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RESEARCH PAPER

CENSORED STABLE SUBORDINATORS AND FRACTIONAL DERIVATIVES

Qiang Du [1], **Lorenzo Toniazzi** [2], **Zirui Xu** [3]

Abstract

Based on the popular Caputo fractional derivative of order β in $(0,1)$, we define the censored fractional derivative on the positive half-line $\mathbb{R}_+$. This derivative proves to be the Feller generator of the censored (or resurrected) decreasing β-stable process in $\mathbb{R}_+$. We provide a series representation for the inverse of this censored fractional derivative. We are then able to prove that this censored process hits the boundary in a finite time τ_∞, whose expectation is proportional to that of the first passage time of the β-stable subordinator. We also show that the censored relaxation equation is solved by the Laplace transform of τ_∞. This relaxation solution proves to be a completely monotone series, with algebraic decay one order faster than its Caputo counterpart, leading, surprisingly, to a new regime of fractional relaxation models. Lastly, we discuss how this work identifies a new sub-diffusion model.

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Key Words and Phrases: fractional initial value problem; censored stable subordinator; fractional relaxation equation; Mittag-Leffler function

1. Introduction

Fractional derivatives, a special class of nonlocal integral and pseudo-differential operators [15, 22, 47, 28], have been successfully employed to model heterogeneities and nonlocal interactions in many applications (see,

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e.g., [39, 42, 44, 11]). They also enjoy an interesting mathematical theory with deep connections to Lévy processes (see, e.g., [41, 7, 33, 34, 35]). For example, the Caputo derivative [14] of order $\beta \in (0,1)$ on the positive half-line $\mathbb{R}^+$, plays important roles in modelling non-exponential relaxation [14, 42] and non-Markovian sub-diffusive dynamics [40, 1, 23]. For a smooth function u vanishing outside $\mathbb{R}_+$, the Caputo derivative equals the Riemann–Liouville (R–L) derivative D_0^β [14] given by

$$D_0^\beta u(x) = \int_0^x \big(u(x)-u(x-r)\big)\frac{r^{-1-\beta}}{\big|\Gamma(-\beta)\big|}\,\mathrm{d}r + u(x)\frac{x^{-\beta}}{\Gamma(1-\beta)}, \quad x>0. \tag{1.1}$$

Probabilistically, $-D_0^\beta$ generates a killed Lévy process, which is the decreasing β-stable process $S^1 = \{S^1_s\}_{s\geq 0}$ killed at time τ_1, the first exit time from $\mathbb{R}_+$ [3, 29]. Intuitively, the first summand in (1.1) describes the decreasing β-stable jumps landing inside $\mathbb{R}_+$, while $x^{-\beta}/\Gamma(1-\beta) = \int_x^\infty r^{-1-\beta}/\big|\Gamma(-\beta)\big|\,\mathrm{d}r$ is the killing coefficient for the jumps landing outside $\mathbb{R}_+$. In this work, we introduce what we call the *censored fractional derivative* ∂_0^β, allowing the representation

$$\partial_0^\beta u(x) = \int_0^x \big(u(x)-u(x-r)\big)\frac{r^{-1-\beta}}{\big|\Gamma(-\beta)\big|}\,\mathrm{d}r, \quad x>0. \tag{1.2}$$

It is intuitively clear that $-\partial_0^\beta$ only allows the decreasing β-stable jumps to land inside $\mathbb{R}_+$, and suppresses those landing outside $\mathbb{R}_+$. Indeed we prove that it is the (Feller) generator of $S^c = \{S^c_s\}_{s\geq 0}$, the *censored* decreasing β-stable process in $\mathbb{R}_+$. We will construct S^c by repeatedly resurrecting in situ the killed decreasing β-stable process, following the canonical Ikeda–Nagasawa–Watanabe (INW) piecing together procedure [26]. (Cf. [37, Remark 3.3] for two other notions of "censoring" a process.)

We initiate the study of the censored fractional derivative, and then apply its theory to derive several new and non-trivial results about the censored stable subordinator, as we now explain. We first prove the well-posedness of the basic initial value problem (IVP)

$$\begin{cases} \partial_0^\beta u(x) = g(x), & x\in(0,T],\\ \quad u(x) = u_0, & x=0, \end{cases} \tag{1.3}$$

for any $T>0$, $u_0\in\mathbb{R}$ and certain $g\in C(0,T]$. Our proof is based on constructing the candidate solution $u = u_0 + I_0^\beta g$, where I_0^β allows a probabilistic series representation and the expected potential representation:

$$I_0^\beta g(x) = J_0^\beta g(x) + \sum_{j=1}^\infty \mathbb{E}_x\Big[J_0^\beta g(X_j)\Big] \tag{1.4}$$

$$= \mathbb{E}_x\left[\int_0^{\tau_\infty} g(S^c_s)\,\mathrm{d}s\right]. \tag{1.5}$$

Here, J^β_0 is the R–L integral, i.e. the inverse of D^β_0, given by

$$J^\beta_0 g(x) = \int_0^x g(y)\frac{(x-y)^{\beta-1}}{\Gamma(\beta)}\,\mathrm{d}y = \mathbb{E}_x\left[\int_0^{\tau_1} g(S^1_s)\,\mathrm{d}s\right], \tag{1.6}$$

where the second identity is the known potential representation for J^β_0; the discrete-time process $X \,|\, X_0 = x$ is defined as $X_j := x\prod_{i=1}^j B_i$, where $\{B_i\}_{i\in\mathbb{N}}$ is an *i.i.d.* collection of beta-distributed random variables with parameters $1-\beta$ and β; and τ_∞ is the lifetime of S^c. The equivalence of (1.4) and (1.5) is due to the equality in law between X_j and S^c at its j-th resurrection time, combined with the second identity in (1.6) (see Remark 4.3 for more details). The way we solve (1.3) is to regard it as a linear R–L IVP $D^\beta_0 u = ku + g$, $u(0) = 0$ with the coefficient $k(x) = x^{-\beta}/\Gamma(1-\beta)$. It turns out that the formula given in [14, Theorem 7.10] for bounded k still converges for this specific unbounded k, allowing us to construct the solution. (As for more general k that may diverge as $O(x^{-\beta})$, [38, Example 3.4] gave a non-constructive proof of the existence result.) As we show in [16, Section 3.2], this explicit solution allows us to establish the (global) well-posedness of general IVPs $\partial^\beta_0 u = f(x,u)$, $u(0) = u_0$, for certain Lipschitz data f.

Using the results above, we are able to solve the linear IVP $\partial^\beta_0 u = \lambda u$, $u(0) = u_0$, for any $\lambda \in \mathbb{R}$. We obtain the Mittag-Leffler-type representation for its solution

$$u(x) = u_0\sum_{N=0}^\infty \lambda^N x^{\beta N}\prod_{n=1}^N\left(\frac{\Gamma(1+n\beta)}{\Gamma(n\beta+1-\beta)} - \frac{1}{\Gamma(1-\beta)}\right)^{-1}, \tag{1.7}$$

where an empty product equals 1 by convention (also, $u(x) = u_0$ if $\lambda = 0$) and each factor of the indexed product is positive by (2.1). Surprisingly, for $\lambda < 0$, this solution decays at the fast algebraic rate $x^{-1-\beta}$ (Theorem 3.2), which we believe is a new regime for fractional relaxation models. Indeed the Caputo fractional relaxation solution $u_0E_\beta(\lambda x^\beta)$ decays at the rate $x^{-\beta}$ [14, Theorem 7.3], where $E_\beta(x) = \sum_{n=0}^\infty x^n/\Gamma(n\beta+1)$ is the Mittag-Leffler function. Moreover, the lagging and leading coupled fractional relaxation equations in [2, 50] model the decay rate $x^{-\gamma}$ for some $\gamma \in (0,1)$. Our proof (inspired by [18, Theorem 3.2]) is based on maximum principle and turns out to be versatile, albeit elementary. Indeed the same argument proves the decay rate $x^{-1-\alpha}$ of the solution to $\partial^\beta_0 u = \lambda x^{\alpha-\beta}u$ ($\lambda < 0$, $\alpha > 0$) (see Proposition 3.2), which is again one order faster than its Caputo counterpart (expressed by the Kilbas–Saigo function [43]). Moreover, we will show how to adapt this argument to the Caputo setting to give new and simple proofs of the two-sided uniform bounds of E_β and more generally, a class of Kilbas–Saigo functions, which are the recent results in [46, Theorem 4] and [10, Proposition 4.12], respectively. This very argument may have even broader applications, e.g., in general Caputo-type relaxation problems (corresponding to general killed subordinators), see Remark 3.4-(iii).

As a special case of (1.5), we have the identity

$$\mathbb{E}_x[\tau_\infty] = \mathbb{E}_x[\tau_1]\frac{\beta\pi}{\beta\pi - \sin(\beta\pi)}, \quad \text{where } \mathbb{E}_x[\tau_1] = \frac{x^\beta}{\Gamma(\beta+1)}, \tag{1.8}$$

which implies that S^c hits 0 in finite time, a fact that we believe has not been shown before. This is fundamental and not obvious, especially in view of [6, Theorem 1.1-(1)], which proves that the censored *symmetric* β-stable Lévy process never hits the boundary, whether censored in an interval or $\mathbb{R}_+$. (Also, censored decreasing compound Poisson processes do not hit the barrier in finite time, and our numerical simulations suggest neither do censored gamma subordinators.) We are then able to show several more connections between the analytic and probabilistic aspects of ∂_0^β. That is, we will prove that S^c is indeed a Feller process generated by $-\partial_0^\beta$, and that the exit problem for τ_∞ is solved by (1.7), i.e.

$$u_0\mathbb{E}_x\big[\exp\{\lambda\tau_\infty\}\big] \text{ equals the series (1.7), for all } x>0 \text{ and } \lambda\in\mathbb{R}. \tag{1.9}$$

As a consequence of (1.9), we can obtain all the moments of τ_∞ and confirm the complete monotonicity of (1.7). We emphasise that (1.9) is significantly harder to prove than Caputo's counterpart $\mathbb{E}_x\big[\exp\{\lambda\tau_1\}\big] = E_\beta(\lambda x^\beta)$. This is mainly due to the inapplicability of Laplace transforms to S^c and the complexity of the coefficients in (1.7) (see [16, Remark 4.14] for more detail). Nonetheless, we obtain a proof by combining our series solution to the resolvent equation $\partial_0^\beta u = \lambda u + g$ with a simple semigroup theory argument, following [24, Corollary 5.1]. We could alternatively try combining our IVP theory with standard potential theory (see, e.g., [13, Chapter 3]) appplied to the Feynman–Kac semigroup of $-D_0^\beta + (\lambda + k)$, but it would be more involved. (We also remark that (1.9) serves as an efficient alternative to numerically compute (1.7) for $\lambda < 0$.)

Lastly, we discuss how this work sets the foundations for the study of a new time-fractional diffusion equation $\partial_0^\beta u = \Delta u/2$, solved by the process $\{B_{\tau_\infty(t)}\}_{t\ge 0}$. (Here ∂_0^β acts on the time variable, B is a Brownian motion independent of $\tau_\infty(t) := \tau_\infty \,|\, S_0^c = t$.) This is the censored analogue of the Caputo time-fractional diffusion equation $D_0^\beta\big[u - u(0)\big] = \Delta u/2$, which is solved by the fractional kinetic process $\{B_{\tau_1(t)}\}_{t\ge0}$ (with B independent of the inverse stable subordinator $\tau_1(t) := \inf\{s : t < -S_s^1\}$), a non-Markovian sub-diffusion process arising from several central limit theorems [40, 1, 23]. As we discuss in Remark 4.5-(i), although both B_{τ_1} and B_{τ_∞} are sub-diffusion processes (due to (1.8)), their respective characteristic functions, $E_\beta(\lambda t^\beta)$ and (1.7) (for some $\lambda < 0$), display strikingly different decay rates (due to our results on relaxation solutions).

This work is organized as follows: Section 2 introduces notation, recalls basic results on fractional calculus, defines the censored fractional derivative, and studies the solution kernels; Section 3 focuses on the well-posedness and series representation of the solution to (1.3), then addresses linear censored IVPs; in Section 4 we construct the censored decreasing β-stable process and apply our IVP theory to its study.

2. **Preliminary notation and definitions**

Throughout this article, we denote by β $(0 < \beta < 1)$ the order of fractional derivatives, and by $[0, T]$ $(0 < T < \infty)$ the interval of interest. We denote by $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}_+$ the sets of positive integers, real numbers and positive numbers, respectively. For any interval $\Omega \subseteq \mathbb{R}$ we denote by $C(\Omega), C^1(\Omega)$ and $L^1(\Omega)$ the real functions on Ω that are continuous, continuously differentiable and Lebesgue integrable, respectively. We abbreviate $C(\Omega) \cap L^1(\Omega)$ to $C \cap L^1(\Omega)$. For compact Ω we denote by $\|\cdot\|_{C(\Omega)}$ the sup norm. We denote by Γ the gamma function and frequently use without mention the standard identities $\Gamma(2-\alpha) = (1-\alpha)\Gamma(1-\alpha)$ for all $\alpha \in \mathbb{R}\backslash\mathbb{N}$, $\Gamma(\beta+1)\Gamma(1-\beta) = \beta\pi/\sin(\beta\pi)$ and

$$\int_0^x (x-r)^{\gamma-1} r^{\alpha-1}\,\mathrm{d}r = x^{\gamma+\alpha-1}\frac{\Gamma(\gamma)\Gamma(\alpha)}{\Gamma(\gamma+\alpha)} \text{ for all } \alpha,\, \gamma,\, x > 0.$$

We also rely crucially on the inequality (which we prove in Lemma 2.3)

$$\Gamma(\alpha+1-\beta) < \Gamma(1+\alpha)\Gamma(1-\beta) \text{ for all } \alpha > 0. \tag{2.1}$$

2.1. **R–L calculus and fractional function spaces.** We present some basic results about the R–L fractional derivative. We refer to [14] for a general study of Caputo/R–L derivatives.

Definition 2.1. For $\beta \in (0,1)$, $u \in C \cap L^1(0,T]$, define *R–L integral*

$$J_0^\beta u(x) = \int_0^x \frac{(x-r)^{\beta-1}}{\Gamma(\beta)} u(r)\,\mathrm{d}r, \quad x \in (0,T].$$

We define the function spaces

$$C_\beta(0,T] = \Big\{u \in C \cap L^1(0,T] :\, J_0^{1-\beta}u \in C^1(0,T]\Big\},$$
$$C_\beta[0,T] = C[0,T] \cap C_\beta(0,T],$$

and for $u \in C_\beta(0,T]$, $x \in (0,T]$, we define the *R–L derivative*

$$D_0^\beta u(x) = \frac{\mathrm{d}}{\mathrm{d}x} J_0^{1-\beta} u(x) = \frac{\mathrm{d}}{\mathrm{d}x} \int_0^x \frac{(x-r)^{-\beta}}{\Gamma(1-\beta)} u(r)\,\mathrm{d}r.$$

REMARK 2.1. Note that $C_\beta(0,T]$ is chosen so that the image of D_0^β is contained in $C(0,T]$. Moreover, $C_\beta[0,T]$ is chosen to be the solution space, as we will explain in Remark 2.4.

LEMMA 2.1. *The following relations between D_0^β and J_0^β hold.*

(i) If $u \in C \cap L^1(0,T]$, then $J_0^\beta u \in C \cap L^1(0,T]$.
(ii) If $u \in C \cap L^1(0,T]$, then $J_0^\beta u \in C_\beta(0,T]$ and $D_0^\beta J_0^\beta u = u$.
(iii) Assume $g \in C \cap L^1(0,T]$. Then

$$u = J_0^\beta g \quad \text{if and only if} \quad \begin{cases} u \in C_\beta(0,T], \\ D_0^\beta u = g, \\ \lim_{x\to 0} J_0^{1-\beta} u(x) = 0. \end{cases}$$

(iv) If $u \in C_\beta[0,T]$ satisfies $D_0^\beta u = 0$, then $u = 0$.

The proof is straightforward and given in [16, Appendix A.1].

REMARK 2.2. Note that in Lemma 2.1-(iv), the condition $u \in C_\beta[0,T]$ cannot be weakened to $u \in C_\beta(0,T]$, since $D_0^\beta x^{\beta-1}$ is also 0.

2.2. **Censored fractional derivative.**

DEFINITION 2.2. Given $\beta \in (0,1)$, we define the *censored fractional derivative* of any $u \in C_\beta(0,T]$ as

$$\partial_0^\beta u(x) = D_0^\beta u(x) - \frac{x^{-\beta}}{\Gamma(1-\beta)} u(x), \quad \text{for all } \ x \in (0,T].$$

REMARK 2.3.

(i) Like the Caputo derivative, the censored fractional derivative maps constants to 0, and satisfies the scaling property

$$\partial_0^\beta v(x) = c^{-\beta} \partial_0^\beta u(x/c),$$

where $u \in C_\beta(0,T]$, c is a positive constant and $v(x) := u(x/c) \in C_\beta(0,\, cT]$.

(ii) For functions of the form x^α ($\alpha > 0$), the censored fractional derivative equals the R–L derivative up to a constant multiple: $\partial_0^\beta x^\alpha = c_{\alpha,\beta} D_0^\beta x^\alpha$, where

$$c_{\alpha,\beta} = 1 - \frac{\Gamma(\alpha+1-\beta)}{\Gamma(\alpha+1)\Gamma(1-\beta)}, \quad D_0^\beta x^\alpha = x^{\alpha-\beta}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-\beta)}.$$

By (2.1), $c_{\alpha,\beta}$ is in $(0,1)$. In particular, for $\alpha=\beta$, we have $\partial_0^\beta x^\alpha = \Gamma(\beta+1)\big(\beta\pi - \sin(\beta\pi)\big)/(\beta\pi)$. While we can talk about the semigroup property for D_0^β and the Caputo derivative [14, Theorem 2.13 and Lemma 3.13], we cannot for ∂_0^β. For instance,

$$\partial_0^\beta \partial_0^\gamma x^\alpha = c_{\alpha-\gamma,\beta}\, c_{\alpha,\gamma}\, D_0^{\beta+\gamma} x^\alpha,$$
$$\partial_0^\gamma \partial_0^\beta x^\alpha = c_{\alpha-\beta,\gamma}\, c_{\alpha,\beta}\, D_0^{\beta+\gamma} x^\alpha,$$

however $c_{\alpha-\gamma,\beta}\, c_{\alpha,\gamma} \neq c_{\alpha-\beta,\gamma}\, c_{\alpha,\beta}$ unless $\beta=\gamma$.

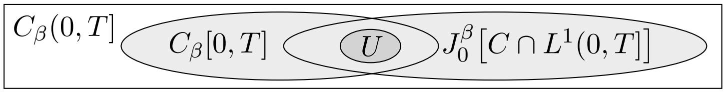
(iii) If $u \in C^1(0,T] \cap L^1(0,T]$, then on $(0,T]$, $\partial_0^\beta u$ allows the representation (1.2), from which it is clear that $-\partial_0^\beta$ satisfies the positive maximum principle [9], and hence it is dissipative in the sense that $\|\lambda u + \partial_0^\beta u\|_{C[0,T]} \geq \lambda \|u\|_{C[0,T]}$ for any $\lambda > 0$ and $u \in C^1[0,T]$.

(iv) The Laplace transform of the censored fractional derivative is

$$\mathcal{L}\big[\partial_0^\beta u\big](k) = k^\beta\left(\mathcal{L}[u](k) - k^{-1}\mathcal{L}\left[\frac{u(x/k)x^{-\beta}}{\Gamma(1-\beta)}\right](1)\right), \quad k>0,$$

which differs from $k^\beta\big(\mathcal{L}[u](k) - k^{-1}u(0)\big)$, the Laplace transform of the Caputo derivative [41, Chapter 2.4]. One can notice that even in Laplace space, it is unclear if the initial conditions can be imposed on the problem $\partial_0^\beta u = g$.

Remark 2.4. We will spend the next few pages establishing the well-posedness of $\partial_0^\beta u = g$ with $u(0)=u_0$, for certain g. With the initial condition imposed, $C_\beta[0,T]$, which equals $\{u \in C[0,T] : J_0^{1-\beta} u \in C^1(0,T]\}$, now becomes a natural function space for solutions. A large part of the Caputo literature (e.g., [14]), however, chose $J_0^\beta\big[C \cap L^1(0,T]\big]$, i.e., the image of J_0^β over $C \cap L^1(0,T]$, as the solution space. This difference seems not to matter, at least to our studies. Indeed, the set U consisting of the solutions to (1.3) (for those g of interest) is contained in the intersection of those two spaces, as shown in the diagram below (see [16, Appendix A.3] for the proof of the diagram)



where we define $U = \left\{u \in C_\beta[0,T] : x^{\beta-\alpha}\partial_0^\beta u \in C[0,T] \text{ for some } \alpha > 0\right\}$. Lastly, let us mention that $J_0^\beta C[0,T] = \left\{u \in C[0,T] : J_0^{1-\beta}u \in C^1[0,T]\right\}$ [48, Proposition 4.1].

2.3. **An integral operator and related kernels.** As we can see from (1.4), the solution to the IVP (1.3) may be seen as a variation of the R–L integral. In this subsection we introduce an integral operator and related kernels for the convergence study of (1.4). This leads to Lemma 2.3, which is a crucial bound in this work. The probabilistic interpretation of the kernels under consideration will be presented in Section 4.

Definition 2.3. For $0 < r < x$, recursively define the following kernels

$$k_j(x,r) = \begin{cases} \dfrac{(x-r)^{\beta-1}r^{-\beta}}{\Gamma(\beta)\Gamma(1-\beta)}, & j = 1, \\ \displaystyle\int_r^x k_1(x,s)k_{j-1}(s,r)\,\mathrm{d}s, & j \geq 2. \end{cases} \tag{2.2}$$

Remark 2.5. Note that for each $x > 0$, $k_1(x,\cdot)$ is a beta distribution on $(0,x)$ with parameters $(1-\beta,\beta)$, and straightforward induction arguments can be used to prove that

$$\int_0^x k_j(x,r)\,\mathrm{d}r = 1 \quad (j \geq 1,\ x > 0)$$

and

$$k_j(x,r) = \int_r^x k_{j-1}(x,s)k_1(s,r)\,\mathrm{d}s \quad (j \geq 2,\ x > r > 0).$$

Definition 2.4. For $\psi \in C[0,T]$, we define

$$\mathcal{K}\psi(x) = \begin{cases} \displaystyle\int_0^x k_1(x,r)\psi(r)\,\mathrm{d}r, & x > 0, \\ \psi(0), & x = 0, \end{cases}$$

where the explicit dependence of $\mathcal{K}$ on β is suppressed to ease notation.

REMARK 2.6. It is easy to see that $\mathcal{K}\psi(x) = J_0^\beta\big[x^{-\beta}\psi(x)/\Gamma(1-\beta)\big]$ for $\psi \in C[0,T]$ and $x \in (0,T]$, and that $\mathcal{K}$ is a linear operator preserving positivity ($\mathcal{K}\psi \geq 0$ if $\psi \geq 0$).

LEMMA 2.2. *For any* $\alpha \geq 0$, *we have*

$$\mathcal{K}x^\alpha = x^\alpha \Gamma(\alpha+1-\beta)\big/\big(\Gamma(1+\alpha)\Gamma(1-\beta)\big).$$

If $\psi \in C[0,T]$ *satisfies* $\big|\psi(x)\big| \leq Mx^\alpha$ *for some* $M > 0$ *and all* $x \in (0,T]$, *then* $\mathcal{K}\psi \in C[0,T]$, *and* $\big|\mathcal{K}\psi(x)\big| \leq M\mathcal{K}x^\alpha$ *for all* $x \in (0,T]$.

P r o o f. The first claim is immediate from the definition of $\mathcal{K}$, and by the assumption on ψ, we have $\big|\mathcal{K}\psi(x)\big| \leq \mathcal{K}|\psi|(x) \leq M\mathcal{K}x^\alpha$. We now prove that $\mathcal{K}\psi$ is continuous on $(0,T]$. For $\varepsilon \in (0,\, x/2)$, define

$$\mathcal{K}_\varepsilon\psi(x) = \int_\varepsilon^{x-\varepsilon} k_1(x,r)\psi(r)\,\mathrm{d}r.$$

Given $T_1 \in (0,T]$, for every $x \in [T_1,\, T]$ and $\varepsilon \in (0,T_1/2)$, we have

$$\begin{aligned}\big|\mathcal{K}_\varepsilon\psi(x) - \mathcal{K}\psi(x)\big| &\leq \int_0^\varepsilon k_1(x,r)\big|\psi(r)\big|\,\mathrm{d}r + \int_{x-\varepsilon}^x k_1(x,r)\big|\psi(r)\big|\,\mathrm{d}r \\ &\leq \frac{\beta(x/\varepsilon-1)^{\beta-1} + (1-\beta)(x/\varepsilon-1)^{-\beta}}{\beta(1-\beta)\Gamma(\beta)\Gamma(1-\beta)}\|\psi\|_{C[0,T]} \\ &\leq \frac{\beta(T_1/\varepsilon-1)^{\beta-1} + (1-\beta)(T_1/\varepsilon-1)^{-\beta}}{\beta(1-\beta)\Gamma(\beta)\Gamma(1-\beta)}\|\psi\|_{C[0,T]},\end{aligned}$$

therefore, as $\varepsilon \to 0$, $\mathcal{K}_\varepsilon\psi \to \mathcal{K}\psi$ uniformly on $[T_1,\, T]$. Because $\mathcal{K}_\varepsilon\psi$ is continuous on $[T_1,\, T]$, $\mathcal{K}\psi$ must be continuous on $[T_1,\, T]$, and thus on $(0,T]$. In addition, by the continuity of ψ at $x = 0$, $\mathcal{K}\psi(x) \to \psi(0)$ as $x \to 0$, and therefore $\mathcal{K}\psi \in C[0,T]$. □

We can now obtain a crucial bound that will help us adapt [14, Theorem 7.10] to the censored IVP (1.3) in order to express the solution as a series.

LEMMA 2.3. *For any* $\alpha > 0$, *we have*

$$\sum_{j=1}^\infty \mathcal{K}^j x^\alpha = x^\alpha \left(\frac{\Gamma(1+\alpha)\Gamma(1-\beta)}{\Gamma(\alpha+1-\beta)} - 1\right)^{-1}. \tag{2.3}$$

If $\psi \in C[0,T]$ *and* $\big|\psi(x)\big| \leq Mx^\alpha$ *for some* $M > 0$ *and all* $x \in (0,T]$, *then*

$$\sum_{j=1}^\infty \mathcal{K}^j\psi \in C[0,T], \text{ and } \left|\sum_{j=1}^\infty \mathcal{K}^j\psi(x)\right| \leq M\sum_{j=1}^\infty \mathcal{K}^j x^\alpha \text{ for all } x \in (0,T].$$

In addition, $\mathcal{K}^j\psi(x) = \int_0^x k_j(x,r)\psi(r)\,\mathrm{d}r$ *for all* $j \in \mathbb{N}$, $x \in (0,T]$.

P r o o f. We first confirm (2.1) using the fact that t^α and $(1-t)^{-\beta}$ strictly increase, so that

$$\frac{1}{\alpha-\beta+1}=\int_0^1(1-t)^\alpha(1-t)^{-\beta}\,\mathrm{d}t<\int_0^1 t^\alpha(1-t)^{-\beta}\,\mathrm{d}t=\frac{\Gamma(1+\alpha)\Gamma(1-\beta)}{\Gamma(1+\alpha+1-\beta)}.$$

Applying Lemma 2.2 for j times, we get $\mathcal{K}^j x^\alpha = x^\alpha\big(\Gamma(1+\alpha)\Gamma(1-\beta)/\Gamma(\alpha+1-\beta)\big)^{-j}$. Then, by summing over j, we obtain (2.3) from (2.1). Meanwhile, we have $\left|\mathcal{K}^j\psi(x)\right|\le M\mathcal{K}^j x^\alpha$ and $\mathcal{K}^j\psi\in C[0,T]$, so $\sum_{j=1}^\infty \mathcal{K}^j\psi$ converges uniformly to a limit in $C[0,T]$, whose absolute value is pointwise bounded by $M\sum_{j=1}^\infty \mathcal{K}^j x^\alpha$. Finally, by induction,

$$\begin{aligned}\mathcal{K}^j\psi(x)=\mathcal{K}\mathcal{K}^{j-1}\psi(x)&=\int_0^x k_1(x,r)\int_0^r k_{j-1}(r,s)\psi(s)\,\mathrm{d}s\,\mathrm{d}r\\&=\int_0^x\int_s^x k_1(x,r)k_{j-1}(r,s)\psi(s)\,\mathrm{d}r\,\mathrm{d}s\\&=\int_0^x k_j(x,s)\psi(s)\,\mathrm{d}s.\end{aligned}$$

□

REMARK 2.7. In Lemma 2.3, we require $\alpha>0$ (though the last statement there holds for all $\alpha\ge 0$), in fact, if $\alpha=0$, let $\psi=1$, then

$$\sum_{j=1}^\infty \mathcal{K}^j\psi(x)=\sum_{j=1}^\infty\int_0^x k_j(x,r)\,\mathrm{d}r=\infty.$$

3. Well-posedness of the censored IVPs

3.1. Inverse of ∂_0^β.

We begin with the basic censored IVP (1.3) with $g\in C(0,T]$ and $u_0\in\mathbb{R}$. Our strategy is to consider the equivalent Caputo/R–L problem for $\bar{u}=u-u_0$ with the unbounded coefficient $x^{-\beta}/\Gamma(1-\beta)$,

$$D_0^\beta\bar{u}(x)=\frac{x^{-\beta}}{\Gamma(1-\beta)}\bar{u}(x)+g(x),\quad x>0,\quad \bar{u}(0)=0, \tag{3.1}$$

and then show that for certain forcing terms g, the formula [14, Theorem 7.10] for bounded coefficients still yields a solution to (3.1), and thus to (1.3). (Note that for (3.1) with $g\in C[0,T]$, the solution is already guaranteed by [38, Example 3.4] to exist but given no explicit expression. See [16, Remark 3.1-(ii)] for more detail.)

REMARK 3.1.

(i) We can solve (3.1) using Picard iteration, i.e.,

$\bar{u}_{m+1}(x)=J_0^\beta\left[x^{-\beta}\bar{u}_m(x)/\Gamma(1-\beta)+g(x)\right]$ $(m=1,2,\cdots)$ with $\bar{u}_1=0$.

By Remark 2.6, the limit equals $I_0^\beta g$ defined in (3.2) if the iteration converges.

(ii) If one replaces the coefficient in the R–L problem (3.1) by $Cx^{-\beta}/\Gamma(1-\beta)$, then the series representation for the solution would be $\bar{u} = \sum_{j=0}^{\infty} C^j \mathcal{K}^j J_0^\beta g$, which does not converge for important data (like $g = 1$) if $|C| \geq \Gamma(1+\beta)\Gamma(1-\beta)$.

We now present a key result concerning IVP (1.3), which serves as the fundamental theorem of calculus for ∂_0^β. Or simply put, I_0^β is to ∂_0^β as J_0^β is to D_0^β.

Theorem 3.1. *Let $u_0 \in \mathbb{R}$ and $g \in C(0,T]$ such that $\big|g(x)\big| \leq Mx^{\alpha-\beta}$ for some $M, \alpha > 0$ and all $x \in (0,T]$. Then there exists a unique function $u \in C_\beta[0,T]$ satisfying (1.3), and it has the series representation*

$$u(x) - u_0 = I_0^\beta g(x) := \sum_{j=0}^{\infty} \mathcal{K}^j J_0^\beta g(x), \tag{3.2}$$

where $\mathcal{K}^0$ is the identity operator by convention. Moreover, u depends on u_0 and g continuously in the sense of Remark 3.3.

Theorem 3.1 is an immediate consequence of Lemmata 3.1 and 3.2.

Remark 3.2. For g satisfying the conditions in Theorem 3.1, $I_0^\beta g$ can be equivalently represented as

$$I_0^\beta g(x) = J_0^\beta g(x) + \sum_{j=1}^{\infty} \int_0^x k_j(x,r) J_0^\beta g(r)\,\mathrm{d}r, \tag{3.3}$$

$$= \sum_{j=1}^{\infty} \mathcal{K}^j \left[\Gamma(1-\beta) x^\beta g(x)\right] \tag{3.4}$$

$$= \sum_{j=0}^{\infty} J_0^\beta \left[\frac{\mathcal{K}^j \left[x^\beta g(x)\right]}{x^\beta}\right], \tag{3.5}$$

where (3.3) is due to Lemma 2.3, while (3.4) and (3.5) are due to Remark 2.6. From any representation, we can see that I_0^β is a linear operator preserving positivity ($I_0^\beta g \geq 0$ if $g \geq 0$).

Remark 3.3. For $g \in C[0,T]$, we can prove the continuous dependence by showing that $\|u - u_0\|_{C[0,T]} \leq C\|g\|_{C[0,T]}$ for some C dependent only on β and T. For a more general g which may diverge at $x = 0$, C will depend also on α, and $\|g\|_{C[0,T]}$ needs to be replaced by $\|g\|_{G^{\alpha-\beta}(0,T]}$, where we define for any $\gamma \in \mathbb{R}$ a Banach space $G^{\gamma}(0,T] = \{h \in C(0,T] : \|h\|_{G^{\gamma}(0,T]} < \infty\}$, with the norm $\|h\|_{G^{\gamma}(0,T]} := \sup\{|x^{-\gamma}h(x)| : x \in (0,T]\}$. In particular, if $g \in C[0,T]$ and $\alpha = \beta$, then $\|g\|_{G^{\alpha-\beta}(0,T]} = \|g\|_{C[0,T]}$. (Note that G^{γ} is the same as $\hat{B}$ defined in [14, Proof of Lemma 5.3].)

Lemma 3.1. *Solutions to problem* (1.3) *are unique in* $C_\beta[0,T]$.

P r o o f. Let $u_1, u_2 \in C_\beta[0,T]$ be two solutions to problem (1.3). By linearity of ∂_0^β, $u := u_1 - u_2 \in C_\beta[0,T]$ satisfies $\partial_0^\beta u = 0$ on $(0,T]$. Therefore for every $x \in (0,T]$, $D_0^\beta u(x) = \Gamma(1-\beta)^{-1}x^{-\beta}u(x)$, where the right-hand side is in $C \cap L^1(0,T]$. Using Lemma 2.1-(ii) as well as Remark 2.6, we obtain

$$D_0^\beta u(x) = \frac{x^{-\beta}}{\Gamma(1-\beta)}u(x) = D_0^\beta J_0^\beta\left[\frac{x^{-\beta}}{\Gamma(1-\beta)}u(x)\right] = D_0^\beta \mathcal{K}u(x),$$

where $\mathcal{K}u \in C_\beta(0,T]$. By Lemma 2.2, $\mathcal{K}u$ is in $C[0,T]$, and so is $u - \mathcal{K}u$. Consequently, $u - \mathcal{K}u \in C_\beta[0,T]$. By the linearity of D_0^β, we know $D_0^\beta[u - \mathcal{K}u]=0$. According to Lemma 2.1-(iv), we obtain $u = \mathcal{K}u$.

Let $\xi \in \arg\max_{r\in[0,T]} |u(r)|$. If $\xi = 0$, then $u = 0$ on $[0,T]$ because $u(0) = 0$. If $\xi > 0$, using the fact that $u(\xi) = \mathcal{K}u(\xi)$, we have $\int_0^\xi k_1(\xi,r)\big(u(\xi) - u(r)\big)\,\mathrm{d}r = 0$, where $u(\xi) - u(r)$ never changes sign for all $r \in [0,\xi]$, according to the definition of ξ. So $u(\xi) = u(r)$ for all $r \in [0,\xi]$, therefore $u(\xi) = u(0) = 0$, and we still obtain $u = 0$ on $[0,T]$. This proves $u_1 = u_2$, and we are done. □

Lemma 3.2. *For* g *satisfying the conditions in Theorem 3.1,* $I_0^\beta g$ *is in* $C_\beta[0,T]$ *with* $I_0^\beta g(0) = 0$ *and* $\partial_0^\beta I_0^\beta g = g$*. In addition,* $I_0^\beta g$ *depends on* g *continuously in the sense of Remark 3.3.*

P r o o f. Using representation (3.4), we can see $I_0^\beta g(0) = 0$ from the assumptions on g and Definition 2.4, then from Lemma 2.3 we obtain $I_0^\beta g \in C[0,T]$ and that for all $x \in (0,T]$

$$\left|I_0^\beta g(x)\right| \le I_0^\beta|g|(x) \le Mx^\alpha\left(\frac{\Gamma(1+\alpha)}{\Gamma(\alpha+1-\beta)} - \frac{1}{\Gamma(1-\beta)}\right)^{-1}. \tag{3.6}$$

Note that in (3.5), the summation commutes with J_0^β, by Fubini's Theorem and the above bound. So,

$$\begin{aligned} I_0^\beta g(x) = J_0^\beta \sum_{j=0}^{\infty} \frac{\mathcal{K}^j\big[x^\beta g(x)\big]}{x^\beta} &= J_0^\beta \Big[g(x) + x^{-\beta}\sum_{j=1}^{\infty}\mathcal{K}^j\big[x^\beta g(x)\big]\Big] \\ &= J_0^\beta\Big[g(x) + \frac{x^{-\beta} I_0^\beta g(x)}{\Gamma(1-\beta)}\Big], \end{aligned} \tag{3.7}$$

with the last equality due to (3.4). Therefore, $I_0^\beta g = J_0^\beta \psi$ for a $\psi \in C(0,T]$ satisfying

$$\big|\psi(x)\big| \le M x^{\alpha-\beta}\left(1 - \frac{\Gamma(\alpha+1-\beta)}{\Gamma(1+\alpha)\Gamma(1-\beta)}\right)^{-1}, \quad \text{for all } \ x\in(0,T], \tag{3.8}$$

so Lemma 2.1-(ii) proves that $I_0^\beta g$ is in $C_\beta(0,T]$ and thus $C_\beta[0,T]$. Lemma 2.1-(ii) also proves that

$$D_0^\beta I_0^\beta g(x) = \psi(x) = g(x) + \frac{x^{-\beta} I_0^\beta g(x)}{\Gamma(1-\beta)}, \qquad \text{for all } x\in(0,T],$$

which rewrites as $\partial_0^\beta I_0^\beta g = g$ by Definition 2.2.

To see the continuity of I_0^β, let the M in (3.6) be $\|g\|_{G^{\alpha-\beta}(0,T]}$ ($G^\gamma(0,T]$ is defined in Remark 3.3), then we obtain

$$\|I_0^\beta g\|_{G^\alpha(0,T]} \le \left(\frac{\Gamma(1+\alpha)}{\Gamma(\alpha+1-\beta)} - \frac{1}{\Gamma(1-\beta)}\right)^{-1}\|g\|_{G^{\alpha-\beta}(0,T]}.$$

Since $\alpha > 0$, we have $\|I_0^\beta g\|_{C[0,T]} \le T^\alpha \|I_0^\beta g\|_{G^\alpha(0,T]} \le C\|g\|_{G^{\alpha-\beta}(0,T]}$ for some C dependent only on α, β and T. □

Example 3.1. Recall that for the Caputo IVP $D_0^\beta\big[u - u(0)\big] = x^\alpha$ ($\alpha > -1$) with $u(0) = u_0$, the solution is $u_0 + J_0^\beta x^\alpha$ [14]. By (3.4) and Lemma 2.3, the solution to (1.3) for $g(x) = x^\alpha$ ($\alpha > -\beta$) is

$$u(x) - u_0 = I_0^\beta x^\alpha = \left(\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)} - \frac{1}{\Gamma(1-\beta)}\right)^{-1} x^{\alpha+\beta} = c_{\alpha+\beta,\,\beta}^{-1}\, J_0^\beta x^\alpha, \tag{3.9}$$

where $c_{\alpha+\beta,\,\beta}$ is defined in Remark 2.3-(ii). In particular, when $\alpha = 0$, $c_{\alpha+\beta,\,\beta}^{-1} = \beta\pi/\big(\beta\pi - \sin(\beta\pi)\big)$. If $\alpha \in (-1,-\beta]$, we may not be able to impose the initial condition in (1.3), since the solution may explode at 0. For example, when $\alpha = -\beta$, one can verify that a particular solution is $-\Gamma(1-\beta)\ln(x)/H_{-\beta}$, where $H_{-\beta}$ is the Harmonic number.

3.2. **Inhomogeneous linear IVPs.** We now consider the linear IVP

$$\begin{cases} \partial_0^\beta u(x) = \lambda x^{\alpha-\beta} u(x) + g(x), & x \in (0,T], \\ u(x) = u_0, & x = 0. \end{cases} \tag{3.10}$$

Like its counterpart in classical ODEs, such IVP can play important roles in more general equations. (Using Picard iteration, we can also show the global well-posedness for general IVPs $\partial_0^\beta u = f(x,u)$, $u(0) = u_0$. See [16, Section 3.2] for more detail.) To solve (3.10), we need the following lemma.

LEMMA 3.3. *For $x, \alpha > 0$ and $N \in \mathbb{N}$, we have*

$$\left(I_0^\beta[x^{\alpha-\beta}\cdot\,]\right)^N 1(x) \le C\frac{2^N x^{N\alpha}}{(N!\alpha^N)^\beta},$$

where $1(x)$ is the constant function 1, C is a positive constant dependent only on α and β, and we denote

$$\left(I_0^\beta[x^{\alpha-\beta}\cdot\,]\right)^N g(x) = \underbrace{I_0^\beta\Big[x^{\alpha-\beta}\cdots I_0^\beta\big[x^{\alpha-\beta}}_{N \text{ times}} g(x)\big]\cdots\Big].$$

P r o o f. From Example 3.1 we know that

$$\left(I_0^\beta[x^{\alpha-\beta}\cdot\,]\right)^N 1(x) = \prod_{n=1}^{N}\left(\frac{\Gamma(1+n\alpha)}{\Gamma(n\alpha+1-\beta)} - \frac{1}{\Gamma(1-\beta)}\right)^{-1} x^{N\alpha}, \tag{3.11}$$

where each factor is positive. Using Stirling's formula for $\Gamma(z)$, i.e.

$$\Gamma(z) = \sqrt{\frac{2\pi}{z}}\left(\frac{z}{e}\right)^z\left(1 + O\left(\frac{1}{z}\right)\right),$$

we have the following approximation

$$\frac{\Gamma(1+n\alpha)}{\Gamma(n\alpha+1-\beta)} = (n\alpha)^\beta\left(1 + O\left(\frac{1}{n}\right)\right),$$

which indicates that there exists $\tilde{n} \in \mathbb{N}$ such that for all $n > \tilde{n}$,

$$\frac{\Gamma(1+n\alpha)}{\Gamma(n\alpha+1-\beta)} - \frac{1}{\Gamma(1-\beta)} \ge \frac{(n\alpha)^\beta}{2},$$

so there exists $C > 0$ such that Lemma 3.3 holds for all $N \in \mathbb{N}$. □

PROPOSITION 3.1. *Let $\lambda, u_0 \in \mathbb{R}$, $\alpha > 0$ and $g \in C(0,T]$ such that $\left|g(x)\right| \le Mx^{\gamma-\beta}$ for some $M, \gamma > 0$ and all $x \in (0,T]$. Then (3.10) has a unique solution in $C_\beta[0,T]$ given by the following series*

$$u(x) = u_0\sum_{N=0}^{\infty}\lambda^N\left(I_0^\beta[x^{\alpha-\beta}\cdot\,]\right)^N 1(x) + \sum_{N=0}^{\infty}\lambda^N\left(I_0^\beta[x^{\alpha-\beta}\cdot\,]\right)^N I_0^\beta g(x), \tag{3.12}$$

which depends on u_0 and g continuously (analogous to Theorem 3.1).

P r o o f. From Lemma 3.2 we know $I_0^\beta g \in C[0,T]$ and thus for $N \in \mathbb{N}$, $\big(I_0^\beta[x^{\alpha-\beta}\cdot]\big)^N I_0^\beta g \in C[0,T]$. By the positivity preserving property of I_0^β,

$$\Big|\big(I_0^\beta[x^{\alpha-\beta}\cdot]\big)^N I_0^\beta g\Big| \le \big(I_0^\beta[x^{\alpha-\beta}\cdot]\big)^N |I_0^\beta g| \le \big(I_0^\beta[x^{\alpha-\beta}\cdot]\big)^N 1\cdot \|I_0^\beta g\|_{C[0,T]}.$$

As by Lemma 3.3 the series $\sum_{N=0}^\infty \lambda^N \big(I_0^\beta[x^{\alpha-\beta}\cdot]\big)^N 1$ converges uniformly on $[0,T]$, so does $\sum_{N=0}^\infty \lambda^N \big(I_0^\beta[x^{\alpha-\beta}\cdot]\big)^N I_0^\beta g$. So the function u given by (3.12) is in $C[0,T]$ and $I_0^\beta[x^{\alpha-\beta}u]$ is well-defined. Therefore,

$$\begin{aligned}
\lambda I_0^\beta[x^{\alpha-\beta}u] + I_0^\beta g &= \lambda u_0\, I_0^\beta\Big[x^{\alpha-\beta}\sum_{N=0}^\infty \lambda^N \big(I_0^\beta[x^{\alpha-\beta}\cdot]\big)^N 1\Big] \\
&\quad + \lambda\, I_0^\beta\Big[x^{\alpha-\beta}\sum_{N=0}^\infty \lambda^N \big(I_0^\beta[x^{\alpha-\beta}\cdot]\big)^N I_0^\beta g\Big] + I_0^\beta g \\
&= \lambda u_0 \sum_{N=0}^\infty \lambda^N \big(I_0^\beta[x^{\alpha-\beta}\cdot]\big)^{N+1} 1 \\
&\quad + \lambda \sum_{N=0}^\infty \lambda^N \big(I_0^\beta[x^{\alpha-\beta}\cdot]\big)^{N+1} I_0^\beta g + I_0^\beta g \\
&= u - u_0,
\end{aligned}$$

where the second equality is due to the continuous dependence in Theorem 3.1. Using Theorem 3.1 again, we know that u solves (3.10). The continuous dependence of u on u_0 and g is clear from the above convergence.

To show the uniqueness, assume that $u_1,\, u_2 \in C_\beta[0,T]$ solve (3.10), then $u := u_1 - u_2 \in C_\beta[0,T]$ satisfies $\partial_0^\beta u = \lambda x^{\alpha-\beta} u$ with $u(0) = 0$. By Theorem 3.1, $u = \lambda I_0^\beta[x^{\alpha-\beta}u] = \cdots = \lambda^N \big(I_0^\beta[x^{\alpha-\beta}\cdot]\big)^N u$. Then, letting $N \to \infty$ and using Lemma 3.3, we obtain $u = 0$. □

3.3. **Homogeneous linear IVPs with constant coefficients.**

COROLLARY 3.1. [of Proposition 3.1] *For any $\lambda, u_0 \in \mathbb{R}$, the IVP*

$$\begin{cases} \partial_0^\beta u(x) = \lambda u(x), & x \in (0,T], \\ u(x) = u_0, & x = 0, \end{cases} \tag{3.13}$$

has a unique solution in $C_\beta[0,T]$ given by $u(x) = u_0 \sum_{N=0}^\infty \lambda^N (I_0^\beta)^N 1(x)$, which is equivalent to (1.7) *by letting $\alpha = \beta$ in* (3.11).

Although the series (1.7) looks cumbersome, it surprisingly decays at the simple algebraic rate $x^{-1-\beta}$ for $\lambda < 0$.

THEOREM 3.2. *For $\lambda < 0$ and $u_0 > 0$, the solution u to* (3.13) *is completely monotone (i.e., $(-1)^n u^{(n)} \geq 0$ on $\mathbb{R}_+$ for $n = 0, 1, 2, \ldots$) and there exists a constant $C > 1$ such that*

$$\frac{C^{-1}}{x^{1+\beta}} \leq u(x) \leq \frac{C}{x^{1+\beta}}, \quad \textit{for all} \ \ x \geq 1.$$

P r o o f. The complete monotonicity will be proved in Corollary 4.1, using a probabilistic argument. The upper and lower bounds are proved in Lemmata 3.5 and 3.6 below, using a maximum principle argument. □

REMARK 3.4.

(i) For Caputo's counterpart of IVP (3.13), i.e., $D_0^\beta\big[u - u(0)\big] = \lambda u,\ u(0) = u_0$, the solution is expressed in terms of the Mittag-Leffler function

$$u(x) = u_0 \sum_{N=0}^{\infty} \frac{(\lambda x^\beta)^N}{\Gamma(N\beta + 1)}. \tag{3.14}$$

For $\lambda < 0$ and $u_0 > 0$, it is completely monotone and decays at the rate $x^{-\beta}$ [14, Theorem 7.3]. By contrast, the censored relaxation equation (3.13) models a new decay regime $x^{-1-\beta}$. (See also [50, page 1623] for related fractional relaxation equations, where the decay rate is $x^{-\gamma}$ for some $\gamma \in (0, 1)$.)

As a side note, for $\lambda, u_0 > 0$, obviously both (1.7) and (3.14) increase in x faster than any polynomial. Indeed, for $\lambda = 1$, the latter grows at the rate e^x [21, Proposition 3.5], and our numerical results suggest $\exp\{x + cx^{1-\beta}\}$ for the former, where c is positive and depends only on β.

(ii) For (3.14) with $\lambda = -1,\ u_0 = 1$, [46, Theorem 4] gave the uniform estimates with optimal constants: $\big(1 + \Gamma(1-\beta)x^\beta\big)^{-1} \leq u(x) \leq \big(1 + \Gamma(1+\beta)^{-1} x^\beta\big)^{-1}$. In [16, Proposition B.1] we give what we believe to be a new and simple proof of those bounds, using the same strategy used for the uniform bounds of (1.7). Recently [10, Proposition 4.12] gave another new proof by showing the generalized results for a class of Kilbas–Saigo functions. Our simple proof can also be applied with few modifications to prove those generalized results (see [16, Proposition B.4]). In Section 3.4 we will use it again, to prove the uniform bounds of (3.19), the solution to $\partial_0^\beta u = \lambda x^{\alpha-\beta} u$.

(iii) The reason why our proof is both simple and versatile is that it involves only maximum principle (mentioned in Remark 2.3-(iii)) and some suitable candidate bounds (e.g. $(1+cx)^{-1-\beta}$), but no specific representation of the solution (e.g. (1.7) or (1.9)). In fact, we expect this strategy to have broader applications. As an example, consider a general Caputo-type derivative D^{ψ}_{*0} (so $-D^{\psi}_{*0}$ generates a non-increasing pure jump Lévy process killed upon leaving $\mathbb{R}^+$ [34]) for a Lévy measure ψ with $\int_0^\infty \min\{r,1\}\psi(\mathrm{d}r)<\infty$,

$$D^{\psi}_{*0}u(x)=\int_0^x \big(u(x)-u(x-r)\big)\psi(\mathrm{d}r)+\big(u(x)-u(0)\big)\psi\big((x,\infty)\big),$$

and its relaxation equation $D^{\psi}_{*0}u=\lambda u$ ($\lambda<0$). The solution is given as an expectation or a series under mild assumptions [31, Lemma 3.4]. It is possible for our strategy to prove two-sided bounds of this solution without those representations of it. Indeed, this has already been done for certain absolutely continuous ψ (so $\psi(\mathrm{d}r)=\psi(r)\,\mathrm{d}r$). For instance, for compactly supported ψ, the solution is given an upper bound of the decay rate x^{-1} [18, Remark 3.5]. A special case is the truncated fractional kernel $\psi(r)=\mathbf{1}_{\{r\in(0,\delta]\}}r^{-1-\beta}$ with $\delta>0$ (see [18, Theorem 3.2], which inspired our proof). Even if ψ is not compactly supported, as long as $\int_0^\infty r\psi(r)\,\mathrm{d}r<\infty$, the same argument applies. Another instance is when $r^{1+\beta}\psi(r)$ is continuous on $\mathbb{R}_+$ and bounded within $[C^{-1},C]$ for some $C>1$, our strategy (in [16, Proposition B.1]) can still prove the two-sided bounds, both of $x^{-\beta}$ decay.

LEMMA 3.4. *If $\lambda<0$ and $v\in C^1(0,T]\cap C[0,T]$ satisfies $\partial^\beta_0 v\ge\lambda v$, then v is nonnegative if $v(0)\ge 0$, and positive if $v(0)>0$.*

P r o o f. If $v(0)\ge 0$ but v is not nonnegative, let x_0 be a minimum point of v on $[0,T]$, then $x_0>0$ and $v(x_0)<0$. So we have $0<\lambda v(x_0)\le \partial^\beta_0 v(x_0)$. However, since $v\in C^1(0,T]$, by Remark 2.3-(iii) we know that

$$\partial^\beta_0 v(x_0)=\int_0^{x_0}\big(v(x_0)-v(x_0-r)\big)\frac{r^{-1-\beta}}{\big|\Gamma(-\beta)\big|}\,\mathrm{d}r\le 0,$$

which is a contradiction. Similarly, we can prove $v>0$ if $v(0)>0$. □

LEMMA 3.5. *For $\lambda<0$ and $u_0=1$, the solution u to (3.13) is positive and can be bounded from above by $v(x)=(1+c|\lambda|^{1/\beta}x)^{-1-\beta}$, where*

$$c=\frac{\big|\Gamma(-\beta)\big|^{1/\beta}}{2}\Big(\frac{2^{1+\beta}-1}{1-\beta}+\frac{2}{\beta}\Big)^{-1/\beta}. \tag{3.15}$$

P r o o f. We know from Corollary 3.1 that $u \in C[0,T]$. We also know that $u \in C^1(0,T]$ from the uniform convergence of the series representation of its derivative on any closed interval not containing 0. Thus u remains positive by Lemma 3.4.

Let $v(x) = (1+x/c)^{-1-\beta}$ and first assume that there is a constant $c > 0$ such that v satisfies the condition in Lemma 3.4. Under this assumption, we get $\partial_0^\beta(v-u) - \lambda(v-u) \geq 0$ with $v(0) - u(0) = 0$. By Lemma 3.4, we have $v \geq u$ on $[0,T]$.

Now, given $\beta \in (0,1)$ and $\lambda < 0$, up to a constant multiple, it remains to find a constant $c > 0$ such that $v(x) = (x+c)^{-1-\beta}$ satisfies $\partial_0^\beta v \geq \lambda v$, i.e., for all $x > 0$,

$$\int_0^x \big(v(x) - v(x-r)\big)\frac{r^{-1-\beta}}{\big|\Gamma(-\beta)\big|}\,\mathrm{d}r \geq \lambda v(x),$$

or equivalently, for all $x > 0$,

$$\big|\lambda\Gamma(-\beta)\big| \geq \int_0^x \left(\frac{(x+c)^{1+\beta}}{(x+c-r)^{1+\beta}} - 1\right)\frac{\mathrm{d}r}{r^{1+\beta}}. \tag{3.16}$$

Let $y = x + c$, then the right-hand side of (3.16) equals

$$\int_0^{y-c}\left(\frac{y^{1+\beta}}{(y-r)^{1+\beta}} - 1\right)\frac{\mathrm{d}r}{r^{1+\beta}} = y^{-\beta}\int_0^{1-c/y}\left(\frac{1}{(1-s)^{1+\beta}} - 1\right)\frac{\mathrm{d}s}{s^{1+\beta}}. \tag{3.17}$$

If $y \leq 2c$, then the right-hand side of (3.17) can be bounded from above by

$$y^{-\beta}\int_0^{1/2}\left(\frac{1}{(1-s)^{1+\beta}} - 1\right)\frac{\mathrm{d}s}{s^{1+\beta}} \leq y^{-\beta}\int_0^{1/2} 2s(2^{1+\beta}-1)\frac{\mathrm{d}s}{s^{1+\beta}} = \frac{2^\beta}{y^\beta}\frac{2^{1+\beta}-1}{1-\beta}.$$

If $y > 2c$, we split the interval $[0, 1-c/y]$ into two parts $[0,1/2]$ and $[1/2, 1-c/y]$. For the second subinterval,

$$\int_{1/2}^{1-c/y}\left(\frac{1}{(1-s)^{1+\beta}} - 1\right)\frac{\mathrm{d}s}{s^{1+\beta}} \leq 2^{1+\beta}\int_{1/2}^{1-c/y}\frac{\mathrm{d}s}{(1-s)^{1+\beta}} \leq \frac{2^{1+\beta}}{\beta}\Big(\frac{y}{c}\Big)^\beta.$$

Therefore the right-hand side of (3.16) can be bounded from above by

$$\frac{2^\beta}{y^\beta}\frac{2^{1+\beta}-1}{1-\beta} + \frac{2^{1+\beta}}{y^\beta\beta}\Big(\frac{y}{c}\Big)^\beta \leq \frac{2^\beta}{c^\beta}\frac{2^{1+\beta}-1}{1-\beta} + \frac{2^{1+\beta}}{c^\beta\beta}.$$

Let

$$c \geq \frac{2}{\big|\lambda\Gamma(-\beta)\big|^{1/\beta}}\Big(\frac{2^{1+\beta}-1}{1-\beta} + \frac{2}{\beta}\Big)^{1/\beta},$$

then (3.16) will be satisfied and we are done. □

Lemma 3.6. *If $\lambda < 0$ and $u_0 = 1$, then the solution to* (3.13) *is bounded from below by $w(x) = (1 + d|\lambda|^{1/\beta}x)^{-1}(1 + d^\beta|\lambda|x^\beta)^{-1}$, where $d = \left|\Gamma(-\beta)\right|^{1/\beta} \max\left\{4,\ (1-\beta)(1+2^\beta)/\beta\right\}^{1/\beta}$.*

The proof is similar to that of Lemma 3.5 and given in [16, Appendix B.2].

3.4. **Homogeneous linear IVPs with nonconstant coefficients.**

Corollary 3.2. *[of Proposition 3.1] For $\lambda, u_0 \in \mathbb{R}$ and $\alpha > 0$, the IVP*

$$\begin{cases} \partial_0^\beta u(x) = \lambda x^{\alpha-\beta} u(x), & x \in (0,T], \\ \quad u(x) = u_0, & x = 0. \end{cases} \tag{3.18}$$

has a unique solution in $C_\beta[0,T]$ given by the following series

$$\begin{aligned} u(x) &= u_0 \sum_{N=0}^{\infty} \lambda^N \left(I_0^\beta[x^{\alpha-\beta}\cdot\,]\right)^N 1(x) \\ &= u_0 \sum_{N=0}^{\infty} (\lambda x^\alpha)^N \prod_{n=1}^{N} \left(\frac{\Gamma(1+n\alpha)}{\Gamma(n\alpha+1-\beta)} - \frac{1}{\Gamma(1-\beta)}\right)^{-1}. \end{aligned} \tag{3.19}$$

Surprisingly, the solution (3.19) has a decay property analogous to what we see in Section 3.3.

Proposition 3.2. *For $\lambda < 0$ and $u_0 > 0$, there exists a constant $C > 1$ such that the solution u to* (3.18) *satisfies*

$$\frac{C^{-1}}{x^{1+\alpha}} \le u(x) \le \frac{C}{x^{1+\alpha}}, \quad \textit{for all} \ \ x \ge 1.$$

The proof is similar to that of Theorem 3.2 and given in [16, Proposition 3.24].

Remark 3.5.

(i) For Caputo's counterpart of IVP (3.18), the solution can be expressed in terms of the Kilbas–Saigo function

$$u(x) = u_0 \sum_{N=0}^{\infty} (\lambda x^\alpha)^N \prod_{n=1}^{N} \left(\frac{\Gamma(1+n\alpha)}{\Gamma(n\alpha+1-\beta)}\right)^{-1}. \tag{3.20}$$

For $\lambda < 0$ and $u_0 > 0$, the solution (3.20) decays at the rate $x^{-\alpha}$ [10, Remark 4.6 (c)] (and is completely monotone [10, Remark 3.1

(d)] if $\alpha \le 1$). On the other hand, the censored IVP (3.18) once again models a new decay regime $x^{-1-\alpha}$.

As a side note, for $\lambda, u_0 > 0$, both (3.19) and (3.20) increase in x faster than any polynomial. Indeed, for $\lambda = 1$, the latter can be bounded by $\exp\big\{(\frac{\beta}{\alpha}+\varepsilon)x^{\alpha/\beta}\big\}$ for any ε positive and x large enough [21, Theorem 5.9], and our numerical results suggest the same for the former.

(ii) For (3.20) with $\lambda = -1$, $u_0 = 1$, [10, Proposition 4.12] proved the uniform bounds

$$\big(1+\Gamma(1-\beta)x^{\alpha}\big)^{-1} \le u(x) \le \big(1+\Gamma(1+\alpha-\beta)\Gamma(1+\alpha)^{-1}x^{\alpha}\big)^{-1}.$$

As mentioned in Remark 3.4-(ii), our maximum principle argument can give a new and simple proof of those bounds.

(iii) For $\alpha = 1$, (3.18) can be seen as a linear equation $\sigma\partial_0^{\beta}u = \lambda u$, where we let $\sigma(x) = x^{\beta-1}$ so that the rescaled fractional derivative $\sigma\partial_0^{\beta}$ acts like the classical first order derivative on linear functions. This kind of rescaling naturally extends to more general nonlocal derivatives, and we refer to [15] for a discusison of nonlocal calculus and rescaling.

4. Censored decreasing β-stable process

In this section we first prove that the hitting time of 0 (or lifetime) for the censored decreasing β-stable process is finite and that I_0^{β} has probabilistic representations (1.4) and (1.5). We then use these results to prove that our censored process is Feller with generator $-\partial_0^{\beta}$, which in turn leads us to show that the Laplace transform of the lifetime equals the series (1.7), and thus they are completely monotone. We denote by $\mathbf{1}_A$ the indicator function of a set A. All our stochastic processes are real-valued right-continuous with left limits (càdlàg), hence we always assume the canonical underlying filtered probability space as in [4, Chapter O]. For a stochastic process $Y = \{Y_s\}_{s\ge 0}$ and a real-valued integrable function f on the probability space of Y, we use the notation $\mathbb{E}_y\big[f(Y)\big] = \mathbb{E}\big[f(Y)\,\big|\,Y_0 = y\big]$, $\mathbb{E}\big[f(Y)\big] = \mathbb{E}_0\big[f(Y)\big]$, and correspondingly $\mathbb{P}_y[A]$, $\mathbb{P}[A]$ when $f = \mathbf{1}_A$. We write $Y_{t-} = \lim_{s\uparrow t} Y_s$. By a *$\beta$-stable subordinator* ($\beta \in (0,1)$) we mean the Lévy process $-S^1 = \{-S_s^1\}_{s\ge 0}$ characterised by the Laplace transform $\mathbb{E}\big[\exp\{kS_s^1\}\big] = \exp\{-sk^{\beta}\}$, $k, s > 0$ [4, Chapter III]. We denote by $B[0,T]$ the set of real-valued bounded Borel measurable functions on $[0,T]$ and define $C_\infty(0,T] = \big\{u \in C[0,T] : u(0) = 0\big\}$, both understood as Banach spaces with the sup norm. We extend the domain of any $f \in B[0,T]$ to a cemetery state ∂ imposing $f(\partial) = 0$. As discussed in Section 1, we treat the censored decreasing stable process in $\mathbb{R}_+$ because it is generated by $-\partial_0^{\beta}$, where ∂_0^β is the "left" censored derivative (at 0). However, it should be clear that all the results in this section translate immediately to the censored stable subordinator in $(-\infty, b)$ when paired with the "right" censored derivative at $b \in \mathbb{R}$.

4.1. **Construction and finite lifetime.** The starting point of the censored process is always assumed to be fixed to some $x > 0$. We define the *censored decreasing β-stable process* S^c by the INW piecing together construction, then [26, Theorem 1.1 and Section 5.i] guarantees us a càdlàg strong (sub-)Markov process. The construction is: run $x + S_t^1$ until τ_1, the time when it first exits $(0, T]$, where $-S^1$ is a β-stable subordinator (started at 0); then kill the process if $x + S^1_{\tau_1-} \le 0$; otherwise piece together an independent copy of S^1 started at $x + S^1_{\tau_1-}$ and repeat the same procedure for at most countably many times.

With Lemma 4.1 we prove that we can directly define the censored decreasing β-stable process $S^c \,|\, S_0^c = x$ as

$$S_t^c := \begin{cases} \widetilde{S}_t^j, & \tau_{j-1} \le t < \tau_j,\ j \in \mathbb{N}, \\ \partial, & t \ge \tau_\infty, \end{cases} \tag{4.1}$$

with

$$\widetilde{S}_t^j := \begin{cases} x + S_t^j, & j = 1, \\ \widetilde{S}^{j-1}_{\tau_{j-1}-} + S^j_{t-\tau_{j-1}}, & j \ge 2, \end{cases} \qquad \tau_j := \begin{cases} 0, & j = 0, \\ \inf\left\{s > \tau_{j-1} : \widetilde{S}_s^j \le 0\right\}, & j \in \mathbb{N}, \\ \lim\limits_{j\to\infty} \tau_j, & j = \infty, \end{cases}$$

where $\{-S^j\}_{j\in\mathbb{N}}$ is an *i.i.d.* collection of β-stable subordinators. Recall [4, Chapter III] the expectation of the inverse stable subordinator

$$\mathbb{E}\big[E_1(y)\big] = y^\beta / \Gamma(\beta + 1), \tag{4.2}$$

where we define $E_j(y) = \inf\left\{s > 0 : \ y < -S_s^j\right\}$ for $j \in \mathbb{N}$ and $y > 0$.

Lemma 4.1. *For any $x > 0$ and $j \in \mathbb{N}$, assuming $S_0^c = x$, we have*

(i) $\mathbb{E}_x[\tau_j] < \infty$, $\mathbb{P}_x\big[S^c_{\tau_j} \in (0, x)\big] = 1$ *and* $S^c_{\tau_j}$ *has the density* $k_j(x, \cdot)$, *as defined in* (2.2);

(ii) $S^c_{\tau_j} > 0$, *and* (4.1) *equals the INW construction of the censored decreasing β-stable process;*

(iii)
$$\mathbb{E}_x[\tau_{j+1} - \tau_j] = \mathbb{E}_x\big[E_{j+1}(S^c_{\tau_j})\big] = \int_0^x \frac{y^\beta}{\Gamma(\beta+1)} k_j(x, y)\,\mathrm{d}y; \tag{4.3}$$

(iv) $\mathbb{P}_x[\tau_\infty < \infty] = 1$ *and* $\mathbb{P}_x\big[S^c_{\tau_\infty -} = 0\big] = 1$.

P r o o f. The statement (ii) follows immediately from (i). We now prove (i) by induction. For $j=1$, $\mathbb{E}_x[\tau_1]=\mathbb{E}\big[E_1(x)\big]=x^\beta/\Gamma(\beta+1)<\infty$, and it is known that $S^c_{\tau_1}=x+S^1_{\tau_1-}$ is beta-distributed on $(0,x)$ with density $k_1(x,\,\cdot\,)$ [4, Chapter III, Proposition 2]. Then we perform induction for each $j\geq 1$: since $\tau_j<\infty$, $S^c_{\tau_j}>0$ and S^{j+1} is independent of $(S^c_{\tau_j},\tau_j)$, we have

$$\begin{aligned}\tau_{j+1}-\tau_j&=\inf\big\{s>\tau_j: S^c_{\tau_j}<-S^{j+1}_{s-\tau_j}\big\}-\tau_j\\&=\inf\big\{r>0: S^c_{\tau_j}<-S^{j+1}_r\big\}\\&=E_{j+1}(S^c_{\tau_j}).\end{aligned}\tag{4.4}$$

Combining (4.4) with $S^c_{\tau_j}<x$ and (4.2), we obtain

$$\mathbb{E}_x[\tau_{j+1}]=\mathbb{E}_x\big[E_{j+1}(S^c_{\tau_j})\big]+\mathbb{E}_x[\tau_j]\leq\mathbb{E}\big[E_{j+1}(x)\big]+\mathbb{E}_x[\tau_j]<\infty.$$

By definition and (4.4), we have

$$S^c_{\tau_{j+1}}=S^c_{\tau_j}+S^{j+1}_{(\tau_{j+1}-\tau_j)-}=S^c_{\tau_j}+S^{j+1}_{E_{j+1}(S^c_{\tau_j})-}\in(0,S^c_{\tau_j})\subseteq(0,x).$$

Therefore for any bounded measurable f, we have

$$\begin{aligned}\mathbb{E}_x\Big[f\big(S^c_{\tau_{j+1}}\big)\Big]&=\mathbb{E}_x\Big[f\Big(S^c_{\tau_j}+S^{j+1}_{E_{j+1}(S^c_{\tau_j})-}\Big)\Big]\\&=\int_0^x\left(\int_0^y f(z)k_1(y,z)\,\mathrm{d}z\right)k_j(x,y)\,\mathrm{d}y\\&=\int_0^x f(z)\left(\int_z^x k_j(x,y)\,k_1(y,z)\,\mathrm{d}y\right)\mathrm{d}z,\end{aligned}$$

where the second equality holds because $S^c_{\tau_j}$ is independent of S^{j+1} and has the density $k_j(x,\,\cdot\,)$; the last equality is due to Fubini's theorem. By Remark 2.5 we know that $S^c_{\tau_{j+1}}$ has the density $k_{j+1}(x,\,\cdot\,)$. The induction step is now complete.

For part (iii), by (4.4) we have $\mathbb{E}_x[\tau_{j+1}-\tau_j]=\mathbb{E}_x\big[E_{j+1}(S^c_{\tau_j})\big]$, meanwhile, since S^{j+1} is independent of $S^c_{\tau_j}$, by (4.2) we have

$$\mathbb{E}_x\Big[E_{j+1}\big(S^c_{\tau_j}\big)\Big]=\int_0^x\mathbb{E}\big[E_{j+1}(y)\big]k_j(x,y)\,\mathrm{d}y=\int_0^x\frac{y^\beta}{\Gamma(\beta+1)}k_j(x,y)\,\mathrm{d}y.$$

We now prove part (iv). The results obtained so far are enough to derive Theorem 4.1 below, which immediately implies that $\mathbb{P}_x[\tau_\infty<\infty]=1$. To prove $\mathbb{P}_x[S^c_{\tau_\infty-}>0]=0$, first, observe that

$$\mathbb{P}_x\big[S^c_{\tau_\infty-}>0\big]\leq\sum_{n=1}^\infty\mathbb{P}_x\big[S^c_{\tau_\infty-}\geq n^{-1}\big],$$

and for each $n\in\mathbb{N}$

$$\mathbb{P}_x\big[S^c_{\tau_\infty -} \geq n^{-1}\big] = \mathbb{P}_x\left[\bigcap_{j=1}^{\infty}\big\{S^c_{\tau_j} \geq n^{-1}\big\}\right] = \lim_{j\to\infty}\mathbb{P}_x\big[S^c_{\tau_j} \geq n^{-1}\big],$$

where we used $\{S^c_{\tau_j} \geq n^{-1}\} \supseteq \{S^c_{\tau_{j+1}} \geq n^{-1}\}$ for each $j \in \mathbb{N}$ and convergence from above of finite measures. Then, Chebyshev's inequality and the above results guarantee that

$$\frac{1}{n}\mathbb{P}_x\big[S^c_{\tau_j} \geq n^{-1}\big] \leq \mathbb{E}_x\big[S^c_{\tau_j}\big] = \int_0^x k_j(x,y)y\,\mathrm{d}y,$$

and the right-hand side goes to 0 as $j \to \infty$ by Lemma 2.3. □

We can now prove our main result of this subsection, which gives (1.8).

Theorem 4.1. *The hitting time of* 0 *of the censored β-stable Lévy process* (4.1) *is finite in expectation, with* $\mathbb{E}_x[\tau_\infty] = \mathbb{E}_x[\tau_1]\big(1-\sin(\beta\pi)/(\beta\pi)\big)^{-1}$.

Remark 4.1. Our key ingredient for proving Theorem 4.1 is the following closed formula for (4.3) (obtained in the proof of Lemma 2.3)

$$\int_0^x y^\beta k_j(x,y)\,\mathrm{d}y = x^\beta\big(\Gamma(\beta+1)\Gamma(1-\beta)\big)^{-j}, \quad \text{for all } j \in \mathbb{N} \text{ and } x > 0.$$

Proof. [of Theorem 4.1] On the one hand, by Monotone Convergence Theorem, $\mathbb{E}_x[\tau_\infty] = \lim_{j\to\infty}\mathbb{E}_x[\tau_{j+1}]$. On the other hand, by (4.2), (4.3) and Remark 4.1, for each $j \in \mathbb{N}$,

$$\mathbb{E}_x[\tau_{j+1}] = \mathbb{E}_x[\tau_1] + \sum_{i=1}^{j}\mathbb{E}_x[\tau_{i+1}-\tau_i] = \frac{x^\beta}{\Gamma(\beta+1)}\sum_{i=0}^{j}\big(\Gamma(\beta+1)\Gamma(1-\beta)\big)^{-i}.$$

As $\Gamma(\beta+1)\Gamma(1-\beta) = \beta\pi/\sin(\beta\pi) > 1$, the claim follows letting $j \to \infty$. □

Remark 4.2.

(i) Theorem 4.1 is not obvious. For instance, the censored *symmetric* β-stable process for $\beta \in (0,1)$ never hits the boundary, whether the censoring is performed in an interval or $\mathbb{R}_+$ [6, Theorem 1.1-(1)].
(ii) Any compound Poisson process in $\mathbb{R}^d$ censored upon exiting an open set must have infinite lifetime, and so does a non-increasing compound Poisson process censored in $(0,T]$. This is because the lifetime can be bounded from below by $\sum_{n=1}^{\infty} e_n = \infty$, where $\{e_n\}_{n\in\mathbb{N}}$ is an infinite subset of the *i.i.d.* exponential waiting times of the process.

(iii) The censored gamma subordinator with Lévy measure $\psi(r) = e^{-r}/r$ [7, Example 5.10] seems to have infinite lifetime, because our numerical simulations indicate pathwise that $\tau_j \approx 2\sqrt{j/3}$ and $S^c_{\tau_j} \approx \exp\{-\sqrt{3j}\}$ for $x = 1$ and $j \gg 1$. We do not know whether other censored (driftless) subordinators hit the barrier in finite time. If they do, it is not clear if our proof strategy can be extended to such cases, as it relies on the closed formula for the potential kernel, which is only available for the stable case.

4.2. **Probabilistic representations of I_0^β.** Firstly, we prove that I_0^β is equal to the potential of the semigroup of the censored process S^c. Secondly, we give a representation of I_0^β in terms of products of *i.i.d.* beta-distributed random variables.

PROPOSITION 4.1. *If g satisfies the assumption in Theorem 3.1 or if $g \in B[0,T]$, it holds that $I_0^\beta g \in C_\infty(0,T]$, and for all $x \in (0,T]$ we have*

$$I_0^\beta g(x) = \mathbb{E}_x\left[\int_0^{\tau_\infty} g(S^c_s)\,\mathrm{d}s\right]. \tag{4.5}$$

P r o o f. For $g \geq 0$ we justify the following equalities

$$\begin{aligned}
\mathbb{E}_x\left[\int_0^{\tau_\infty} g(S^c_s)\,\mathrm{d}s\right] &= \sum_{j=0}^{\infty}\mathbb{E}_x\left[\int_0^{\tau_{j+1}-\tau_j} g(S^c_{\tau_j+s})\,\mathrm{d}s\right]\\
&= \sum_{j=0}^{\infty}\mathbb{E}_x\left[\int_0^{E_{j+1}(S^c_{\tau_j})} g(S^c_{\tau_j}+S^{j+1}_s)\,\mathrm{d}s\right]\\
&= \sum_{j=0}^{\infty}\mathbb{E}_x\left[\mathbb{E}\left[\int_0^{E_{j+1}(S^c_{\tau_j})} g(S^c_{\tau_j}+S^{j+1}_s)\,\mathrm{d}s\,\Big|\,S^c_{\tau_j}\right]\right]\\
&= \sum_{j=0}^{\infty}\mathbb{E}_x\left[J_0^\beta g(S^c_{\tau_j})\right] = \sum_{j=0}^{\infty}\mathcal{K}^j J_0^\beta g(x) = I_0^\beta g(x).
\end{aligned} \tag{4.6}$$

The first equality is an application of Tonelli's Theorem and a simple change of variables; the second follows from (4.4); the third is due to the law of total expectation; the fourth is due to the independence of S^{j+1} and $S^c_{\tau_j}$ along with the known identity (1.6) (which is a straightforward consequence of [7, Eq. (1.38)]); the fifth follows from Lemmata 4.1-(i) and 2.3; the last follows the definition of I_0^β. If $g \in B[0,T]$, recalling that $J_0^\beta|g|(x) \leq \sup\{|g(y)| : y \in [0,x]\}x^\beta/\Gamma(\beta+1)$ and $J_0^\beta g \in C[0,T]$, by Lemma 2.3 we know that $\sum_{j=0}^{\infty}\mathcal{K}^j J_0^\beta g \in C_\infty(0,T]$, and that we can apply

Fubini's Theorem to the above equalities. If g satisfies the condition in Theorem 3.1, then Theorem 3.1 proves $I_0^\beta g \in C_\infty(0,T]$ and justifies the application of Fubini's Theorem. □

Remark 4.3.

(i) The above proof provides the following intuition for how ∂_0^β extends the memory effect of D_0^β. Rewrite the right-hand side of (4.5) as

$$\mathbb{E}\left[\int_0^{E_1(x)} g(x+S_s^1)\,\mathrm{d}s\right] + \mathbb{E}_x\left[\int_{\tau_1}^{\tau_\infty} g(S_s^c)\,\mathrm{d}s\right]. \tag{4.7}$$

Then the first term in (4.7) weights the *past* values of g on the interval $(x+S^1_{E_1(x)-}, x]$, just like (1.6) in the Caputo case (note that (1.6) takes a slightly different form, just because there we assume S^1 starts from x instead of 0). Meanwhile, the second term proceeds on the interval $(0, x+S^1_{E_1(x)-}]$ according to the censored process. This second term can be simplified further using the the distribution of $S^c_{\tau_j}$ and written in terms of products of *i.i.d.* beta-distributed random variables, as we will see in Proposition 4.2.

(ii) Proposition 4.1 proves that $\mathbb{E}_x\big[\int_0^{\tau_\infty}(S_s^c)^\alpha\,\mathrm{d}s\big]$ equals the right-hand side of (3.9). If $\alpha > -\beta$, it is finite and yields Theorem 4.1 (by letting $\alpha = 0$). If $\alpha \le -\beta$, then it is infinite by Remark 2.7. In contrast, $\mathbb{E}\big[\int_0^{E_1(x)}(x+S_s^1)^\alpha\,\mathrm{d}s\big] < \infty$ for all $\alpha > -1$.

Proposition 4.2. *For $x > 0$, $S^c_{\tau_j} \mid S^c_0 = x$ equals $X_j := x\prod_{i=1}^j B_i$ in law for each $j \in \mathbb{N}$, where $\{B_i\}_{i\in\mathbb{N}}$ is an i.i.d. collection of beta-distributed random variables on $(0,1)$ with parameters $(1-\beta,\beta)$. Moreover, under the assumption of Proposition 4.1, I_0^β allows the probabilistic series representation*

$$I_0^\beta g(x) = \sum_{j=0}^\infty \mathbb{E}_x\big[J_0^\beta g(X_j)\big], \quad x \in (0,T]. \tag{4.8}$$

The proof is straightforward and given in [16, Proposition 4.8].

4.3. **Laplace transform of τ_∞.** We recall some definitions adapted to our setting that relate to Feller semigroups [9]. A collection of operators $P = \{P_s\}_{s\ge 0}$ is said to be a *semigroup* on a Banach space X if $P_s : X \to X$ is bounded and linear for any $s > 0$, $P_sP_t = P_{s+t}$ for all $t,s > 0$, and P_0 is the identity operator. We say that P is *strongly continuous on $\mathcal{L} \subseteq X$* if for any $f \in \mathcal{L}$, $P_sf \to f$ in X as $s \to 0$, and that P is *strongly continuous* if P is strongly continuous on X. We define the *generator of P* to be the pair

$(\mathcal{G},\mathcal{D})$, where $\mathcal{D} := \{f \in X : \mathcal{G}f$ converges in $X\}$ with $\mathcal{G}f := \lim_{s\to 0}(P_s f - f)/s$, and we call $\mathcal{D}$ the *domain of the generator of* P. Moreover, P is said to be a *positivity preserving contraction* on $X \subseteq B[0,T]$ if $0 \le P_s f \le 1$ for any $s > 0$ and $f \in X$ such that $0 \le f \le 1$. Finally, a semigroup P on $X = C_\infty(0,T]$ is said to be a *Feller semigroup* if it is a strongly continuous positivity preserving contraction on X. We recall [9, Page 15] that there exists a one-to-one correspondence between Feller semigroups and Markov processes $\{Y_s\}_{s\ge 0}$ such that $f(\,\cdot\,) \mapsto P_s f(\,\cdot\,) := \mathbb{E}\big[f(Y_s)\,|\,Y_0 = \cdot\,\big]$, $s \ge 0$, is a Feller semigroup [9, Page 15].

Proposition 4.3. *For any $T > 0$, the censored decreasing β-stable process S^c induces a Feller semigroup on $C_\infty(0,T]$, whose generator is*

$$\left(-\partial_0^\beta, I_0^\beta C_\infty(0,T]\right).$$

P r o o f. Let $P_t^c f(x) = \mathbb{E}_x\big[f(S_t^c)\big]$ for $t \ge 0$ and $f \in B[0,T]$ (defining $S_t^c = \partial$ for all $t > 0$ if $S_0^c = 0$). Then $P^c = \{P_t^c\}_{t\ge 0}$ is a positivity preserving contraction semigroup on $B[0,T]$, due to S^c being a Markov process. We denote by $\mathcal{L}^c$ the largest subset of $B[0,T]$ on which P^c is strongly continuous and by $\mathcal{D}^c$ the domain of the generator of P^c. First we prove $\mathcal{L}^c \supseteq C_\infty(0,T]$. For $f \in C_\infty(0,T]$, let $\tilde f(x) := f\big(\max\{x,0\}\big)$ for any $x \in (-\infty,T]$, and compute

$$\begin{aligned}
\big|P_t^c f(x) - f(x)\big| &\le \Big|\mathbb{E}_x\big[\mathbf{1}_{\{t<\tau_1\}}(f(S_t^c) - f(x))\big]\Big| + \mathbb{E}_x\big[\mathbf{1}_{\{t\ge\tau_1\}}|f(S_t^c) - f(x)|\big]\\
&\le \Big|\mathbb{E}\big[\mathbf{1}_{\{t<E_1(x)\}}(\tilde f(x+S_t^1) - \tilde f(x))\big]\Big| + 2\|f\|_{C[0,x]}\mathbb{P}\big[t \ge E_1(x)\big]\\
&\le \Big|\mathbb{E}\big[\tilde f(x+S_t^1) - \tilde f(x)\big]\Big| + 3\|f\|_{C[0,x]}\mathbb{P}\big[t \ge E_1(x)\big],
\end{aligned}$$

where the first summand vanishes uniformly in x as $t \to 0$ because S^1 is a Feller process on $\big\{g \in C(-\infty,T] : \lim_{x\to-\infty} g(x) = 0\big\}$ [9]. Meanwhile for the second summand, for any $\varepsilon > 0$ we can choose $\delta > 0$ so that $\|f\|_{C[0,x]} \le \varepsilon$ for all $x \in (0,\delta]$ and then we choose $\tilde t$ small so that

$$\mathbb{P}\big[t \ge E_1(x)\big] = \mathbb{P}[x + S_t^1 \le 0] \le \mathbb{P}[\delta \le -S_t^1] \le \varepsilon, \quad \text{for all } x \ge \delta \text{ and } t \le \tilde t,$$

so for all $t \le \tilde t$

$$3\|f\|_{C[0,x]}\mathbb{P}\big[t \ge E_1(x)\big] \le \begin{cases} 3\varepsilon, & 0 \le x \le \delta,\\ 3\varepsilon\|f\|_{C[0,T]}, & \delta < x \le T. \end{cases} \tag{4.9}$$

Therefore we have proved the strong continuity of P^c on $C_\infty(0,T]$ and thus $C_\infty(0,T] \subseteq \mathcal{L}^c$. We now prove that $C_\infty(0,T]$ is invariant under P^c. The key ingredients are Theorem 4.1 and Proposition 4.1, which prove that I_0^β

equals the potential $\int_0^\infty P_s^c \,\mathrm{d}s$ and is a bounded operator from $B[0,T]$ to $C_\infty(0,T]$. Then [19, Theorem 1.1'] implies $\mathcal{D}^c = I_0^\beta \mathcal{L}^c$, and we have

$$I_0^\beta C_\infty(0,T] \subseteq I_0^\beta \mathcal{L}^c \subseteq I_0^\beta B[0,T] \subseteq C_\infty(0,T].$$

Since Stone–Weierstrass Theorem and Example 3.1 prove that $I_0^\beta C_\infty(0,T]$ is dense in $C_\infty(0,T]$, by [19, Property 1.3.B] and the above inclusions we obtain $\mathcal{L}^c = C_\infty(0,T]$. Because $P^c\mathcal{L}^c \subseteq \mathcal{L}^c$ [19, Property 1.3.A], we have proved that P^c is a Feller semigroup on $C_\infty(0,T]$. Since its potential is I_0^β and a bounded potential determines the generator [19, Theorem 1.1'], Theorem 3.1 implies that the generator of P^c is $\big(-\partial_0^\beta, I_0^\beta C_\infty(0,T]\big)$. □

We are now ready to prove a Mittag-Leffler-type representation for $\mathbb{E}_x\big[e^{\lambda\tau_\infty}\big]$, whose analogue in the Caputo setting is the probabilistic identity

$$\mathbb{E}_x\big[e^{\lambda\tau_1}\big] = \sum_{j=0}^{\infty} \frac{\lambda^j x^{\beta j}}{\Gamma(j\beta+1)}, \tag{4.10}$$

first proved in [5]. Our proof follows the approach of [24, Corollary 5.1] to (4.10). This approach allows one to solve exit problems by computing the Laplace transform of the lifetime of a killed Markov process when one knows the analytical solution to the resolvent equation $-\mathcal{G}u = \lambda u + g$ ($\mathcal{G}$ being the generator of the process). In our case, the analytical solution is given by Proposition 3.1.

THEOREM 4.2. *For every $\lambda < 0$, $T > 0$ and $g \in C[0,T]$,*

$$\mathbb{E}_x\left[\int_0^{\tau_\infty} e^{\lambda s} g(S_s^c)\,\mathrm{d}s\right] = \sum_{j=0}^{\infty} \lambda^j (I_0^\beta)^{j+1} g(x), \quad x \in (0,T]. \tag{4.11}$$

Moreover, the Mittag-Leffler-type series (1.7) *($u_0 = 1$) equals $\mathbb{E}_x\big[e^{\lambda\tau_\infty}\big]$ for all $\lambda \in \mathbb{R}$ and $x > 0$.*

P r o o f. For the first claim, if $g \in C_\infty(0,T]$, recalling [19, Theorem 1.1], the equality (4.11) holds by Propositions 4.3 and 3.1, as both sides of it are the unique solution in $C_\beta[0,T]$ to the resolvent equation

$$\partial_0^\beta u = \lambda u + g, \quad u(0) = 0, \quad g \in C_\infty(0,T],$$

where we used $I_0^\beta C_\infty(0,T] \subseteq C_\beta[0,T]$ given by Lemma 3.2. Now, for any $g \in C[0,T]$, take $g_n \in C_\infty(0,T]$ so that $g_n \to g$ uniformly on every compact subset of $(0,T]$ and $\sup_n \|g_n\|_{C[0,T]} < \infty$. Fix $x \in (0,T]$, then for any $s > 0$,

$$\mathbb{E}_x\big[g_n(S_s^c)\big] \to \mathbb{E}_x\big[g(S_s^c)\big], \quad \text{as } n \to \infty,$$

by Dominated Convergence Theorem. Applying Dominated Convergence Theorem again, with the dominating function $\sup_n \|g_n\|_{C[0,T]} e^{\lambda s}$, we obtain

$$\int_0^\infty e^{\lambda s}\mathbb{E}_x\big[g_n(S_s^c)\big]\,\mathrm{d}s \to \int_0^\infty e^{\lambda s}\mathbb{E}_x\big[g(S_s^c)\big]\,\mathrm{d}s, \quad \text{as } n\to\infty.$$

On the other hand, by the continuous dependence in Proposition 3.1 (let the (α,γ) there be $(\beta,\beta/2)$, for example),

$$\sum_{j=0}^\infty \lambda^j (I_0^\beta)^{j+1} g_n(x) \to \sum_{j=0}^\infty \lambda^j (I_0^\beta)^{j+1} g(x),\ \text{as } n\to\infty.$$

Therefore we have proved (4.11) for all $g\in C[0,T]$.

To prove the second claim for $\lambda<0$, in (4.11) let $g=\lambda$, so that on the left-hand side

$$\mathbb{E}_x\bigg[\int_0^{\tau_\infty} e^{\lambda s}\lambda\,\mathrm{d}s\bigg] = \lambda\frac{\mathbb{E}_x[e^{\lambda\tau_\infty}]-1}{\lambda} = \mathbb{E}_x\big[e^{\lambda\tau_\infty}\big]-1,$$

and on the right-hand side

$$\sum_{j=0}^\infty \lambda^{j+1}(I_0^\beta)^{j+1}1(x) = \sum_{j=0}^\infty \lambda^j (I_0^\beta)^j 1(x) - 1.$$

Hence by (3.11) (with $\alpha=\beta$) we have proved the second claim for $\lambda\le 0$ (with $\lambda=0$ being a trivial case), which combined with Lemma 3.3 allows us to compute the moments by differentiating $\mathbb{E}_x\big[e^{\lambda\tau_\infty}\big]$ in λ ($\lambda\le 0$) for $n\in\mathbb{N}$ times, i.e.

$$\mathbb{E}_x\big[(\tau_\infty)^n\big] = x^{\beta n} n! \prod_{j=1}^n \left(\frac{\Gamma(1+j\beta)}{\Gamma(j\beta+1-\beta)} - \frac{1}{\Gamma(1-\beta)}\right)^{-1}.$$

Those moments in turn allow us to prove the second claim also for $\lambda>0$, since we have

$$\mathbb{E}_x\big[e^{\lambda\tau_\infty}\big] = \sum_{j=0}^\infty \frac{\lambda^j}{j!}\mathbb{E}_x\big[(\tau_\infty)^j\big], \quad \lambda, x>0,$$

where the series in the right-hand side converges to (1.7) by Lemma 3.3. □

Corollary 4.1. *For any $\lambda<0$, the Mittag-Leffler-type series* (1.7) *is completely monotone. More generally, for any Bernstein function f the series* (1.7) *composed with $f^{1/\beta}$ is completely monotone.*

P r o o f. Denote by μ_1 the law of τ_∞ for $S_0^c=1$ and by $M_\lambda(x)$ the series (1.7) ($u_0=1$). Then

$$M_\lambda(x) = M_{\lambda x^\beta}(1) = \mathbb{E}_1\big[e^{(\lambda x^\beta)\tau_\infty}\big] = \int_{[0,\infty)} e^{\lambda x^\beta y}\mu_1(\mathrm{d}y),$$

where the second equality is due to Theorem 4.2. The second claim now follows from [45, Theorem 3.7] because M_λ composed with $f^{1/\beta}$ equals

$$x \mapsto \int_{[0,\infty)} e^{\lambda f(x) y} \mu_1(\mathrm{d}y),$$

the composition of $x \mapsto \int_{[0,\infty)} e^{\lambda xy}\mu_1(\mathrm{d}y)$ (which is completely monotone [45, Theorem 1.4]) with the Bernstein function f. The first claim corresponds to the Bernstein function $f(x) = x^\beta$. □

Remark 4.4. The proof of (1.9) in Theorem 4.2 suits well our IVP theory, but is rather indirect, especially when compared to the standard proofs of (4.10). However, we cannot adapt those standard proofs from the Caputo setting to our censored setting (see [16, Remark 4.14] for details).

Remark 4.5.

(i) Let $\tau_1(t)$, $\tau_\infty(t)$ and B denote $E_1(t)$ (i.e. the inverse stable subordinator), $\tau_\infty \,|\, S_0^c{=}t$ and an independent Brownian motion, respectively. It is known (e.g. [40]) that the Caputo time-fractional diffusion equation $D_0^\beta\big[u - u(0)\big] = \Delta u/2$ is solved by the fractional kinetic process $\{B_{\tau_1(t)}\}_{t\ge 0}$. This process is well-known as sub-diffusion since (4.2) implies $\mathbb{E}\big[|B_{\tau_1(t)}|^2\big] = \mathbb{E}\big[\tau_1(t)\big] = t^\beta/\Gamma(\beta{+}1)$, which is slower than normal diffusion $\mathbb{E}\big[|B_t|^2\big] = t$. Our work suggests that the censored counterpart $\partial_0^\beta u = \Delta u/2$ is solved by a new sub-diffusion process $\{B_{\tau_\infty(t)}\}_{t\ge 0}$. Indeed, Theorem 4.1 shows that $\mathbb{E}\big[|B_{\tau_\infty(t)}|^2\big] = ct^\beta$ $(c > 0)$, and we expect the time-fractional evolution equation $\partial_0^\beta u = \mathcal{G}u + g$, $u(0) = \phi$ to have a unique (generalised) solution

$$u(t,x) = \mathbb{E}\left[\phi(X_{\tau_\infty}) + \int_0^{\tau_\infty} g(S_s^c, X_s)\,\mathrm{d}s \;\middle|\; (S_0^c, X_0) = (t,x)\right],$$

where $(t,x) \in (0,T] \times \mathbb{R}^d$, $\phi \in \mathrm{Dom}(\mathcal{G})$, $g \in C\big([0,T]\times\mathbb{R}^d\big)$, and $\big(\mathcal{G}, \mathrm{Dom}(\mathcal{G})\big)$ is the generator of any Feller process X on $\mathbb{R}^d$ independent of S^c. (We think the last claim can be proved using the techniques from [17, 25], in the light of Proposition 4.3.) Let us also mention that to find strong solutions to $\partial_0^\beta u = \Delta u/2$, Theorem 4.2 opens up the possibility of applying the spectral decomposition method of [12].

(ii) Although both B_{τ_1} and B_{τ_∞} spread like t^β, their respective Fourier modes model entirely different relaxation regimes. Namely, we have

$$\mathbb{E}\big[\exp\{i\lambda\cdot B_{\tau_1(t)}\}\big] = \mathbb{E}\big[\exp\{-|\lambda|^2\tau_1(t)/2\}\big] \asymp t^{-\beta},$$
$$\mathbb{E}\big[\exp\{i\lambda\cdot B_{\tau_\infty(t)}\}\big] = \mathbb{E}\big[\exp\{-|\lambda|^2\tau_\infty(t)/2\}\big] \asymp t^{-1-\beta},$$

for any $\lambda \in \mathbb{R}^d$, by (4.10), Theorem 4.2 and Remark 3.4-(i). Here $f \asymp g$ means $C^{-1}g \leq f \leq Cg$ for some constant $C > 1$.

(iii) There are several interesting questions revolving around B_{τ_∞}, a new example of anomalous diffusion. For instance, it is natural to ask if there is a continuous-time-random-walk-type framework which scales to B_{τ_∞}, as is the case for B_{τ_1} [1, 40] and several other anomalous diffusion processes [2, 50] related to Caputo derivatives. Moreover, it is challenging and interesting to study the difference in path regularity between B_{τ_∞} and B_{τ_1}, in particular because the latter can be "trapped" [40].

(iv) We mention that sub-diffusion and fractional relaxation equations are widely used to model anomalous (non-Debye) relaxation in dielectrics, see [30, 49, 32, 50] and references therein. Their role is to provide a probabilistic theoretic explanation of the empirical (Havriliak–Negami) formula $\chi(\omega) = \big(1 + (i\omega)^\alpha\big)^{-\gamma}$. (Here ω is the electric field's frequency and χ is the electric susceptibility. This formula fits well a majority of experimental data.) A typical example (Cole–Cole) is $\alpha = \beta \in (0, 1)$ and $\gamma = 1$, which is modelled by the sub-diffusion B_{τ_1} [49, page 3]. On the other hand, we expect B_{τ_∞} to model a new regime with $\alpha = 1 + \beta \in (1, 2)$ and $\gamma = \beta/(1 + \beta)$ (by [49, Eq. (1) and (5)]), although in the literature (e.g. [32, 49]) we have not seen the parameter range $\alpha > 1$.

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[1] *Department of Applied Physics and Applied Mathematics,*
and Data Science Institute
Columbia University, New York 10027, USA
e-mail: qd2125@columbia.edu

[2] *Department of Mathematics and Statistics*
Otago University, Dunedin 9054, NEW ZEALAND
e-mail: ltoniazzi@maths.otago.ac.nz (Corresponding author)
Received: December 31, 2020

[3] *Department of Applied Physics and Applied Mathematics*
Columbia University, New York 10027, USA
e-mail: zx2250@columbia.edu

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