Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME 24, NUMBER 4 (2021)
(Print) ISSN 1311-0454
(Electronic) ISSN 1314-2224

RESEARCH PAPER

FRACTIONAL DIFFUSION-WAVE EQUATIONS: HIDDEN REGULARITY FOR WEAK SOLUTIONS

Paola Loreti [1], **Daniela Sforza** [2]

Abstract

We prove a "hidden" regularity result for weak solutions of time fractional diffusion-wave equations where the Caputo fractional derivative is of order $\alpha \in (1,2)$. To establish such result we analyse the regularity properties of the weak solutions in suitable interpolation spaces.

MSC 2010: Primary 35R11; Secondary 26A33, 35L05

Key Words and Phrases: Caputo fractional derivative; fractional diffusion-wave equations; hidden regularity; Riemann–Liouville fractional integral

1. Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a bounded open set with sufficiently smooth boundary $\partial\Omega$. Our target is to show some regularity properties for the weak solutions of the time fractional diffusion-wave equation

$$\partial_t^\alpha u(t,x) = \triangle u(t,x), \qquad t \geq 0, \quad x \in \Omega, \tag{1.1}$$

where the symbol $\partial_t^\alpha u$ denotes the Caputo fractional derivative of order $\alpha \in (1,2)$, defined by

$$\partial_t^\alpha u(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-\tau)^{1-\alpha} \frac{d^2 u}{d\tau^2}(\tau)\, d\tau\,,$$

(Γ is the Euler Gamma function), see e.g. [4, 8, 20, 21, 23].

pp. 1015–1034 , DOI: 10.1515/fca-2021-0044 DE GRUYTER

It is well known that (1.1) interpolates the heat equation and the wave equation, because the weak solutions of (1.1) exhibit some typical properties of the heat equation and, at the same time, others which are characteristic of the wave equation (see e.g. [3, 16]). From a mathematical point of view one can obtain (1.1) from the heat equation or from the wave equation by replacing the first order time-derivative ∂_t or the second order time-derivative ∂_t^2 with the fractional derivative ∂_t^α. For a general discussion about the type of regularity required for solutions to fractional differential equations, see [24].

In this paper we concentrate our study into establishing direct inequalities and hidden regularity for weak solutions of (1.1), that are peculiar results for the wave equations as shown by the existing literature. Indeed, hidden regularity results have been proved for wave equations, for direct PDEs methods see e.g. [10], while, as regards the Hilbert Uniqueness Method, see e.g. [11, 12].

Hidden regularities do not follow from classical trace theorems. As well known, by the trace theory in Sobolev spaces one can define for any function $u \in H^2(\Omega)$ the normal derivative $\partial_\nu u$. On the other hand, in general, the weak solution u of a Cauchy problem for the wave equation with Dirichlet boundary conditions does not satisfy such regularity: this is why the condition

$$\partial_\nu u \in L^2_{loc}(\mathbb{R}; L^2(\partial\Omega))$$

is known as a "hidden" regularity property of the weak solution.

In control theory the hidden regularity follows by the direct inequality, that is a fundamental step to get exact controllability for distributed system by means of the Hilbert Uniqueness Method of J.-L. Lions. Indeed, one has to prove that for all $T > 0$ there exists a positive constant $C = C(T)$ such that

$$\int_0^T \int_{\partial\Omega} \left|\partial_\nu u\right|^2 d\sigma dt \le C\Big(\|\nabla u_0\|^2_{L^2(\Omega)} + \|u_1\|^2_{L^2(\Omega)}\Big).$$

For further results concerning the hidden regularity for wave equations and wave equations with memory the interested reader can see e.g. [1, 13, 14, 18] and references therein.

As regards the time-fractional diffusion-wave equation (1.1), first the analysis requires a detailed study of the existence and regularity of weak solutions in suitable spaces. To this end, we borrow from [22] the existing theory, that we have to integrate in order to state and prove the result about the hidden regularity for weak solutions of (1.1).

In detail we establish the following regularity results for a time-fractional diffusion-wave Cauchy problem with Dirichlet boundary conditions. Recall that A denotes the operator in $L^2(\Omega)$ defined by

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega)$$
$$(Au)(x) = -\triangle u(x), \quad u \in D(A),\ x \in \Omega.$$

The fractional powers A^θ of the operator A are defined for $\theta > 0$, see e.g. [19]. Moreover, $D(A^{-\theta}) := (D(A^\theta))'$.

Theorem 1.1. *Let $\alpha \in (1,2)$ and $T > 0$. If $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$, then the unique weak solution u of problem*

$$\begin{cases} \partial_t^\alpha u(t,x) = \triangle u(t,x)\,, & t \geq 0,\ x \in \Omega, \\ u(t,x) = 0 & t \geq 0,\ x \in \partial\Omega, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in \Omega, \end{cases} \tag{1.2}$$

belongs to $C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; D(A^{-\theta}))$, with $\theta \in \big(\frac{2-\alpha}{2\alpha}, \frac{1}{2}\big]$,

$$\lim_{t\to 0} \|\nabla u(t,\cdot) - \nabla u_0\|_{L^2(\Omega)} = \lim_{t\to 0} \|\partial_t u(t,\cdot) - u_1\|_{D(A^{-\theta})} = 0\,,$$

and for some constant $C > 0$

$$\|\nabla u\|_{C([0,T];L^2(\Omega))} + \|\partial_t u\|_{C([0,T];D(A^{-\theta}))} \leq C\big(\|\nabla u_0\|_{L^2(\Omega)} + \|u_1\|_{L^2(\Omega)}\big).$$

In addition, for any $\theta_1 \in \big(0, \frac{1}{2\alpha}\big)$, $\theta_2 \in \big(\frac{\alpha-1}{2\alpha}, \frac{1}{2}\big)$ and some $C > 0$ we have

$$\|\nabla u\|_{L^2(0,T;D(A^{\theta_1}))} + \|\partial_t^\alpha u\|_{L^2(0,T;D(A^{-\theta_2}))} \leq C\big(\|\nabla u_0\|_{L^2(\Omega)} + \|u_1\|_{L^2(\Omega)}\big). \tag{1.3}$$

It is noteworthy to observe that the assumption $1 < \alpha < 2$ on the order α of the fractional derivative is essential in Theorem 1.1. Indeed, in the estimate (1.3) the available intervals $\big(0, \frac{1}{2\alpha}\big)$ and $\big(\frac{\alpha-1}{2\alpha}, \frac{1}{2}\big)$ of the exponents θ_1 and θ_2 make sense just thanks to the condition $\alpha \in (1,2)$.

Moreover, if we assume $\nabla u_0 \in D(A^\theta)$ with $\theta \in \big(\frac{2-\alpha}{2\alpha}, \frac{1}{2}\big)$, then

$$\lim_{t\to 0} \|\partial_t u(t,\cdot) - u_1\|_{L^2(\Omega)} = 0\,,$$
$$\|\partial_t u\|_{C([0,T];L^2(\Omega))} \leq C(\|\nabla u_0\|_{D(A^\theta)} + \|u_1\|_{L^2(\Omega)}).$$

We observe that to assure a regularity of $\partial_t u$ in $L^2(\Omega)$, we have to assume the datum u_0 belonging to a proper subset of $H_0^1(\Omega) = D(A^{\frac{1}{2}})$, that is $\nabla u_0 \in D(A^\theta)$ with $\theta \in \big(\frac{2-\alpha}{2\alpha}, \frac{1}{2}\big)$.

The properties of the weak solutions proved in Theorem 1.1 are fundamental to obtain the following hidden regularity result. Precisely, in the proof of Theorem 1.2 the crucial point will be to have, at the same time,

$\nabla u \in L^2(0,T;D(A^\theta))$ and $\partial_t^\alpha u \in L^2(0,T;D(A^{-\theta}))$, for $\theta \in (0,1)$. Indeed, thanks to (1.3) it will be possible to choose such exponent θ, since the intersection of the available intervals $\big(0, \frac{1}{2\alpha}\big)$ and $\big(\frac{\alpha-1}{2\alpha}, \frac{1}{2}\big)$ is non empty for $\alpha \in (1,2)$.

THEOREM 1.2. *Let $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$ and $T > 0$. If u is the weak solution of* (1.2), *then we define the normal derivative $\partial_\nu u$ of u such that we have*

$$\int_0^T \int_{\partial\Omega} \big|\partial_\nu u\big|^2 d\sigma dt \le C\Big(\|\nabla u_0\|^2_{L^2(\Omega)} + \|u_1\|^2_{L^2(\Omega)}\Big), \tag{1.4}$$

for some constant $C = C(T)$ independent of the initial data u_0 and u_1.

The paper consists of four sections. In Section 2 we list some notations, definitions and known results that we use to prove Theorems 1.1 and 1.2. Section 3 is devoted to show the regularity of the weak solution for initial data $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$, that is the proof of Theorem 1.1 is given. In Section 4 first we state and prove some technical results. Finally, we demonstrate Theorem 1.2, that establish the hidden regularity for weak solutions.

2. **Preliminaries**

In this section we get together some notations, definitions and known results that we need to introduce and prove Theorems 1.1 and 1.2.

Let $\Omega \subset \mathbb{R}^N$, $N \ge 1$, be a bounded open set with C^2 boundary $\partial\Omega$. As usual, we consider $L^2(\Omega)$ endowed with the inner product and norm defined by

$$\langle u, v\rangle = \int_\Omega u(x)v(x)\, dx, \quad \|u\|_{L^2(\Omega)} = \left(\int_\Omega |u(x)|^2\, dx\right)^{1/2}, \quad u, v \in L^2(\Omega).$$

DEFINITION 2.1. For any $f \in L^1(0,T)$, $T > 0$, we denote the Riemann–Liouville fractional integral operator I^β of order β, $\beta > 0$, by

$$I^\beta(f)(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} f(\tau)\, d\tau, \qquad \text{a.e. } t \in (0,T). \tag{2.1}$$

We define the operator A in $L^2(\Omega)$ by

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega)$$
$$(Au)(x) = -\triangle u(x), \quad u \in D(A),\ x \in \Omega.$$

The fractional powers A^θ are defined for $\theta > 0$, see e.g. [19] and [15, Example 4.34]. We recall that the spectrum of A consists of a sequence of positive eigenvalues tending to $+\infty$ and there exists an orthonormal basis of $L^2(\Omega)$ consisting of eigenvectors of A. Moreover, we assume that the eigenvalues are distinct numbers. We denote such a basis by $\{e_n\}_{n\in\mathbb{N}}$ and by λ_n the eigenvalue with eigenvector e_n, that is $Ae_n = \lambda_n e_n$. Then, for $\theta > 0$ the domain $D(A^\theta)$ of A^θ consists of those functions $u \in L^2(\Omega)$ such that

$$\sum_{n=1}^{\infty} \lambda_n^{2\theta} |\langle u, e_n\rangle|^2 < +\infty$$

and

$$A^\theta u = \sum_{n=1}^{\infty} \lambda_n^{\theta} \langle u, e_n\rangle e_n, \quad u \in D(A^\theta).$$

Moreover, $D(A^\theta)$ is a Hilbert space with the norm given by

$$\|u\|_{D(A^\theta)} = \|A^\theta u\|_{L^2(\Omega)} = \left(\sum_{n=1}^{\infty} \lambda_n^{2\theta} |\langle u, e_n\rangle|^2\right)^{1/2}, \quad u \in D(A^\theta). \tag{2.2}$$

We have $D(A^\theta) \subset H^{2\theta}(\Omega)$. In particular, $D(A^{\frac{1}{2}}) = H_0^1(\Omega)$.

If we identify the dual $(L^2(\Omega))'$ with $L^2(\Omega)$ itself, then we have $D(A^\theta) \subset L^2(\Omega) \subset (D(A^\theta))'$. From now on we set

$$D(A^{-\theta}) := (D(A^\theta))', \tag{2.3}$$

whose elements are bounded linear functionals on $D(A^\theta)$. If $\varphi \in D(A^{-\theta})$ and $u \in D(A^\theta)$ the value $\varphi(u)$ is denoted by

$$\langle \varphi, u\rangle_{-\theta,\theta} := \varphi(u)\,. \tag{2.4}$$

In addition, $D(A^{-\theta})$ is a Hilbert space with the norm given by

$$\|\varphi\|_{D(A^{-\theta})} = \left(\sum_{n=1}^{\infty} \lambda_n^{-2\theta} |\langle \varphi, e_n\rangle_{-\theta,\theta}|^2\right)^{1/2}, \quad \varphi \in D(A^{-\theta})\,. \tag{2.5}$$

We also recall that

$$\langle \varphi, u\rangle_{-\theta,\theta} = \langle \varphi, u\rangle \qquad \text{for } \varphi \in L^2(\Omega)\,, u \in D(A^\theta), \tag{2.6}$$

see e.g. [2, Chapitre V].

For arbitrary $\alpha, \beta > 0$, we denote the Mittag–Leffler function by

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C}. \tag{2.7}$$

The power series $E_{\alpha,\beta}(z)$ defines an entire function of $z \in \mathbb{C}$. The Mittag–Leffler function $E_{\alpha,1}(z)$ is usually denoted by $E_\alpha(z)$.

The proof of the following result can be found in [20, p. 35], see also [22, Lemma 3.1].

LEMMA 2.1. *Let $\alpha \in (1,2)$ and $\beta > 0$. Then for any $\mu \in \mathbb{R}$ such that $\pi\alpha/2 < \mu < \pi$ there exists a constant $C = C(\alpha, \beta, \mu) > 0$ such that*

$$\big|E_{\alpha,\beta}(z)\big| \le \frac{C}{1+|z|}, \qquad z \in \mathbb{C},\ \mu \le |\arg(z)| \le \pi. \tag{2.8}$$

We also exhibit an elementary result that will be useful in the estimates.

LEMMA 2.2. *For any $0 < \beta < 1$ the function $x \to \frac{x^\beta}{1+x}$ gains its maximum on $[0, +\infty[$ at point $\frac{\beta}{1-\beta}$ and the maximum value is given by*

$$\max_{x\ge 0} \frac{x^\beta}{1+x} = \beta^\beta (1-\beta)^{1-\beta}, \qquad \beta \in (0,1)\,. \tag{2.9}$$

Now we recall the definition of fractional vector-valued Sobolev spaces. For $\beta \in (0,1)$, $T > 0$ and a Hilbert space H, endowed with the norm $\|\cdot\|_H$, $H^\beta(0,T;H)$ is the space of all $u \in L^2(0,T;H)$ such that

$$[u]_{H^\beta(0,T;H)} := \left(\int_0^T \int_0^T \frac{\|u(t)-u(\tau)\|_H^2}{|t-\tau|^{1+2\beta}}\, dt d\tau\right)^{1/2} < +\infty\,, \tag{2.10}$$

i.e. $[u]_{H^\beta(0,T;H)}$ is the so-called Gagliardo semi-norm of u. $H^\beta(0,T;H)$ is endowed with the norm

$$\|u\|_{H^\beta(0,T;H)} := \|u\|_{L^2(0,T;H)} + [u]_{H^\beta(0,T;H)}, \qquad u \in H^\beta(0,T;H). \tag{2.11}$$

The following extension of a known result (see [6, Theorem 2.1]) to the case of vector valued functions will be relevant in the proof of Theorem 1.2. We will use the symbol $\sim$ between norms to indicate two equivalent norms.

THEOREM 2.1. *Let H be a separable Hilbert space.*

(i) *The Riemann–Liouville operator $I^\beta : L^2(0,T;H) \to L^2(0,T;H)$, $0 < \beta \le 1$, is injective and the range $\mathcal{R}(I^\beta)$ of I^β is given by*

$$\mathcal{R}(I^\beta) = \begin{cases} H^\beta(0,T;H), & 0 < \beta < \frac{1}{2}, \\ \Big\{v \in H^{\frac{1}{2}}(0,T;H) : \int_0^T t^{-1}|v(t)|^2 dt < \infty\Big\}, & \beta = \frac{1}{2}, \\ {}_0H^\beta(0,T;H), & \frac{1}{2} < \beta \le 1, \end{cases}$$

where ${}_0H^\beta(0,T) = \{u \in H^\beta(0,T) : u(0) = 0\}$.

(ii) *For the Riemann–Liouville operator I^β and its inverse operator $I^{-\beta}$ the norm equivalences*

$$\begin{aligned} &\|I^\beta(u)\|_{H^\beta(0,T;H)} \sim \|u\|_{L^2(0,T;H)}, \qquad u \in L^2(0,T;H), \\ &\|I^{-\beta}(v)\|_{L^2(0,T;H)} \sim \|v\|_{H^\beta(0,T;H)}, \qquad v \in \mathcal{R}(I^\beta), \end{aligned} \tag{2.12}$$

hold true.

The Caputo fractional derivative of order $\alpha \in (1,2)$ is defined by

$$\partial_t^\alpha f(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-\tau)^{1-\alpha} \frac{d^2 f}{d\tau^2}(\tau)\, d\tau = I^{2-\alpha}\Big(\frac{d^2 f}{dt^2}\Big)(t)\,, \tag{2.13}$$

involving the Riemann–Liouville fractional integral $I^{2-\alpha}$, see (2.1).

For the sake of completeness, we recall the notion of weak solutions for fractional diffusion-wave equations, see [22, Definition 2.1].

Definition 2.2. Let $\alpha \in (1,2)$ and $T > 0$. We define u as a weak solution to the problem

$$\begin{cases} \partial_t^\alpha u(t,x) = \triangle u(t,x) & t \in (0,T),\ x \in \Omega, \\ u(t,x) = 0 & t \in (0,T),\ x \in \partial\Omega, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in \Omega, \end{cases} \tag{2.14}$$

if $\partial_t^\alpha u(t,\cdot) = \triangle u(t,\cdot)$ holds in $L^2(\Omega)$, $u(t,\cdot) \in H_0^1(\Omega)$ for almost all $t \in (0,T)$ and for some $\theta > 0$, depending on the initial data u_0, u_1, one has $u, \partial_t u \in C([0,T]; D(A^{-\theta}))$ and

$$\lim_{t\to 0} \|u(t,\cdot) - u_0\|_{D(A^{-\theta})} = \lim_{t\to 0} \|\partial_t u(t,\cdot) - u_1\|_{D(A^{-\theta})} = 0\,.$$

We also need to recall some existence results given in [22, Theorem 2.3], that we have integrated with other essential regularity properties of the solution, see (2.15) below.

Theorem 2.2. *(i) Let $u_0 \in L^2(\Omega)$ and $u_1 \in D(A^{-\frac{1}{\alpha}})$. Then there exists a unique weak solution $u \in C([0,T]; L^2(\Omega)) \cap C((0,T]; H^2(\Omega) \cap H_0^1(\Omega))$ to (2.14) with $\partial_t^\alpha u \in C((0,T]; L^2(\Omega))$ and satisfying*

$$\lim_{t\to 0} \|u(t,\cdot) - u_0\|_{L^2(\Omega)} = 0\,,$$

$$\|u\|_{C([0,T];L^2(\Omega))} \le C\big(\|u_0\|_{L^2(\Omega)} + \|u_1\|_{D(A^{-\frac{1}{\alpha}})}\big)\,,$$

$$\begin{aligned} &\lim_{t\to 0} \|\partial_t u(t,\cdot) - u_1\|_{D(A^{-\theta})} = 0\,, \qquad \theta \in \Big(\frac{1}{\alpha}, 1\Big), \\ &\|\partial_t u\|_{C([0,T];D(A^{-\theta}))} \le C\big(\|u_0\|_{L^2(\Omega)} + \|u_1\|_{D(A^{-\frac{1}{\alpha}})}\big)\,, \end{aligned} \tag{2.15}$$

for some constant $C > 0$.

Moreover, if $u_1 \in L^2(\Omega)$ we have

$$u(t,x) = \sum_{n=1}^\infty \big[\langle u_0, e_n\rangle E_\alpha(-\lambda_n t^\alpha) + \langle u_1, e_n\rangle t E_{\alpha,2}(-\lambda_n t^\alpha)\big] e_n(x), \tag{2.16}$$

$$\partial_t u(t,x) = \sum_{n=1}^{\infty} \big[-\lambda_n \langle u_0, e_n\rangle t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha) + \langle u_1, e_n\rangle E_\alpha(-\lambda_n t^\alpha)\big] e_n(x), \tag{2.17}$$

$$\partial_t^\alpha u(t,x) = -\sum_{n=1}^{\infty} \lambda_n \big[\langle u_0, e_n\rangle E_\alpha(-\lambda_n t^\alpha) + \langle u_1, e_n\rangle t E_{\alpha,2}(-\lambda_n t^\alpha)\big] e_n(x)\,, \tag{2.18}$$

$$\|\partial_t u(t,\cdot)\|_{L^2(\Omega)} \le C\big(t^{-1}\|u_0\|_{L^2(\Omega)} + \|u_1\|_{L^2(\Omega)}\big) \qquad (C>0)\,.$$

(ii) If $u_0 \in H^2(\Omega)\cap H_0^1(\Omega)$ and $u_1 \in H_0^1(\Omega)$, then the unique weak solution u to (2.14) *given by* (2.16) *belongs to $C([0,T];H^2(\Omega)\cap H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega))$ and $\partial_t^\alpha u \in C([0,T];L^2(\Omega))$.*

P r o o f. We refer to [22, Theorem 2.3] for the proof of all statements, except for the proof of (2.15). We first observe that, since $u_1 \in D(A^{-\frac{1}{\alpha}})$, thanks to the duality (2.4) the expression (2.17) for $\partial_t u$ has to be written in the form

$$\partial_t u(t,x) = \sum_{n=1}^{\infty} \big[-\lambda_n \langle u_0, e_n\rangle t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha) + \langle u_1, e_n\rangle_{-\frac{1}{\alpha},\frac{1}{\alpha}} E_\alpha(-\lambda_n t^\alpha)\big] e_n(x)\,.$$

For $\theta\in(0,1)$ to choose suitably later, taking into account (2.5) we have

$$\begin{aligned}\|\partial_t u(t,\cdot) - u_1\|^2_{D(A^{-\theta})} &= \sum_{n=1}^{\infty} \lambda_n^{-2\theta} \big| -\lambda_n \langle u_0, e_n\rangle t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha) + \langle u_1, e_n\rangle_{-\frac{1}{\alpha},\frac{1}{\alpha}} \big(E_\alpha(-\lambda_n t^\alpha) - 1\big)\big|^2 \\ &\le 2t^{2(\alpha-1)} \sum_{n=1}^{\infty} \lambda_n^{2(1-\theta)}\ |\langle u_0, e_n\rangle E_{\alpha,\alpha}(-\lambda_n t^\alpha)|^2 \\ &\quad + 2\sum_{n=1}^{\infty} \lambda_n^{-2\theta} \big|\langle u_1, e_n\rangle_{-\frac{1}{\alpha},\frac{1}{\alpha}} \big(E_\alpha(-\lambda_n t^\alpha) - 1\big)\big|^2\,. \end{aligned} \tag{2.19}$$

To estimate the first sum we use (2.8) and (2.9) to get

$$t^{2(\alpha-1)} \lambda_n^{2(1-\theta)}\ |\langle u_0, e_n\rangle E_{\alpha,\alpha}(-\lambda_n t^\alpha)|^2 \le C t^{2(\alpha\theta-1)} \Big(\frac{(\lambda_n t^\alpha)^{1-\theta}}{1+\lambda_n t^\alpha}\Big)^2 |\langle u_0, e_n\rangle|^2 \le C t^{2(\alpha\theta-1)} |\langle u_0, e_n\rangle|^2\,,$$

while, regarding the second sum, we have

$$\lambda_n^{-2\theta}\big|\langle u_1, e_n\rangle_{-\frac{1}{\alpha},\frac{1}{\alpha}}\big(E_\alpha(-\lambda_n t^\alpha)-1\big)\big|^2 = \lambda_n^{-2(\theta-\frac{1}{\alpha})}\lambda_n^{-\frac{2}{\alpha}}|\langle u_1, e_n\rangle_{-\frac{1}{\alpha},\frac{1}{\alpha}}|^2\big|E_\alpha(-\lambda_n t^\alpha)-1\big|^2 .$$

Therefore, plugging the above two estimates into (2.19) we obtain

$$\|\partial_t u(t,\cdot)-u_1\|^2_{D(A^{-\theta})} \le Ct^{2(\alpha\theta-1)}\|u_0\|^2_{L^2(\Omega)} + 2\sum_{n=1}^{\infty}\lambda_n^{-2(\theta-\frac{1}{\alpha})}\lambda_n^{-\frac{2}{\alpha}}|\langle u_1, e_n\rangle_{-\frac{1}{\alpha},\frac{1}{\alpha}}|^2\big|E_\alpha(-\lambda_n t^\alpha)-1\big|^2 ,$$

whence it follows that for $\theta > \frac{1}{\alpha}$ (2.15) holds true. □

3. Regularity in the case $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$

We establish a result about the regularity of the weak solutions assuming on the data u_0 and u_1 a degree of regularity intermediate between those assumed in (i) and (ii) of Theorem 2.2.

For further results about existence and regularity of solutions, see [7, 17].

THEOREM 3.1. *If $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$, then the unique weak solution u to (2.14) given by (2.16)–(2.18) belongs to $C([0,T];H_0^1(\Omega)) \cap C^1([0,T];D(A^{-\theta}))$, $\theta \in \big(\frac{2-\alpha}{2\alpha},\frac{1}{2}\big]$, and*

$$\begin{aligned}
&\lim_{t\to 0}\|\nabla u(t,\cdot)-\nabla u_0\|_{L^2(\Omega)} = 0\\
&\lim_{t\to 0}\|\partial_t u(t,\cdot)-u_1\|_{D(A^{-\theta})} = 0 ,\\
&\|\nabla u\|_{C([0,T];L^2(\Omega))} + \|\partial_t u\|_{C([0,T];D(A^{-\theta}))}\\
&\qquad\qquad \le C(\|\nabla u_0\|_{L^2(\Omega)} + \|u_1\|_{L^2(\Omega)}),
\end{aligned} \tag{3.1}$$

for some constant $C > 0$. In addition, for any $\theta \in \big(0,\frac{1}{2\alpha}\big)$ we have

$$\|\nabla u\|_{L^2(0,T;D(A^\theta))} \le C\big(\|\nabla u_0\|_{L^2(\Omega)} + \|u_1\|_{L^2(\Omega)}\big), \tag{3.2}$$

and for any $\theta \in \big(\frac{\alpha-1}{2\alpha},\frac{1}{2}\big)$ we have

$$\|\partial_t^\alpha u\|_{L^2(0,T;D(A^{-\theta}))} \le C\big(\|\nabla u_0\|_{L^2(\Omega)} + \|u_1\|_{L^2(\Omega)}\big) , \tag{3.3}$$

for some constants $C > 0$.

If we assume, in addition, that $\nabla u_0 \in D(A^\theta)$ with $\theta \in \big(\frac{2-\alpha}{2\alpha},\frac{1}{2}\big)$, then

$$\begin{aligned}
&\lim_{t\to 0}\|\partial_t u(t,\cdot)-u_1\|_{L^2(\Omega)} = 0 ,\\
&\|\partial_t u\|_{C([0,T];L^2(\Omega))} \le C(\|\nabla u_0\|_{D(A^\theta)} + \|u_1\|_{L^2(\Omega)}).
\end{aligned} \tag{3.4}$$

P r o o f. In virtue of the expression (2.16) for the solution u and (2.8) we have

$$\begin{aligned}&\|\nabla u(t,\cdot)-\nabla u_0\|^2_{L^2(\Omega)}\\&\quad=\sum_{n=1}^\infty \lambda_n\big|\langle u_0,e_n\rangle\big(E_\alpha(-\lambda_n t^\alpha)-1\big)+\langle u_1,e_n\rangle tE_{\alpha,2}(-\lambda_n t^\alpha)\big|^2\\&\qquad\le 2\sum_{n=1}^\infty \lambda_n\big|\langle u_0,e_n\rangle\big|^2\big|E_\alpha(-\lambda_n t^\alpha)-1\big|^2\\&\qquad\quad+t^{2-\alpha}2C^2\sum_{n=1}^\infty\big|\langle u_1,e_n\rangle\big|^2\Big(\frac{(\lambda_n t^\alpha)^{\frac12}}{1+\lambda_n t^\alpha}\Big)^2. \qquad (3.5)\end{aligned}$$

We observe that for any $n\in\mathbb{N}$ $\lim_{t\to0}\big(E_\alpha(-\lambda_n t^\alpha)-1\big)=0$. Moreover, again by (2.8), we get for $n\in\mathbb{N}$ and $0\le t\le T$:

$$\begin{aligned}&\lambda_n\big|\langle u_0,e_n\rangle\big|^2\big|E_\alpha(-\lambda_n t^\alpha)-1\big|^2\\&\qquad\le 2\lambda_n\big|\langle u_0,e_n\rangle\big|^2\Big(\frac{C}{(1+\lambda_n t^\alpha)^2}+1\Big)\le C\lambda_n\big|\langle u_0,e_n\rangle\big|^2,\end{aligned}$$

hence by (3.5) we deduce $\lim_{t\to0}\|\nabla u(t,\cdot)-\nabla u_0\|_{L^2(\Omega)}=0$ and for any $t\in[0,T]$

$$\|\nabla u(t,\cdot)\|^2_{L^2(\Omega)}\le C(\|\nabla u_0\|^2_{L^2(\Omega)}+\|u_1\|^2_{L^2(\Omega)}).$$

To complete the proof of (3.1), we fix $\theta\in\big(\frac{2-\alpha}{2\alpha},\frac12\big]$ and use formula (2.17) to note that

$$\begin{aligned}&\|\partial_t u(t,\cdot)-u_1\|^2_{D(A^{-\theta})}\\&\quad=\sum_{n=1}^\infty\lambda_n^{-2\theta}\big|-\lambda_n\langle u_0,e_n\rangle t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n t^\alpha)+\langle u_1,e_n\rangle\big(E_\alpha(-\lambda_n t^\alpha)-1\big)\big|^2\\&\qquad\le Ct^{\alpha-2+2\alpha\theta}\sum_{n=1}^\infty\lambda_n\big|\langle u_0,e_n\rangle\big|^2\left(\frac{(\lambda_n t^\alpha)^{\frac{1-2\theta}{2}}}{1+\lambda_n t^\alpha}\right)^2\\&\qquad\quad+2\sum_{n=1}^\infty\lambda_n^{-2\theta}\big|\langle u_1,e_n\rangle\big|^2\big|E_\alpha(-\lambda_n t^\alpha)-1\big|^2, \qquad (3.6)\end{aligned}$$

thanks also to (2.8). Since $0<\frac{1-2\theta}{2}<1$ we can apply (2.9) to have

$$\begin{aligned}&\|\partial_t u(t,\cdot)-u_1\|^2_{D(A^{-\theta})}\\&\qquad\le Ct^{\alpha-2+2\alpha\theta}\|\nabla u_0\|^2_{L^2(\Omega)}+2\sum_{n=1}^\infty\big|\langle u_1,e_n\rangle\big|^2\big|E_\alpha(-\lambda_n t^\alpha)-1\big|^2.\end{aligned}$$

Since $\alpha - 2 + 2\alpha\theta > 0$, by analogous argumentations to those done before we deduce $\lim_{t\to 0} \|\partial_t u(t,\cdot) - u_1\|_{D(A^{-\theta})} = 0$ and for any $t \in [0,T]$

$$\|\partial_t u(t,\cdot)\|^2_{D(A^{-\theta})} \le C(\|\nabla u_0\|^2_{L^2(\Omega)} + \|u_1\|^2_{L^2(\Omega)}),$$

that ends the proof of (3.1).

Now, we fix $\theta \in \big(0, \frac{1}{2\alpha}\big)$. Thanks to (2.16) we get

$$\begin{aligned}\|\nabla u(t,\cdot)\|^2_{D(A^{\theta})} &= \sum_{n=1}^{\infty} \lambda_n^{1+2\theta} \big|\langle u_0, e_n\rangle E_\alpha(-\lambda_n t^\alpha) + \langle u_1, e_n\rangle t E_{\alpha,2}(-\lambda_n t^\alpha)\big|^2 \\ &\le C\sum_{n=1}^{\infty} \lambda_n |\langle u_0, e_n\rangle|^2 \frac{\lambda_n^{2\theta}}{(1+\lambda_n t^\alpha)^2} + C\sum_{n=1}^{\infty} |\langle u_1, e_n\rangle|^2 \frac{\lambda_n^{1+2\theta} t^2}{(1+\lambda_n t^\alpha)^2}\,.\end{aligned}$$

Since

$$\begin{aligned}\frac{\lambda_n^{2\theta}}{(1+\lambda_n t^\alpha)^2} &= \Big(\frac{(\lambda_n t^\alpha)^\theta}{1+\lambda_n t^\alpha}\Big)^2 t^{-2\alpha\theta}, \\ \frac{\lambda_n^{1+2\theta} t^2}{(1+\lambda_n t^\alpha)^2} &= \Big(\frac{(\lambda_n t^\alpha)^{\frac{1+2\theta}{2}}}{1+\lambda_n t^\alpha}\Big)^2 t^{2-\alpha(1+2\theta)},\end{aligned}$$

and $0 < \theta < \frac{1}{2}$, we can apply (2.9) to have

$$\|\nabla u(t,\cdot)\|^2_{D(A^{\theta})} \le Ct^{-2\alpha\theta} \|\nabla u_0\|^2_{L^2(\Omega)} + Ct^{2-\alpha(1+2\theta)} \|u_1\|^2_{L^2(\Omega)}.$$

Taking into account that $\theta \in \big(0, \frac{1}{2\alpha}\big)$ we have $\nabla u \in L^2(0,T; D(A^\theta))$ and (3.2) follows.

To prove (3.3) we have to fix $\theta \in \big(\frac{\alpha-1}{2\alpha}, \frac{1}{2}\big)$. Thanks to (2.18), (2.5) and (2.8) we obtain

$$\begin{aligned}&\|\partial_t^\alpha u(t,\cdot)\|^2_{D(A^{-\theta})} \\ &\quad= \sum_{n=1}^{\infty} \lambda_n^{-2\theta} \big|\lambda_n \langle u_0, e_n\rangle E_\alpha(-\lambda_n t^\alpha) + \lambda_n \langle u_1, e_n\rangle t E_{\alpha,2}(-\lambda_n t^\alpha)\big|^2 \\ &\quad\le C\sum_{n=1}^{\infty} \lambda_n |\langle u_0, e_n\rangle|^2 \frac{\lambda_n^{1-2\theta}}{(1+\lambda_n t^\alpha)^2} + C\sum_{n=1}^{\infty} |\langle u_1, e_n\rangle|^2 \frac{\lambda_n^{2(1-\theta)} t^2}{(1+\lambda_n t^\alpha)^2}\,. \qquad (3.7)\end{aligned}$$

Observing that

$$\begin{aligned}\frac{\lambda_n^{1-2\theta}}{(1+\lambda_n t^\alpha)^2} &= \Big(\frac{(\lambda_n t^\alpha)^{\frac{1-2\theta}{2}}}{1+\lambda_n t^\alpha}\Big)^2 t^{\alpha(2\theta-1)}, \\ \frac{\lambda_n^{2(1-\theta)} t^2}{(1+\lambda_n t^\alpha)^2} &= \Big(\frac{(\lambda_n t^\alpha)^{1-\theta}}{1+\lambda_n t^\alpha}\Big)^2 t^{2+2\alpha(\theta-1)},\end{aligned}$$

and being $0 < \theta < \frac{1}{2}$, from (2.9) and (3.7) we deduce

$$\|\partial_t^\alpha u(\cdot,t)\|^2_{D(A^{-\theta})} \le Ct^{\alpha(2\theta-1)}\ \|\nabla u_0\|^2_{L^2(\Omega)} + Ct^{2+2\alpha(\theta-1)}\|u_1\|^2_{L^2(\Omega)}\,.$$

Therefore, since $\theta > \frac{\alpha-1}{2\alpha}$ we have $\partial_t^\alpha u \in L^2(0,T;D(A^{-\theta}))$ and (3.3) holds.

To prove the last point, we assume $\nabla u_0 \in D(A^\theta)$ with $\theta \in \big(\frac{2-\alpha}{2\alpha},\frac{1}{2}\big)$. Thanks to formula (2.17) we have

$$\begin{aligned}
&\|\partial_t u(t,\cdot) - u_1\|^2_{L^2(\Omega)}\\
&\quad = \sum_{n=1}^{\infty}\big| -\lambda_n\langle u_0,e_n\rangle t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n t^\alpha) + \langle u_1,e_n\rangle\big(E_\alpha(-\lambda_n t^\alpha)-1\big)\big|^2\\
&\qquad\qquad \le Ct^{\alpha-2+2\alpha\theta}\sum_{n=1}^{\infty}\lambda_n^{1+2\theta}\big|\langle u_0,e_n\rangle\big|^2\left(\frac{(\lambda_n t^\alpha)^{\frac{1-2\theta}{2}}}{1+\lambda_n t^\alpha}\right)^2\\
&\qquad\qquad\qquad + 2\sum_{n=1}^{\infty}\big|\langle u_1,e_n\rangle\big|^2\big|E_\alpha(-\lambda_n t^\alpha)-1\big|^2. \qquad (3.8)
\end{aligned}$$

By using (2.9) with $\beta = \frac{1-2\theta}{2}$ we obtain

$$\begin{aligned}
&\|\partial_t u(t,\cdot) - u_1\|^2_{L^2(\Omega)}\\
&\qquad \le Ct^{\alpha-2+2\alpha\theta}\|\nabla u_0\|_{D(A^\theta)} + 2\sum_{n=1}^{\infty}\big|\langle u_1,e_n\rangle\big|^2\big|E_\alpha(-\lambda_n t^\alpha)-1\big|^2,
\end{aligned}$$

and hence, since $\alpha - 2 + 2\alpha\theta > 0$, we deduce (3.4). □

REMARK 3.1. If we compare the regularity results concerning $\partial_t u$ given in Theorem 3.1 with the analogous ones in Theorem 2.2, then we have to observe that if $\theta \in \big(\frac{2-\alpha}{2\alpha},\frac{1}{2}\big]$, then $D(A^{-\theta}) \subset D(A^{-\eta})$ for any $\eta \in \big(\frac{1}{\alpha},1\big]$. Therefore Theorem 3.1 effectively improves the regularity of the weak solution.

Moreover, we note that to secure a regularity of $\partial_t u$ in $L^2(\Omega)$, taking into account the argumentations used to get (3.6), we have to assume the datum u_0 belonging to a proper subset of $H_0^1(\Omega) = D(A^{\frac{1}{2}})$, that is $\nabla u_0 \in D(A^\theta)$ with $\theta \in \big(\frac{2-\alpha}{2\alpha},\frac{1}{2}\big)$, see (3.4).

4. **Hidden regularity results**

Our approach follows the argumentations developed in [9] for wave equations. To begin with, we single out some technical results that we will use later in the main theorem.

LEMMA 4.1. *For $w \in H^2(\Omega)$ and a vector field $h : \overline{\Omega} \to \mathbb{R}^N$ of class C^1 one has*

$$\int_\Omega \triangle w \; h \cdot \nabla w \; dx = \int_{\partial\Omega} \Big[\partial_\nu w \; h \cdot \nabla w - \frac{1}{2} h \cdot \nu |\nabla w|^2\Big] \; d\sigma \\ - \sum_{i,j=1}^N \int_\Omega \partial_i h_j \partial_i w \partial_j w \; dx + \frac{1}{2} \int_\Omega \sum_{j=1}^N \partial_j h_j \; |\nabla w|^2 \; dx \,. \quad (4.1)$$

P r o o f. We integrate by parts to get

$$\int_\Omega \triangle w \; h \cdot \nabla w \; dx = \int_{\partial\Omega} \partial_\nu w \; h \cdot \nabla w \; d\sigma - \int_\Omega \nabla w \cdot \nabla\big(h \cdot \nabla w\big) \; dx \,. \quad (4.2)$$

Since

$$\int_\Omega \nabla w \cdot \nabla\big(h \cdot \nabla w\big) \; dx = \sum_{i,j=1}^N \int_\Omega \partial_i w \; \partial_i (h_j \partial_j w) \; dx \\ = \sum_{i,j=1}^N \int_\Omega \partial_i w \; \partial_i h_j \partial_j w \; dx + \sum_{i,j=1}^N \int_\Omega h_j \; \partial_i w \partial_j (\partial_i w) \; dx,$$

we evaluate the last term on the right-hand side again by an integration by parts, so we obtain

$$\sum_{i,j=1}^N \int_\Omega h_j \; \partial_i w \partial_j (\partial_i w) \; dx = \frac{1}{2} \sum_{j=1}^N \int_\Omega h_j \; \partial_j \Big(\sum_{i=1}^N (\partial_i w)^2 \Big) \; dx \\ = \frac{1}{2} \int_{\partial\Omega} h \cdot \nu |\nabla w|^2 \; d\sigma - \frac{1}{2} \int_\Omega \sum_{j=1}^N \partial_j h_j \; |\nabla w|^2 \; dx \,.$$

Therefore, if we merge the two previous identities with (4.2), then we have (4.1). □

In the next lemma we need a strong regularity for the weak solution that is guaranteed by Theorem 2.2–(ii). Moreover, we recall the following notations: I^β is the Riemann–Liouville operator of order $\beta > 0$, see (2.1), and for $\theta \in (0,1)$ $\langle \cdot, \cdot \rangle_{-\theta,\theta}$ is the duality brought in (2.4).

LEMMA 4.2. *Assume $\alpha \in (1,2)$ and the weak solution u of*

$$\partial_t^\alpha u(t,x) = \triangle u(t,x) \quad \text{in} \;\; (0,\infty) \times \Omega \quad (4.3)$$

belonging to $C([0,+\infty); H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0,+\infty); L^2(\Omega))$ with $\partial_t^\alpha u \in C([0,+\infty); L^2(\Omega))$. Then, for a vector field $h : \overline{\Omega} \to \mathbb{R}^N$ of class C^1 and $\beta, \theta \in (0,1)$ the following identities hold true:

$$\int_{\partial\Omega}\Big[I^\beta(\partial_\nu u)(t)\ h\cdot I^\beta(\nabla u)(t)-\frac{1}{2}h\cdot\nu\big|I^\beta(\nabla u)(t)\big|^2\Big]\ d\sigma$$
$$=\langle I^\beta(\partial_t^\alpha u)(t),h\cdot I^\beta(\nabla u)(t)\rangle_{-\theta,\theta}+\sum_{i,j=1}^N\int_\Omega\partial_i h_j I^\beta(\partial_i u)(t)I^\beta(\partial_j u)(t)\ dx$$
$$-\frac{1}{2}\int_\Omega\sum_{j=1}^N\partial_j h_j\ |I^\beta(\nabla u)(t)|^2\ dx\,,\qquad t>0,\quad (4.4)$$

$$\int_{\partial\Omega}\big(I^\beta(\partial_\nu u)(t)-I^\beta(\partial_\nu u)(\tau)\big)\ h\cdot\big(I^\beta(\nabla u)(t)-I^\beta(\nabla u)(\tau)\big)d\sigma$$
$$-\frac{1}{2}\int_{\partial\Omega}h\cdot\nu\big|I^\beta(\nabla u)(t)-I^\beta(\nabla u)(\tau)\big|^2 d\sigma$$
$$=\langle I^\beta(\partial_t^\alpha u)(t)-I^\beta(\partial_t^\alpha u)(\tau),h\cdot\big(I^\beta(\nabla u)(t)-I^\beta(\nabla u)(\tau)\big)\rangle_{-\theta,\theta}$$
$$+\sum_{i,j=1}^N\int_\Omega\partial_i h_j\big(I^\beta(\partial_i u)(t)-I^\beta(\partial_i u)(\tau)\big)\big(I^\beta(\partial_j u)(t)-I^\beta(\partial_j u)(\tau)\big)\ dx$$
$$-\frac{1}{2}\sum_{j=1}^N\int_\Omega\partial_j h_j\ |I^\beta(\nabla u)(t)-I^\beta(\nabla u)(\tau)|^2\ dx\,,\qquad t,\tau>0\,.\quad (4.5)$$

P r o o f. First, we apply the operator I^β, $\beta\in(0,1)$, to equation (4.3):

$$I^\beta(\partial_t^\alpha u)(t)=I^\beta(\triangle u)(t)\qquad t>0.\quad (4.6)$$

Fix $\theta\in(0,1)$, by means of the duality $\langle\cdot,\cdot\rangle_{-\theta,\theta}$ brought in (2.4) we multiply the terms of the previous equation by

$$h\cdot\nabla I^\beta(u)(t),$$

that is,

$$\langle I^\beta(\partial_t^\alpha u)(t),h\cdot\nabla I^\beta(u)(t)\rangle_{-\theta,\theta}=\langle\triangle I^\beta(u)(t),h\cdot\nabla I^\beta(u)(t)\rangle_{-\theta,\theta}.$$

Thanks to the regularity of data and (2.6) the term on the right-hand side of the previuos equation can be written as a scalar product in $L^2(\Omega)$, so we have

$$\langle I^\beta(\partial_t^\alpha u)(t),h\cdot\nabla I^\beta(u)(t)\rangle_{-\theta,\theta}=\int_\Omega\triangle I^\beta(u)(t)h\cdot\nabla I^\beta(u)(t)\ dx.\quad (4.7)$$

To evaluate the term

$$\int_\Omega\triangle I^\beta(u)(t)h\cdot\nabla I^\beta(u)(t)\ dx\,,$$

we apply Lemma 4.1 to the function $w(t,x) = I^\beta(u)(t)$, so from (4.1) we deduce

$$\int_\Omega \triangle I^\beta(u)(t) h \cdot \nabla I^\beta(u)(t)\, dx$$
$$= \int_{\partial\Omega} \Big[I^\beta(\partial_\nu u)(t)\; h \cdot I^\beta(\nabla u)(t) - \frac{1}{2} h \cdot \nu \big| I^\beta(\nabla u)(t) \big|^2 \Big] d\sigma$$
$$- \sum_{i,j=1}^N \int_\Omega \partial_i h_j I^\beta(\partial_i u)(t) I^\beta(\partial_j u)(t)\, dx + \frac{1}{2} \int_\Omega \sum_{j=1}^N \partial_j h_j\; |I^\beta(\nabla u)(t)|^2\, dx\,.$$

In conclusion, plugging the above formula into (4.7), we obtain (4.4).

The proof of (4.5) is similar to that of (4.4). Indeed, starting from

$$I^\beta(\partial_t^\alpha u)(t) - I^\beta(\partial_t^\alpha u)(\tau) = I^\beta(\triangle u)(t) - I^\beta(\triangle u)(\tau), \qquad t, \tau > 0,$$

by means of the duality $\langle \cdot, \cdot \rangle_{-\theta,\theta}$ one multiplies both terms by

$$h \cdot \nabla \big(I^\beta(u)(t) - I^\beta(u)(\tau) \big)\,.$$

Then, applying Lemma 4.1 to the function $w(t,\tau,x) = I^\beta(u)(t) - I^\beta(u)(\tau)$ one can get the identity (4.5). We omit the details. □

Remark 4.1. We observe that the proof of the identities (4.4) and (4.5) cannot be done for a general function w and then applied to $w = I^\beta(u)$, since

$$\partial_t^\alpha I^\beta(u) \neq I^\beta(\partial_t^\alpha u),$$

as one easily deduces from (2.13).

Theorem 4.1. *Let $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $u_1 \in H_0^1(\Omega)$ and u the weak solution of*

$$\begin{cases} \partial_t^\alpha u(t,x) = \triangle u(t,x)\,, & t \geq 0,\ x \in \Omega, \\ u(t,x) = 0 & t \geq 0,\ x \in \partial\Omega, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in \Omega. \end{cases} \tag{4.8}$$

Then, for any $T > 0$ there exists a constant $C = C(T)$ such that u satisfies the inequality

$$\int_0^T \int_{\partial\Omega} \big|\partial_\nu u\big|^2 d\sigma dt \leq C \big(\|\nabla u_0\|^2_{L^2(\Omega)} + \|u_1\|^2_{L^2(\Omega)} \big)\,. \tag{4.9}$$

P r o o f. First, we note that by Theorem 2.2–(ii) the unique weak solution u to (2.14) given by (2.16) belongs to $C([0,T]; H^2(\Omega) \cap H_0^1(\Omega)) \cap$

$C^1([0,T];L^2(\Omega))$ and $\partial_t^\alpha u \in C([0,T];L^2(\Omega))$, so the normal derivative $\partial_\nu u$ is well defined.

To prove the statement we use Theorem 2.1. Indeed, for $H = L^2(\partial\Omega)$ and $\beta \in (0,1)$ we can apply (2.12) to get

$$\|\partial_\nu u\|_{L^2(0,T;L^2(\partial\Omega))} \sim \|I^\beta(\partial_\nu u)\|_{H^\beta(0,T;L^2(\partial\Omega))}\,,$$

whence the inequality (4.9) is equivalent to

$$\|I^\beta(\partial_\nu u)\|_{H^\beta(0,T;L^2(\partial\Omega))} \le C\big(\|\nabla u_0\|_{L^2(\Omega)} + \|u_1\|_{L^2(\Omega)}\big)\,. \tag{4.10}$$

Therefore, our goal is to prove (4.10). Thanks to (2.11), we have to evaluate $\big\|I^\beta(\partial_\nu u)\big\|_{L^2(0,T;L^2(\partial\Omega))}$ and $\big[I^\beta(\partial_\nu u)\big]_{H^\beta(0,T;L^2(\partial\Omega))}$. To this end we employ the two identities in Lemma 4.2 with a suitable choice of the vector field h. Indeed, we take a vector field $h \in C^1(\overline{\Omega};\mathbb{R}^N)$ satisfying the condition

$$h = \nu \qquad \text{on} \quad \partial\Omega \tag{4.11}$$

(see e.g. [9] for the existence of such vector field h). First we consider the identity (4.4). Since

$$\nabla u = (\partial_\nu u)\nu \quad \text{on} \quad (0,T)\times\partial\Omega\,, \tag{4.12}$$

(see e.g. [18, Lemma 2.1] for a detailed proof) the left-hand side of (4.4) becomes

$$\frac{1}{2}\int_{\partial\Omega}\big|I^\beta(\partial_\nu u)\big|^2 d\sigma\,.$$

If we integrate (4.4) over $[0,T]$, then we obtain

$$\begin{aligned}\int_0^T\int_{\partial\Omega}\big|I^\beta(\partial_\nu u)\big|^2\,d\sigma dt &= 2\int_0^T\langle I^\beta(\partial_t^\alpha u)(t), h\cdot I^\beta(\nabla u)(t)\rangle_{-\theta,\theta}\,dt\\ &\quad + 2\sum_{i,j=1}^N\int_0^T\int_\Omega \partial_i h_j I^\beta(\partial_i u)(t) I^\beta(\partial_j u)(t)\,dxdt\\ &\quad - \int_0^T\int_\Omega\sum_{j=1}^N \partial_j h_j\,|I^\beta(\nabla u)(t)|^2\,dxdt\,.\end{aligned}$$

Since $h \in C^1(\overline{\Omega};\mathbb{R}^N)$ from the above inequality we get

$$\begin{aligned}&\big\|I^\beta(\partial_\nu u)\big\|_{L^2(0,T;L^2(\partial\Omega))}\\ &\qquad \le C\Big(\big\|I^\beta(\partial_t^\alpha u)\big\|_{L^2(0,T;D(A^{-\theta}))} + \big\|I^\beta(\nabla u)\big\|_{L^2(0,T;D(A^\theta))}\Big)\,,\end{aligned} \tag{4.13}$$

for some constant $C > 0$.

We have to estimate the Gagliardo semi-norm $\big[I^\beta(\partial_\nu u)\big]_{H^\beta(0,T;L^2(\partial\Omega))}$, see (2.10). Thanks again to the condition (4.12) the left-hand side of (4.5) becomes

$$\frac{1}{2}\int_{\partial\Omega}\left|I^{\beta}(\partial_\nu u)(t)-I^{\beta}(\partial_\nu u)(\tau)\right|^2 d\sigma\,.$$

Therefore, if we multiply both terms of (4.5) by $\frac{1}{|t-\tau|^{1+2\beta}}$ and integrate over $[0,T]\times[0,T]$, then we have

$$\begin{aligned}
&\frac{1}{2}\left[I^{\beta}(\partial_\nu u)\right]^2_{H^{\beta}(0,T;L^2(\partial\Omega))}\\
&=\int_0^T\!\!\int_0^T\frac{\langle I^{\beta}(\partial_t^{\alpha}u)(t)-I^{\beta}(\partial_t^{\alpha}u)(\tau),h\cdot\left(I^{\beta}(\nabla u)(t)-I^{\beta}(\nabla u)(\tau)\right)\rangle_{-\theta,\theta}}{|t-\tau|^{1+2\beta}}dtd\tau\\
&+\int_0^T\!\!\int_0^T\frac{\sum_{i,j=1}^N\langle\partial_i h_j\left(I^{\beta}(\partial_i u)(t)-I^{\beta}(\partial_i u)(\tau)\right),I^{\beta}(\partial_j u)(t)-I^{\beta}(\partial_j u)(\tau)\rangle}{|t-\tau|^{1+2\beta}}dtd\tau\\
&\qquad-\frac{1}{2}\int_0^T\!\!\int_0^T\frac{\int_\Omega\sum_{j=1}^N\partial_j h_j\ |I^{\beta}(\nabla u)(t)-I^{\beta}(\nabla u)(\tau)|^2dx}{|t-\tau|^{1+2\beta}}dtd\tau\,. \qquad (4.14)
\end{aligned}$$

We estimate the first term on the right-hand side of the above identity as follows

$$\begin{aligned}
&\int_0^T\!\!\int_0^T\frac{\langle I^{\beta}(\partial_t^{\alpha}u)(t)-I^{\beta}(\partial_t^{\alpha}u)(\tau),h\cdot\left(I^{\beta}(\nabla u)(t)-I^{\beta}(\nabla u)(\tau)\right)\rangle_{-\theta,\theta}}{|t-\tau|^{1+2\beta}}dtd\tau\\
&\qquad\le C\Big(\left[I^{\beta}(\partial_t^{\alpha}u)\right]^2_{H^{\beta}(0,T;D(A^{-\theta}))}+\left[I^{\beta}(\nabla u)\right]^2_{H^{\beta}(0,T;D(A^{\theta}))}\Big)\,,
\end{aligned}$$

and hence from (4.14) we deduce

$$\begin{aligned}
&\left[I^{\beta}(\partial_\nu u)\right]_{H^{\beta}(0,T;L^2(\partial\Omega))}\\
&\qquad\le C\Big(\left[I^{\beta}(\partial_t^{\alpha}u)\right]_{H^{\beta}(0,T;D(A^{-\theta}))}+\left[I^{\beta}(\nabla u)\right]_{H^{\beta}(0,T;D(A^{\theta}))}\Big)\,. \qquad (4.15)
\end{aligned}$$

Putting together (4.13) and (4.15) we obtain

$$\begin{aligned}
&\left\|I^{\beta}(\partial_\nu u)\right\|_{H^{\beta}(0,T;L^2(\partial\Omega))}\\
&\qquad\le C\Big(\left\|I^{\beta}(\partial_t^{\alpha}u)\right\|_{H^{\beta}(0,T;D(A^{-\theta}))}+\left\|I^{\beta}(\nabla u)\right\|_{H^{\beta}(0,T;D(A^{\theta}))}\Big)\,. \qquad (4.16)
\end{aligned}$$

Since $1<\alpha<2$, we can choose $\theta\in\left(\frac{\alpha-1}{2\alpha},\frac{1}{2\alpha}\right)$ to apply Theorem 1.1. So we get $\partial_t^{\alpha}u\in L^2(0,T;D(A^{-\theta}))$ and $\nabla u\in L^2(0,T;D(A^{\theta}))$. Thanks again to Theorem 2.1 we have

$$\begin{aligned}
\left\|I^{\beta}(\partial_t^{\alpha}u)\right\|_{H^{\beta}(0,T;D(A^{-\theta}))}&\sim\|\partial_t^{\alpha}u\|_{L^2(0,T;D(A^{-\theta}))}\,,\\
\left\|I^{\beta}(\nabla u)\right\|_{H^{\beta}(0,T;D(A^{\theta}))}&\sim\|\nabla u\|_{L^2(0,T;D(A^{\theta}))}\,,
\end{aligned}$$

and hence from (1.3) and (4.16) we deduce (4.10). The proof is complete.

□

Theorem 4.2. *Let $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$. If u is the weak solution of*

$$\begin{cases} \partial_t^\alpha u(t,x) = \triangle u(t,x)\,, \quad t \geq 0,\ x \in \Omega, \\ u(t,x) = 0 \qquad t \geq 0,\ x \in \partial\Omega, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \qquad x \in \Omega, \end{cases} \tag{4.17}$$

then we define the normal derivative $\partial_\nu u$ of u such that for any $T > 0$ we have

$$\int_0^T \int_{\partial\Omega} \left|\partial_\nu u\right|^2 d\sigma dt \leq C\Big(\|\nabla u_0\|^2_{L^2(\Omega)} + \|u_1\|^2_{L^2(\Omega)}\Big)\,, \tag{4.18}$$

for some constant $C = C(T)$ independent of the initial data.

P r o o f. For $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $u_1 \in H_0^1(\Omega)$, if we denote by u the weak solution of problem (4.17), then, thanks to Theorem 4.1, the inequality (4.18) holds for any $T > 0$. By density there exists a unique continuous linear map

$$\mathcal{L} : H_0^1(\Omega) \times L^2(\Omega) \to L^2_{loc}((0,\infty); L^2(\partial\Omega))$$

such that

$$\mathcal{L}(u_0,u_1) = \partial_\nu u \qquad \forall (u_0,u_1) \in \big(H^2(\Omega) \cap H_0^1(\Omega)\big) \times H_0^1(\Omega)$$

and

$$\int_0^T \int_{\partial\Omega} \left|\mathcal{L}(u_0,u_1)\right|^2 d\sigma dt \leq C\Big(\|\nabla u_0\|^2_{L^2(\Omega)} + \|u_1\|^2_{L^2(\Omega)}\Big),$$

for any $(u_0,u_1) \in H_0^1(\Omega) \times L^2(\Omega)$.

Finally, given $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$ for the weak solution u of (4.17) we use the notation $\partial_\nu u$ instead of $\mathcal{L}(u_0,u_1)$, and in addition (4.18) holds. □

Remark 4.2. Theorem 4.2 does not follow from the classical trace theorems of the Sobolev spaces. For this reason it can be called a hidden regularity result. The corresponding inequality (4.18) is often called a direct inequality.

References

[1] L. Bociu, J.-P. Zolésio, A pseudo-extractor approach to hidden boundary regularity for the wave equation with mixed boundary conditions. *J. Differential Equations* **259**, No 11 (2015), 5688–5708.

[2] H. Brezis, *Analyse Fonctionnelle. Théorie et Applications.* Masson, Paris (1983).

[3] Y. Fujita, Integrodifferential equation which interpolates the heat equation and the wave equation. *Osaka J. Math.* **27**, No 2 (1990), 309–321.
[4] R. Gorenflo, F. Mainardi, Fractional calculus: Integral and differential equations of fractional order. In: *Fractals and Fractional Calculus in Continuum Mechanics*, Springer-Verlag, New York (1997), 223–276.
[5] R. Gorenflo, M. Yamamoto, Operator theoretic treatment of linear Abel integral equations of first kind. *Japan J. Indust. Appl. Math.* **16**, No 1 (1999), 137–161.
[6] R. Gorenflo, Y. Luchko, M. Yamamoto, Time-fractional diffusion equation in the fractional Sobolev spaces. *Fract. Calc. Appl. Anal.* **18**, No 3 (2015), 799–820; DOI: 10.1515/fca-2015-0048; https://www.degruyter.com/journal/key/FCA/18/3/html.
[7] Y. Kian, M. Yamamoto, Well-posedness for weak and strong solutions of non-homogeneous initial boundary value problems for fractional diffusion equations. *Fract. Calc. Appl. Anal.* **24**, No 1 (2021), 168–201; DOI: 10.1515/fca-2021-0008; https://www.degruyter.com/journal/key/FCA/24/1/html.
[8] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*. Elsevier, Amsterdam (2006).
[9] V. Komornik, *Exact Controllability and Stabilization. The Multiplier Method*. Masson, Paris; John Wiley and Sons, Ltd., Chichester (1994).
[10] I. Lasiecka, R. Triggiani, Regularity of hyperbolic equations under $L_2(0,T;L_2(\Gamma))$-Dirichlet boundary terms. *Appl. Math. Optim.* **10**, No 3, (1983), 275–286.
[11] J.-L. Lions, Hidden regularity in some nonlinear hyperbolic equations. *Mat. Apl. Comput.* **6**, No 1 (1987), 7–15.
[12] J.-L. Lions, *Contrôlabilité Exacte, Perturbations et Stabilisation de Systèmes Distribués I-II*. Masson, Paris (1988).
[13] P. Loreti, D. Sforza, Hidden regularity for wave equations with memory. *Riv. Math. Univ. Parma (N.S.)* **7**, No 2 (2016), 391–405.
[14] P. Loreti, D. Sforza, A semilinear integro-differential equation: global existence and hidden regularity. In: *Trends in Control Theory and Partial Differential Equations*, Springer, Cham (2019), 157–180.
[15] A. Lunardi, *Interpolation Theory*. Edizioni della Normale, Pisa (2018).
[16] Y. Luchko, F. Mainardi, Y. Povstenko, Propagation speed of the maximum of the fundamental solution to the fractional diffusion-wave equation. *Comput. Math. Appl.* **66**, No 5 (2013), 774–784.
[17] F. Mainardi, The fundamental solutions for the fractional diffusion-wave equation. *Appl. Math. Lett.* **9**, No 6 (1996), 23–28.

[18] M. Milla Miranda, L.A. Medeiros, Hidden regularity for semilinear hyperbolic partial differential equations. *Ann. Fac. Sci. Toulouse Math. (5)* **9**, No 1 (1988), 103–120.

[19] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer-Verlag, Berlin (1983).

[20] I. Podlubny, *Fractional Differential Equations*. Academic Press, San Diego (1999).

[21] Y. Povstenko, *Linear Fractional Diffusion–Wave Equation for Scientists and Engineers*. Birkhäuser, Cham (2015).

[22] K. Sakamoto, M. Yamamoto, Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems. *J. Math. Anal. Appl.* **382**, No 1 (2011), 426–447.

[23] S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional Integrals and Derivatives*. Gordon and Breach Science Publishers, Philadelphia (1993).

[24] M. Stynes, Too much regularity may force too much uniqueness. *Fract. Calc. Appl. Anal.* **19**, No 6 (2016), 1554–1562; DOI: 10.1515/fca-2016-0080; https://www.degruyter.com/journal/key/FCA/19/6/html.

[1,2] *Dipartimento di Scienze di Base e Applicate per l'Ingegneria*
Sapienza Università di Roma
Via Antonio Scarpa 16, 00161 Roma, ITALY

[1] *e-mail: paola.loreti@uniroma1.it*

[2] *e-mail: daniela.sforza@uniroma1.it (Corresponding author)*

Received: September 25, 2020

Please cite to this paper as published in:

Fract. Calc. Appl. Anal., Vol. **24**, No 4 (2021), pp. 1015–1034,
DOI: 10.1515/fca-2021-0044