1}}{n^{-2}\theta_n^N}\right)^{\frac{2}{\alpha N}}\right]^{1+\gamma+\frac{\alpha N}{2}}\right)
\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{1}{1+\gamma+\alpha N/2}\left(\frac{n^{-2}}{\rho_k}\right)^{\frac{2}{\alpha N}(1+\gamma)+1}\left(\left[1-\left(\frac{\rho_{k+1}}{n^{-2}\theta_n^N}\right)^{\frac{2}{\alpha N}}\right]^{1+\gamma+\frac{\alpha N}{2}}\right.\\
&\quad\left.-\left(\frac{\rho_k}{\rho_{k+1}}\right)^{\frac{2}{\alpha N}(1+\gamma)+1}\left[1-\left(\frac{\rho_{k+1}}{n^{-2}\theta_n^N}\right)^{\frac{2}{\alpha N}}\right]^{1+\gamma+\frac{\alpha N}{2}}\right)\\
&\geq \frac{1}{1+\gamma+\alpha N/2}\left(\frac{n^{-2}}{\rho_k}\right)^{\frac{2}{\alpha N}(1+\gamma)+1}\left[1-\left(\frac{\rho_{k+1}}{n^{-2}\theta_n^N}\right)^{\frac{2}{\alpha N}}\right]^{1+\gamma+\frac{\alpha N}{2}}\\
&\quad\times\left[1-\left(\frac{\rho_k}{\rho_{k+1}}\right)^{\frac{2}{\alpha N}(1+\gamma)+1}\right]\\
&\geq C\left(\frac{n^{-2}}{\rho_k}\right)^{\frac{2}{\alpha N}(1+\gamma)+1}. \qquad (4.4)
\end{aligned}
$$

Thus, from (4.3) and (4.4) we have

$$
\begin{aligned}
\int_\Omega u(t)dx &\geq C\sum_k f(\tilde{c}_0\rho_k)\int_{t_{k+1}}^{t_k}\sigma^{\gamma+\frac{\alpha N}{2}}d\sigma\\
&\geq C\sum_k f(\tilde{c}_0\rho_k)\left(\frac{n^{-2}}{\rho_k}\right)^{\frac{2}{\alpha N}(1+\gamma)+1}\\
&= Cn^{-2[\frac{2}{\alpha N}(1+\gamma)+1]}\sum_k s_k^{-[\frac{2}{\alpha N}(1+\gamma)+1]}f(s_k). \qquad (4.5)
\end{aligned}
$$

Note that, by all the constraints above, the previous sum is taken over the set

$$
\left\{k:\frac{2}{\theta_n^N}\leq\frac{n^{-2}}{\rho_{k+1}}<\frac{n^{-2}}{\rho_k}\leq\left(t+\frac{1}{\theta_n^{2/\alpha}}\right)^{\alpha N/2}\right\},
$$

also, for any fixed $0<t<\delta_0^2$ and n sufficiently large that $tn^{4/\alpha N}\geq 1$, previous set contain the following set

$$
\left\{k:1\leq\rho_k \text{ and } \rho_{k+1}\leq\frac{1}{2}\rho_{\xi_n}\right\}=\{k:k_0\leq k\leq k_n\}
$$

where k_0 is the smallest value of k for which $\rho_k\geq 1$, and k_n is the maximum value. By choosing ξ_n such that $\rho_{k_n+1}\leq\frac{1}{2}\rho_{\xi_n}$ we can achieve any desired sequence k_n, and since that $\sum_{k=1}^\infty s_k^{-[\frac{2}{\alpha N}(1+\gamma)+1]}f(s_k)=\infty$, it is possible to obtain

$$n^{-2[\frac{2}{\alpha N}(1+\gamma)+1]}\sum_{k=k_0}^{k_n} s_k^{-[\frac{2}{\alpha N}(1+\gamma)+1]} f(s_k)\to\infty,$$

as $n\to\infty$. This and (4.5) imply a contradiction. □

Recall the following characterization by Laister et al. [14, Lemma 4.2].

LEMMA 4.1. *Suppose that $f:[0,\infty)\to[0,\infty)$ is non-decreasing and $q>1$. Then the following two conditions are equivalent*:

(i) *There exists a sequence $\{s_k\}_{k\in\mathbb{N}}$ such that $s_{k+1}\geq\theta s_k$, $\theta>1$ and*

$$\sum_{k=1}^{\infty} s_k^{-q} f(s_k)=\infty.$$

(ii) $\int_1^\infty s^{-q}F(s)ds=\infty$, *where* $F(s)=\sup_{1\leq\tau\leq s} f(\tau)/\tau$.

Now, we use Proposition 4.1 and Lemma 4.1 to conclude Theorem 1.1.

P r o o f o f T h e o r e m 1.2. In fact, assume that $\int_1^\infty s^{-p^*}F(s)ds=\infty$. Using Lemma 4.1 and Proposition 4.1 the result follows. The same argument can be used in the case $\int_0^\infty s^{-p^*}\tilde{F}(s)ds=\infty$, with small modifications in the proof of Proposition 4.1.

4.2. **Existence.** Here, we will discuss the existence of solutions for $u_0\in L^1(\Omega)$. We start by mentioning that $L^1\to L^\infty$ bounds for $S_\alpha(t)$, with $N\geq 2$, are not available in the literature, and it is a sensitive subject indeed. For instance, if $N\geq 2$, we cannot take $(q,r)=(1,\infty)$ in Lemma 2.2, nor in previous results such that those in [9, 13]. Thus, we study the existence in the case $N=1$.

THEOREM 4.1. *Assume that $N=1$, and*

$$\int_1^\infty \tau^{-\left(1+\frac{2(\gamma+1)}{\alpha}\right)}\sup_{1\leq t\leq\tau}\frac{f(t)}{t}d\tau<\infty. \tag{4.6}$$

Then, for every $u_0\in L^1(\Omega)$, there exists a local L^1-solution for (1.1).

P r o o f. The proof follows the idea from [14], except for the extra effort we have to circumvent the use of the semigroup property there. If $u_0\equiv 0$, we have that $v(t)=\chi_\Omega$ is a supersolution. If $u_0\not\equiv 0$, we assume that $\tau\mapsto f(\tau)/\tau$ is a non-decreasing function, for $\tau\geq 1$. Otherwise we define $\tilde{f}(\tau)=f(\tau)$ for $\tau\in[0,1]$ and $\tilde{f}(\tau)=\tau F(\tau)$ for $\tau\geq 1$, where F is defined in Lemma 4.1 (ii), and look for a supersolution of

$$u_t + \partial_t \int_0^t g_\alpha(t-s)\mathcal{A}u(s)ds = \tilde{f}(u).$$

Let $v(t) = bS_1(t^\alpha)u_0 + \chi_\Omega$, where $b > 0$ will be chosen later. Recall the assumption (4.6). Then,

$$\begin{aligned}
&\mathcal{F}(v)(t)\\
&= S_\alpha(t)u_0 + \int_0^t S_\alpha(t-\sigma)\sigma^\gamma f(v(\sigma))d\sigma\\
&= S_\alpha(t)u_0 + \int_0^t S_\alpha(t-\sigma)\sigma^\gamma \frac{f(bS_1(\sigma^\alpha)u_0+\chi_\Omega)}{bS_1(\sigma^\alpha)u_0+\chi_\Omega}\cdot(bS_1(\sigma^\alpha)u_0+\chi_\Omega)\,d\sigma\\
&\le S_\alpha(t)u_0 + \int_0^t \left\|\frac{f\,(bS_1(\sigma^\alpha)u_0+\chi_\Omega)}{bS_1(\sigma^\alpha)u_0+\chi_\Omega}\right\|_{L^\infty}\sigma^\gamma S_\alpha(t-\sigma)\,(bS_1(\sigma^\alpha)u_0+\chi_\Omega)\,d\sigma.
\end{aligned}$$

We notice that

$$\begin{aligned}
S_\alpha(t-\sigma)S_1(\sigma^\alpha)u_0 &= \int_0^\infty M_\alpha(\eta)S_1\left(\eta(t-\sigma)^2+\sigma^\alpha\right)u_0 d\eta\\
&\le \int_0^\infty M_\alpha(\eta)\left[\eta(t-\sigma)^\alpha+\sigma^\alpha\right]^{-\frac{N}{2}}\|u_0\|_{L^1}d\eta\\
&\le \sigma^{-\frac{\alpha}{2}}\|u_0\|_1. \qquad (4.7)
\end{aligned}$$

Then,

$$\mathcal{F}(v)(t) \le S_\alpha(t)u_0 + \int_0^t \left\|\frac{f\,(bS_1(\sigma^\alpha)u_0+\chi_\Omega)}{bS_1(\sigma^\alpha)u_0+\chi_\Omega}\right\|_{L^\infty}\sigma^\gamma\left[b\sigma^{-\frac{\alpha}{2}}\|u_0\|_{L^1}+\chi_\Omega\right]d\sigma.$$

From $bS_1(\sigma^\alpha)u_0+\chi_\Omega \le b\sigma^{-\frac{\alpha}{2}}\|u_0\|_{L^1}+1$, we have

$$\left\|\frac{f\,(bS_1(\sigma^\alpha)u_0+\chi_\Omega)}{bS_1(\sigma^\alpha)u_0+\chi_\Omega}\right\|_{L^\infty} \le \frac{f(b\sigma^{-\frac{\alpha}{2}}\|u_0\|_{L^1}+1)}{b\sigma^{-\frac{\alpha}{2}}\|u_0\|_{L^1}+1}, \qquad (4.8)$$

since $\tau \mapsto \frac{f(\tau)}{\tau}$ is non-decreasing. Hence

$$\begin{aligned}
&\mathcal{F}(v)(t) \qquad (4.9)\\
&\le S_\alpha(t)u_0 + \int_0^t \frac{f(b\sigma^{-\frac{\alpha}{2}}\|u_0\|_{L^1}+1)}{b\sigma^{-\frac{\alpha}{2}}\|u_0\|_{L^1}+1}\sigma^\gamma\left[b\sigma^{-\frac{\alpha}{2}}\|u_0\|_{L^1}+1\right]\\
&= S_\alpha(t)u_0 + (2b\|u_0\|_{L^1})^{\frac{2(\gamma+1)}{\alpha}}\cdot\frac{2}{\alpha}\int_{2b\|u_0\|_{L^1}t^{-\frac{\alpha}{2}}}^\infty \tau^{-\left(1+\frac{2(\gamma+1)}{\alpha}\right)}f(\tau)d\tau, \quad (4.10)
\end{aligned}$$

where we took $t > 0$ so small that $b\sigma^{-\frac{\alpha}{2}}\|u_0\|_{L^1} \ge 1$, for all $\sigma \in (0,t)$, and that the second portion in (4.10) is less than 1. From Lemma 2.2, we have

$$S_\alpha(t)u_0 \le t^{-\frac{\alpha}{2}}\|u_0\|_{L^1}. \qquad (4.11)$$

Moreover, from (4.11) and (1.5), we have

$$\begin{aligned} S_1(t^\alpha) &\geq C_2 t^{-\frac{\alpha}{2}} \int_{\Omega'} e^{-\frac{\lambda_2|x-y|^2}{t^\alpha}} u_0(y)dy \\ &\geq \frac{C_2}{\|u_0\|_{L^1}} \int_{\Omega'} e^{-\frac{\lambda_2|x-y|^2}{t^\alpha}} u_0(y)\, dy\ S_\alpha(t)u_0, \end{aligned} \tag{4.12}$$

for $0 < t < d^2(\Omega', \partial\Omega)/8$, where $x \in \Omega'$. For $T > 0$ sufficiently small, we can take $b > 0$ such that

$$\frac{bC_2}{\|u_0\|_{L^1}} \int_{\Omega'} e^{-\frac{\lambda_2|x-y|^2}{t^\alpha}} u_0(y)dy \geq 1,$$

and $S_\alpha(t) \leq bS_1(t^\alpha)$, for all $t \in (0, T)$. Accordingly,

$$\mathcal{F}(v)(t) \leq v(t), \quad t \in (0, T).$$

We proved that v is a supersolution of (1.1). Then, Lemma 2.4 concludes the proof. □

5. **Examples**

In this section, we gather some results on Gaussian bounds for the heat kernels associated with a general elliptic operator, and then we apply Theorem 1.1 yielding new results of existence and non-existence of solutions in Lebesgue spaces.

5.1. **Fractional diffusion with general elliptic operators in $\mathbb{R}^N$.** As commented in the introduction, we consider the operator $\mathcal{A}$ given by

$$\mathcal{A}u = -\sum_{i,j=1}^{N} a_{ij}(x)u_{x_i x_j} - \sum_{j=1}^{N} b_j(x)u_{x_j} - c(x)u.$$

Assume that:

(i) for all $\xi \in \mathbb{R}^N$ and for almost all $x \in \mathbb{R}^N$, $a_{ij}(x,t)\xi_j\xi_j \geq v|\xi|^2$;
(ii) the coefficients of $\mathcal{A}$ are bounded measurable functions.

From [3], we have that the fundamental solution K of

$$u_t + \mathcal{A}u = 0$$

satisfies (1.4)-(1.5), and from [2], $\mathcal{A}$ generates a semigroup written in terms of the fundamental solution K through (1.2). Therefore, Theorem 1.1 can be applied to obtain the following result.

Theorem 5.1. *Let $\mathcal{A}$ fulfill the above assumptions (i)–(ii), $f : [0, \infty) \to [0, \infty)$ be a continuous non-decreasing function, $\gamma > -1$ and p^* be as in (1.12). Then, for $p^\star > (1 + \gamma)/\alpha$, $r > 1$, we have*

$$\limsup_{\tau\to\infty} \tau^{-p^\star} f(\tau) < \infty \quad \textit{and} \quad \limsup_{\tau\to 0} f(\tau)/\tau < \infty$$

if, and only if,

$$\begin{cases} u_t + \partial_t \int_0^t g_\alpha(s)\mathcal{A}u(t-s)ds &= t^\gamma f(u), \quad \textit{in } \mathbb{R}^N \times (0,T), \\ u(0) &= u_0 \geq 0, \quad \textit{in } \mathbb{R}^N, \end{cases} \tag{5.1}$$

has a local solution for every $u_0 \in L^r(\mathbb{R}^N)$. For the non-existence part, the condition $p^\star > (1+\gamma)/\alpha$ is dropped and $r = 1$ is included. For $r = 1$, existence holds with $N = 1$.

5.2. **Fractional diffusion with general elliptic operators with Robin boundary conditions.** Now, we consider the operator $\mathcal{A}$ given by

$$\mathcal{A}u = -\sum_{i,j=1}^N a_{ij}(x)u_{x_i x_j} - \sum_{j=1}^N b_j(x)u_{x_j} - c(x)u$$

with Robin boundary conditions:

$$\mathcal{B}u(x,t) = \beta(x)\frac{\partial u}{\partial \nu}(x,t) + [1-\beta(x)]u(x,t),$$

where $0 \leq \beta(x) \leq 1$ and $\partial u/\partial \nu$ is given by

$$\frac{\partial u}{\partial \nu}(x) = -\sum_{i,j=1}^N u_{x_i} a_{i,j}(x) n_j(x)$$

with $n(x) = (n_1(x), ..., n_N(x))$ being the unit outer normal at $x \in \Omega$. Here, $= a_{ij}$ is symmetric and satisfies the uniform ellipticity condition

$$k|y|^2 \leq \sum_{i,j=1}^N a_{ij}(x)y_i y_j \leq |y|^2/k, \quad \forall x \in \Omega, \forall y \in \mathbb{R}^N,$$

for some $k > 0$. Moreover, the coefficients have the following regularity: $a_{ij} \in C^{2+\alpha}(\bar{\Omega}), b_j \in C^{1+\alpha}(\bar{\Omega}), c \in C^\alpha(\bar{\Omega})$, an.d $\beta \in C^{2+\alpha}(\partial\Omega)$.

From [2], the realization A of $(\mathcal{A}, \mathcal{B})$ generates a semigroup $(S(t))_{t\geq 0}$ in $L^2(\Omega)$ with kernel K satisfying (1.4). For the lower estimates, we recall that Laister *et al* [15] noted that a combination of [12, Lemma 2.4] and [4, Theorems 8,9] implies (1.6).

Alternatively, we could consider the following assumptions:

(i) the matrix $(a_{ij}(x,t))$ is symmetric for any $(x,t) \in \Omega$;
(ii) $a_{ij} \in W^{1,\infty}(\Omega), b_k, c \in C^1(\bar{\Omega})$;
(iii) $a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, (x,t) \in \bar{\Omega}, \xi \in \mathbb{R}^n$;
(iv) $\|a_{ij}\|_{W^{1,\infty}(\Omega)} + \|b_k\|_{L^\infty(\Omega)} + \|c\|_{L^\infty(\Omega)} \leq A$;
(v) $\beta \in C(\partial\Omega)$;

where $\lambda > 0$ and $A > 0$ are two given constants. In particular $a_{ij} \in L^\infty(\Omega)$, $\beta, c \in L^\infty(\Omega)$, and $b_i \in W^{1,\infty}(\Omega)$. We still have that the realization A of $(\mathcal{A}, \mathcal{B})$ generates a semigroup $(S(t))_{t\geq 0}$ in $L^2(\Omega)$ with kernel K satisfying

(1.4). If further $\beta \in C(\partial\Omega)$, $c \in C^1(\bar{\Omega})$, and Ω is bounded, smooth and convex (or, more generally, satisfies the chain condition in [8]), then (1.5) holds, by [8]. These latter assumptions on $\mathcal{A}$ are weaker than the earlier, but have a stronger assumption on the domain.

In both cases, Theorem 1.1 applies, and we have the following result.

Theorem 5.2. *Let $(\mathcal{A}, \mathcal{B})$ and Ω be as above, $f : [0, \infty) \to [0, \infty)$ be a continuous non-decreasing function, $\gamma > -1$ and p^* be given by (1.12). For $r > 1$ and $p^\star > (1+\gamma)/\alpha$,*

$$\limsup_{\tau\to\infty} \tau^{-p^\star} f(\tau) < \infty$$

if, and only if,

$$\begin{cases} u_t + \partial_t \int_0^t g_\alpha(s)\mathcal{A}u(t-s)ds & = t^\gamma f(u), \quad \text{in } \Omega \times (0,T), \\ \beta(x)\frac{\partial u}{\partial \nu}(x,t) + [1-\beta(x)]u(x,t) & = 0, \quad \text{on } \partial\Omega \times (0,T), \\ u(0) & = u_0 \geq 0, \quad \text{in } \Omega, \end{cases} \tag{5.2}$$

has a local solution for every $u_0 \in L^r(\Omega)$. For the non-existence part, the condition $p^\star > (1+\gamma)/\alpha$ is dropped and $r = 1$ is included. For $r = 1$, existence holds with $N = 1$.

5.3. **Nonlinearities.** Besides $f(t,\tau) = t^\gamma \tau^p$, the same $f(t,\tau) = \frac{t^\gamma \tau^{p^*}}{[\log(\mathrm{e}+\tau)]^\beta}$, similar to that in Laister *et al* in [14, Sec. 4.4], can be considered as an interesting example here.

On the other hand, consider $f(t,\tau) = t^\gamma e^{k\tau}$. It is seen that, for any $r \geq 1$ and $k > 0$, there exists an initial condition $u_0 \in L^r$ that does not admit the existence of a local L^r-mild solution of (1.1), no matter what value γ takes. Nevertheless, solutions of (1.1) with exponential nonlinearities can be considered in Orlicz spaces or uniformly local Lebesgue spaces, see e.g. [11]. In contrast, if $k < 0$, (1.1) always admits a local L^r-mild solution, for a initial datum $u_0 \in L^r$.

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