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RESEARCH PAPER

GLOBAL STABILITY OF FRACTIONAL DIFFERENT ORDERS NONLINEAR FEEDBACK SYSTEMS WITH POSITIVE LINEAR PARTS AND INTERVAL STATE MATRICES

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Abstract

The global stability of continuous-time fractional orders nonlinear feedback systems with positive linear parts and interval state matrices is investigated. New sufficient conditions for the global stability of this class of positive feedback nonlinear systems are established. The effectiveness of these new stability conditions is demonstrated on simple example.

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Key Words and Phrases: global stability; fractional order; nonlinear system; feedback system; positive fractional system; interval state matrix

1. Introduction

In positive systems inputs, state variables and outputs take only nonnegative values for any nonnegative inputs and nonnegative initial conditions [1, 4, 7, 19]. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollutions models. A variety of models having positive behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc. An overview of state of the art in positive systems theory is given in the monographs [1, 4, 7, 10, 13].

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Mathematical fundamentals of the fractional calculus are given in the monographs [10, 13, 20, 21]. The positive fractional linear systems have been investigated in [3, 6, 10]. Positive linear systems with different fractional orders have been addressed in [8, 9, 25]. Descriptor positive systems have been analyzed in [2, 24, 25]. Linear positive electrical circuits with state feedbacks have been addressed in [2, 13]. The stability of nonlinear standard and fractional positive feedback systems has been considered in [5, 6, 11, 12, 15, 17, 18, 22, 23]. A relation between controllability and observability of standard and fractional different orders systems was discussed in [14], and stability and stabilization of fractional-order linear systems with convex polytopic uncertainties in [16].

In this paper the global stability of nonlinear different fractional orders feedback systems with positive linear parts and interval sate matrices will be addressed.

The paper is organized as follows. In Section 2 the basic definitions and theorems concerning the positive different fractional orders linear systems are recalled. The stability of positive linear different fractional orders system with interval state matrices is investigated in Section 3. Main result of the paper the new sufficient conditions for the global stability feedback nonlinear systems with positive linear parts are established in Section 4. Concluding remarks are given in Section 5.

The following notations will be used: $\mathbb{R}$ - the set of real numbers, $\mathbb{R}^{n\times m}$ - the set of $n\times m$ real matrices, $\mathbb{R}_+^{n\times m}$ - the set of $n\times m$ real matrices with nonnegative entries and $\mathbb{R}_+^{n}=\mathbb{R}_+^{n\times 1}$, $\mathbb{M}_n$ - the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), $\mathbb{I}_n$- the $n\times n$ identity matrix.

2. **Positive different fractional orders linear systems**

Consider the fractional continuous-time linear system

$$\frac{d^\alpha x(t)}{dt^\alpha}=Ax(t)+Bu(t), \tag{2.1a}$$

$$y(t)=Cx(t), \tag{2.1b}$$

where $x(t)\in\mathbb{R}^n$, $u(t)\in\mathbb{R}^m$, $y(t)\in\mathbb{R}^p$ are the state, input and output vectors, $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times m}$, $C\in\mathbb{R}^{p\times n}$. In this paper the following Caputo definition of the fractional derivative of α order will be used [10, 13, 20, 21],

$${}_0D_t^\alpha f(t)=\frac{d^\alpha f(t)}{dt^\alpha}=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{\dot f(\tau)}{(t-\tau)^\alpha}d\tau,\quad 0<\alpha<1, \tag{2.2}$$

where $\dot f(\tau)=\dfrac{df(\tau)}{d\tau}$ and $\Gamma(x)$ is the Euler gamma function.

Definition 2.1. ([10, 13]) The fractional system (2.1) is called (internally) positive if $x(t) \in \mathbb{R}_+^n$ and $y(t) \in \mathbb{R}_+^p$, $t \geq 0$ for any initial conditions $x(0) \in \mathbb{R}_+^n$ and all inputs $u(t) \in \mathbb{R}_+^m$, $t \geq 0$.

Theorem 2.1. ([10, 13]) *The fractional system* (2.1) *is positive if and only if*

$$A \in \mathbb{M}_n, \quad B \in \mathbb{R}_+^{n\times m}, \quad C \in \mathbb{R}_+^{p\times n}. \tag{2.3}$$

Definition 2.2. The fractional positive linear system (2.1) is called asymptotically stable (and the matrix A Hurwitz) if

$$\lim_{t\to\infty} x(t) = 0 \quad \text{for all} \quad x(0) \in \mathbb{R}_+^n. \tag{2.4}$$

The positive fractional system (2.1) is asymptotically stable if and only if the real parts of all eigenvalues s_k of the matrix A are negative, i.e. $\mathrm{Re} s_k < 0$ for $k = 1, \ldots, n$, [10, 13].

Theorem 2.2. *The positive fractional system (2.1) is asymptotically stable if and only if one of the following equivalent conditions is satisfied:*

1) *All coefficients of the characteristic polynomial*

$$\det[\mathbb{I}_n s - A] = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 \tag{2.5}$$

are positive, i.e. $a_i < 0$ *for* $i = 0, 1, \ldots, n-1$.

2) *There exists strictly positive vector* $\lambda = [\ \lambda_1 \quad \ldots \quad \lambda_n\ ]$, $\lambda_k > 0$, $k = 1, \ldots, n$ *such that*

$$A\lambda < 0 \quad \text{or} \quad \lambda^T A < 0. \tag{2.6}$$

The transfer matrix of the system (2.1) is given by

$$T(s^\alpha) = C[\mathbb{I}_n s^\alpha - A]^{-1}B. \tag{2.7}$$

Now consider the fractional linear system with two different fractional orders

$$\begin{bmatrix} \frac{d^\alpha x_1(t)}{dt^\alpha} \\ \frac{d^\beta x_2(t)}{dt^\beta} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t), \tag{2.8a}$$

$$y(t) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \tag{2.8b}$$

where $0 < \alpha, \beta < 1$, $x_1(t) \in \mathbb{R}^{n_1}$ and $x_2(t) \in \mathbb{R}^{n_2}$ are the state vectors, $A_{ij} \in \mathbb{R}^{n_i\times n_j}$, $B_i \in \mathbb{R}^{n_i\times m}$, $C_i \in \mathbb{R}^{p\times n_i}$; $i, j = 1, 2$; $u(t) \in \mathbb{R}^m$ is the input vector and $y(t) \in \mathbb{R}^p$ is the output vector. Initial conditions for (2.8) have the form

$$x_1(0) = x_{10}, \quad x_2(0) = x_{20} \quad \text{for} \quad x_0 = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}. \tag{2.9}$$

REMARK 2.1. The state equation (2.8a) of fractional continuous-time linear systems with two different fractional orders has similar structure as the 2D Roeesser type models.

DEFINITION 2.3. The fractional system (2.8) is called positive if $x_1(t) \in \mathbb{R}_+^{n_1}$ and $x_2(t) \in \mathbb{R}_+^{n_2}$, $t \geq 0$ for any initial conditions $x_{10} \in \mathbb{R}_+^{n_1}$, $x_{20} \in \mathbb{R}_+^{n_2}$ and all input vectors $u \in \mathbb{R}_+^m$, $t \geq 0$.

THEOREM 2.3. *The fractional system* (2.8) *for* $0 < \alpha < 1$; $0 < \beta < 1$ *is positive if and only if*

$$\begin{gathered} \bar{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in \mathbb{M}_N, \quad \bar{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \in \mathbb{R}_+^{N \times m}, \\ \bar{C} = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \in \mathbb{R}_+^{p \times n}, \quad N = n_1 + n_2. \end{gathered} \tag{2.10}$$

THEOREM 2.4. *The positive fractional system* (2.8) *is asymptotically stable if and only if one of the following equivalent conditions is satisfied:*

1) *All coefficients of the characteristic polynomial*

$$\det[\mathbb{I}_n s - \bar{A}] = s^n + \bar{a}_{n-1}s^{n-1} + \ldots + \bar{a}_1 s + \bar{a}_0 \tag{2.11}$$

are positive, i.e. $\bar{a}_i > 0$ *for* $i = 0, 1, \ldots, n-1$.

2) *There exists strictly positive vector* $\lambda = [\,\lambda_1 \quad \ldots \quad \lambda_n\,]$, $\lambda_k > 0$, $k = 1, \ldots, n$ *such that*

$$\bar{A}\lambda < 0 \quad \text{or} \quad \lambda^T \bar{A} < 0. \tag{2.12}$$

THEOREM 2.5. *The solution of the equation* (2.8a) *for* $0 < \alpha < 1$; $0 < \beta < 1$ *with initial conditions* (2.9) *has the form*

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \Phi_0(t)x_0 + \int_0^t M(t-\tau)u(\tau)d\tau, \tag{2.13}$$

where

$$\begin{aligned} M(t) &= \Phi_1(t)B_{10} + \Phi_2(t)B_{01} \\ &= \begin{bmatrix} \Phi_{11}^1(t) & \Phi_{12}^1(t) \\ \Phi_{21}^1(t) & \Phi_{22}^1(t) \end{bmatrix} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} + \begin{bmatrix} \Phi_{11}^2(t) & \Phi_{12}^2(t) \\ \Phi_{21}^2(t) & \Phi_{22}^2(t) \end{bmatrix} \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \\ &= \begin{bmatrix} \Phi_{11}^1(t)B_1 + \Phi_{12}^2(t)B_2 \\ \Phi_{21}^1(t)B_1 + \Phi_{22}^2(t)B_2 \end{bmatrix} = \begin{bmatrix} \Phi_{11}^1(t) & \Phi_{12}^2(t) \\ \Phi_{21}^1(t) & \Phi_{22}^2(t) \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \end{aligned} \tag{2.14a}$$

and

$$\Phi_0(t) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} T_{kl} \frac{t^{k\alpha + l\beta}}{\Gamma(k\alpha + l\beta + 1)}, \tag{2.14b}$$

$$\Phi_1(t) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} T_{kl} \frac{t^{(k+1)\alpha + l\beta - 1}}{\Gamma[(k+1)\alpha + l\beta]}, \tag{2.14c}$$

$$\Phi_2(t) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} T_{kl} \frac{t^{k\alpha+(l+1)\beta-1}}{\Gamma[k\alpha+(l+1)\beta]}. \tag{2.14d}$$

In the above,

$$T_{kl} = \begin{cases} \mathbb{I}_n & \text{for} \quad k = l = 0, \\ \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix} & \text{for} \quad k = 1, l = 0, \\ \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix} & \text{for} \quad k = 0, l = 1, \\ T_{10}T_{k-1,l} + T_{01}T_{k,l-1} & \text{for} \quad k + l > 1. \end{cases} \tag{2.14e}$$

and $\Phi_0(t), \Phi_1(t), \Phi_2(t)$ *are the two-parameter Mittag-Leffler functions* $E_{\alpha,\mu}$, *respectively with* $\mu = l\beta + 1$, $\mu = l\beta + \alpha$, $\mu = l\beta + \beta$, *see for example* [21].

P r o o f. A proof is given in [8, 9]. □

Note that if $\alpha = \beta$, then from (2.13) we have the 1-parameter Mittag-Leffler function

$$\Phi_0|_{\alpha=\beta}(t) = \sum_{k=0}^{\infty} \frac{\bar{A}^k t^{k\alpha}}{\Gamma(k\alpha+1)} = E_\alpha(\bar{A}t^\alpha). \tag{2.15}$$

The transfer matrix of the system (2.8) is given by

$$T(s^\alpha, s^\beta) = \bar{C}\left[\begin{bmatrix} \mathbb{I}_{n_1}s^\alpha & 0 \\ 0 & \mathbb{I}_{n_2}s^\beta \end{bmatrix} - \bar{A}\right]^{-1}\bar{B}. \tag{2.16}$$

3. Stability of positive different fractional orders linear systems with interval state matrices

Consider the positive different fractional orders linear system

$$\begin{aligned} \begin{bmatrix} \frac{d^\alpha x_1(t)}{dt^\alpha} \\ \frac{d^\beta x_2(t)}{dt^\beta} \end{bmatrix} &= \bar{A}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad n = n_1 + n_2, \quad 0 < \alpha, \beta < 1 \\ \bar{A} &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad A_{11} \in \mathbb{M}_{n_1}, \quad A_{22} \in \mathbb{M}_{n_2}, \end{aligned} \tag{3.1}$$

where $x_1(t) \in \mathbb{R}_+^{n_1}$, $x_2(t) \in \mathbb{R}_+^{n_2}$ are the state vectors and the interval state matrix $\bar{A} \in \mathbb{M}_n$ is defined by

$$A_1 \leq \bar{A} \leq A_2, \quad \text{or equivalently} \quad \bar{A} \in [A_1, A_2]. \tag{3.2}$$

DEFINITION 3.1. The interval positive system (3.1) is called asymptotically stable if the system is asymptotically stable for all matrices $A \in \mathbb{M}_n$ satisfying the condition (3.2).

The matrix

$$A = (1-q)A_1 + qA_2, \quad 0 < q < 1, \tag{3.3}$$

is called the convex linear combination of the matrices $A_1 \in \mathbb{M}_{n_1}$ and $A_2 \in \mathbb{M}_{n_2}$.

THEOREM 3.1. *If the matrices $A_1 \in \mathbb{M}_{n_1}$ and $A_2 \in \mathbb{M}_{n_2}$ of positive system* (3.1) *are asymptotically stable, then their convex linear combination* (3.3) *is also asymptotically stable.*

P r o o f. By condition (2.12) of Theorem 2.4, if the positive linear system (3.1) is asymptotically stable, then there exists strictly positive vector $\bar{\lambda} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \in \mathbb{R}_+^n$ such that (2.12) holds.

Using (2.6) and (3.3) we obtain

$$\bar{A}\bar{\lambda} = [(1-q)A_1 + qA_2]\bar{\lambda} = (1-q)A_1\bar{\lambda} + qA_2\bar{\lambda} < (1-q)\bar{\lambda} + q\bar{\lambda} = \bar{\lambda}, \quad \text{for} \quad 0 < q < 1. \tag{3.4}$$

Therefore, if the positive linear system (3.1) is asymptotically stable and (3.4) holds, then their convex linear combination is also asymptotically stable. □

THEOREM 3.2. *The interval positive system* (3.1) *is asymptotically stable if and only if the positive systems* (3.2) *are asymptotically stable.*

P r o o f. By condition (2.12) of Theorem 2.4 the matrices $A_1 \in \mathbb{M}_{n_1}$, $A_2 \in \mathbb{M}_{n_2}$ are asymptotically stable if and only if there exists a strictly positive vector $\lambda \in \mathbb{R}_+^n$ such that (3.4) holds. The convex linear combination (3.3) satisfies the condition $\bar{A}\bar{\lambda} < 0$ if and only if (3.4) holds. Therefore, the interval positive system (3.1) is asymptotically stable if and only if the positive systems (3.2) are asymptotically stable. □

EXAMPLE 3.1. Consider the interval positive linear system (3.1) with the matrices

$$A_1 = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -3 & 1 \\ 0 & 1 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -3 & 1.5 & 1 \\ 1.5 & -4 & 1.5 \\ 1 & 2 & -3 \end{bmatrix}, \quad n_1 = 2, \quad n_2 = 1. \tag{3.5}$$

For the matrices (3.5) we choose $\bar{\lambda}^T = [\,1 \quad 0.8 \quad 1\,]^T$ and we obtain

$$
\begin{aligned}
A_1\bar{\lambda} &= \begin{bmatrix} -2 & 1 & 0 \\ 1 & -3 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.8 \\ 1 \end{bmatrix} = \begin{bmatrix} -1.2 \\ -0.4 \\ -1.2 \end{bmatrix}, \\
A_2\bar{\lambda} &= \begin{bmatrix} -3 & 1.5 & 1 \\ 1.5 & -4 & 1.5 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.8 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.6 \\ -0.2 \\ -1.4 \end{bmatrix}.
\end{aligned} \tag{3.6}
$$

Therefore, by Theorem 3.2 the interval positive system (3.1) with (3.5) is asymptotically stable.

4. Different orders nonlinear feedback systems with positive linear parts and interval state matrices

Consider the nonlinear feedback system shown in Fig. 4.1 which consists of the positive linear part, the nonlinear element with characteristic $u = f(e)$, the positive scalar feedback with gain h and interval state matrix $\bar{A}$. The positive linear part is described by the equations

$$
\begin{aligned}
\begin{bmatrix} \frac{d^\alpha x_1(t)}{dt^\alpha} \\ \frac{d^\beta x_2(t)}{dt^\beta} \end{bmatrix} &= \bar{A} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \bar{B}u(t), \\
y(t) &= \bar{C} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},
\end{aligned} \tag{4.1}
$$

where $0 < \alpha, \beta < 1$, $x_1 = x_1(t) \in \mathbb{R}_+^{n_1}$ and $x_1 = x_2(t) \in \mathbb{R}_+^{n_2}$ are the state vectors, $u = u(t) \in \mathbb{R}$ is the input vector, $y = y(t) \in \mathbb{R}$ is the input vector, matrices $\bar{A}$, $\bar{B}$, $\bar{C}$ for $p = m = 1$ are defined by (2.10).

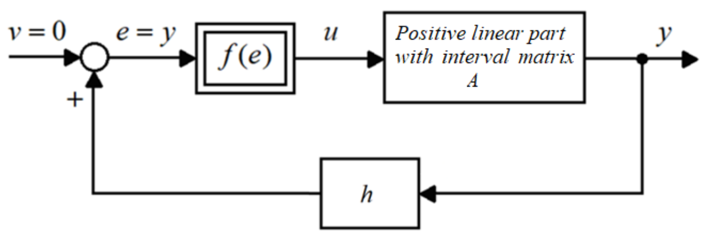

Fig. 4.1: The nonlinear feedback system

The characteristic of the nonlinear element is shown in Fig. 4.2 and it satisfies the condition

$$0 < f(e) < ke, \quad 0 < k < \infty. \tag{4.2}$$

It is assumed that the positive linear part is asymptotically stable (the matrix $\bar{A} \in \mathbb{M}_n$ is Hurwitz) for all $\bar{A}$ satisfying (3.2).

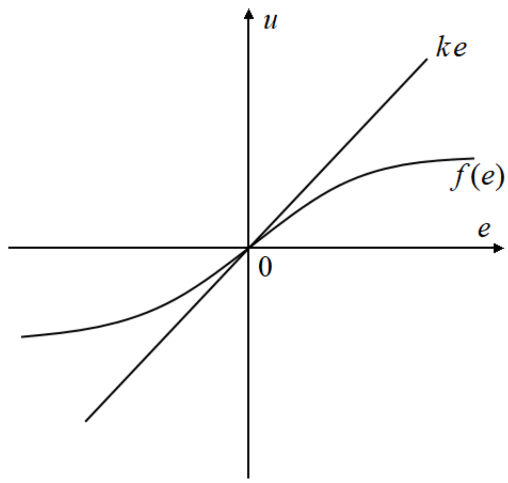

Fig. 4.2: The characteristic of the nonlinear element

Definition 4.1. The nonlinear positive system is called globally stable if it is asymptotically stable for all nonnegative initial conditions $\begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \in \mathbb{R}_+^n$ and all state matrices satisfying (3.2).

The following theorem gives sufficient conditions for the global stability of the positive nonlinear system with interval state matrix $\bar{A}$ satisfying (3.2).

Theorem 4.1. *The nonlinear system consisting of the positive linear part, the nonlinear element satisfying the condition* (4.2), *interval matrix* (3.2) *and the positive scalar feedback with gain h is globally stable if the matrix*

$$A_1 + k_i h \bar{B}\bar{C} \in \mathbb{M}_n \quad \text{for} \quad i = 1, 2 \tag{4.3}$$

is asymptotically stable (Hurwitz matrix), where $\bar{B}$, $\bar{C}$ are given by (2.10).

P r o o f. The proof will be accomplished by the use of the Lyapunov method [17, 18]. As the Lyapunov function $\bar{V}_i(x)$, $i = 1, 2$ we choose

$$\begin{gathered} \bar{V}_i(x) = V_{i,1}(x) + V_{i,2}(x) = \lambda_{i,1}^T x_{i,1} + \lambda_{i,2}^T x_{i,2} \geq 0 \\ \text{for} \quad \bar{x}_i = \begin{bmatrix} x_{i,1} \\ x_{i,2} \end{bmatrix} \in \mathbb{R}_+^n, \quad \bar{\lambda}_i = \begin{bmatrix} \lambda_{i,1} \\ \lambda_{i,2} \end{bmatrix} \in \mathbb{R}_+^n, \end{gathered} \tag{4.4}$$

where $\bar{\lambda}_i$, $i = 1, 2$ 2 are strictly positive vectors with all positive components.

Using (4.4) and (4.2) we obtain

$$\begin{aligned} \frac{d^\alpha V_{i,1}(x)}{dt^\alpha} + \frac{d^\beta V_{i,2}(x)}{dt^\beta} &= \begin{bmatrix} \lambda_{i,1}^T & \lambda_{i,2}^T \end{bmatrix} \begin{bmatrix} \frac{d^\alpha x_{i,1}}{dt^\alpha} \\ \frac{d^\beta x_{i,2}}{dt^\beta} \end{bmatrix} \\ &= \bar{\lambda}_i^T (A_i \bar{x}_i + \bar{B} u) \le \bar{\lambda}_i^T (A_i + k_i h \bar{B}\bar{C}) \bar{x}_i, \end{aligned} \tag{4.5}$$

since $u = f(e) \le ke = kh_i \bar{C}\bar{x}_i$ for $i = 1, 2$.

From (4.5) it follows that $\frac{d^\alpha V_{i,1}(x)}{dt^\alpha} + \frac{d^\beta V_{i,2}(x)}{dt^\beta} < 0$ if the matrix (4.3) is Hurwitz and nonlinear system is globally stable. □

To find the maximal value of k for which the fractional positive nonlinear system with interval state matrices is globally stable the following procedure can be used.

Procedure 4.1.

Step 1. Find the value of k_1 for which the matrix

$$A_1 + k_1 h \bar{B}\bar{C} \in \mathbb{M}_{n_1} \tag{4.6}$$

is asymptotically stable.

Step 2. Find the maximal value of k_2 for which the matrix

$$A_2 + k_2 h \bar{B}\bar{C} \in \mathbb{M}_{n_2} \tag{4.7}$$

is asymptotically stable.

Step 3. Find the desired value of k as

$$k = \min(k_1, k_2). \tag{4.8}$$

Note that to check the global stability of the fractional positive nonlinear system it suffices to check the condition (4.3) only for the matrix $A_1(A_2)$ with greater sum of all its entries.

Example 4.1. Consider the fractional ($\alpha = 0.4$, $\beta = 0.6$) nonlinear system with the positive linear part with the interval state matrices

$$A_1 = \begin{bmatrix} -2.5 & 0.4 & 0.2 & 0.1 \\ 0.8 & -1.8 & 0.2 & 0.3 \\ 0.2 & 0.3 & -4.5 & 0.3 \\ 0.3 & 0.4 & 0.4 & -3.5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -3 & 0.5 & 0.2 & 0.1 \\ 1 & -2 & 0.2 & 0.3 \\ 0.2 & 0.3 & -5 & 0.4 \\ 0.3 & 0.4 & 0.5 & -4 \end{bmatrix},$$

$$\bar{B} = \begin{bmatrix} 0.5 \\ 0.2 \\ 0.6 \\ 0.4 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} 0.2 & 0.4 & 0.5 & 0.3 \end{bmatrix}, \quad n_1 = n_2 = 2, \tag{4.9}$$

the nonlinear element satisfying the condition (4.2) and the positive feedback with gain $h = 0.5$. Find k satisfying (4.2) for which the fractional positive nonlinear system is globally stable.

Using Procedure 4.1 and (4.9) we obtain the following:

Step 1. Using (4.6) and (4.9) for $h = 0.5$ we obtain

$$\hat{A}_1 = A_1 + k_1 h\bar{B}\bar{C} = \begin{bmatrix} -2.5 & 0.4 & 0.2 & 0.1 \\ 0.8 & -1.8 & 0.2 & 0.3 \\ 0.2 & 0.3 & -4.5 & 0.3 \\ 0.3 & 0.4 & 0.4 & -3.5 \end{bmatrix}$$

$$+ 0.5k_1 \begin{bmatrix} 0.5 \\ 0.2 \\ 0.6 \\ 0.4 \end{bmatrix} \begin{bmatrix} 0.2 & 0.4 & 0.5 & 0.3 \end{bmatrix}$$

$$= \begin{bmatrix} -2.5 + 0.05k_1 & 0.4 + 0.1k_1 & 0.2 + 0.125k_1 & 0.1 + 0.075k_1 \\ 0.8 + 0.02k_1 & -1.8 + 0.04k_1 & 0.2 + 0.05k_1 & 0.3 + 0.03k_1 \\ 0.2 + 0.06k_1 & 0.3 + 0.12k_1 & -4.5 + 0.15k_1 & 0.3 + 0.09k_1 \\ 0.3 + 0.04k_1 & 0.4 + 0.08k_1 & 0.4 + 0.1k_1 & -3.5 + 0.06k_1 \end{bmatrix}. \tag{4.10}$$

The characteristic polynomial of the matrix (4.10) has the form

$$\begin{aligned} \det(\mathbb{I}_4 s - \hat{A}_1) = s^4 &+ (12.3 - 0.3k_1)s^3 + (53.96 - 2.9k_1)s^2 \\ &+ (98.83 - 9.21k_1)s + (62.19 - 9.66k_1) \end{aligned} \tag{4.11}$$

and its coefficients are positive, which implies that the nonlinear system with (4.10) is globally stable, for $k_1 < 6.43$.

Step 2. Using (4.3) and (4.9) we obtain

$$\hat{A}_2 = A_2 + k_2 h\bar{B}\bar{C} = \begin{bmatrix} -3 & 0.5 & 0.2 & 0.1 \\ 1 & -2 & 0.2 & 0.3 \\ 0.2 & 0.3 & -5 & 0.4 \\ 0.3 & 0.4 & 0.5 & -4 \end{bmatrix}$$

$$+ 0.5k_1 \begin{bmatrix} 0.5 \\ 0.2 \\ 0.6 \\ 0.4 \end{bmatrix} \begin{bmatrix} 0.2 & 0.4 & 0.5 & 0.3 \end{bmatrix}$$

$$= \begin{bmatrix} -3 + 0.05k_2 & 0.5 + 0.1k_2 & 0.2 + 0.125k_2 & 0.1 + 0.075k_2 \\ 1 + 0.02k_2 & -2 + 0.04k_2 & 0.2 + 0.05k_2 & 0.3 + 0.03k_2 \\ 0.2 + 0.06k_2 & 0.3 + 0.12k_2 & -5 + 0.15k_2 & 0.4 + 0.09k_2 \\ 0.3 + 0.04k_2 & 0.4 + 0.08k_2 & 0.5 + 0.1k_2 & -4 + 0.06k_2 \end{bmatrix}. \tag{4.12}$$

The characteristic polynomial of the matrix (4.12) has the form

$$\begin{aligned} \det(\mathbb{I}_4 s - \hat{A}_2) = s^4 &+ (14 - 0.3k_2)s^3 + (70.05 - 3.31k_2)s^2 \\ &+ (146.39 - 11.99k_2)s + (104.64 - 14.28k_2) \end{aligned} \tag{4.13}$$

and its coefficients are positive, which implies that the nonlinear system with (4.12) is globally stable, for $k_2 < 7.32$.

Step 3. Using (4.8) and the results of Steps 1 and Step 2 we obtain

$$k = \min(k_1, k_2) = \min(6.43, 7.32) = 6.43. \tag{4.14}$$

Therefore, the fractional positive nonlinear system is globally stable for $k < 6.43$.

5. Different orders nonlinear feedback systems with positive linear parts and interval state matrices

The global stability of continuous-time different fractional orders nonlinear systems with positive linear parts and interval state matrices and scalar positive feedback with gain h has been investigated. New sufficient conditions for the global stability of this class of positive nonlinear systems are established (Theorem 4.1). The effectiveness of these new stability conditions has been demonstrated on simple example of positive nonlinear different fractional orders system. The considerations can be extended to discrete-time fractional different orders nonlinear systems with positive linear parts with interval state matrices and positive scalar feedbacks. An open problem is an extension of the considerations to nonlinear different orders fractional systems with all interval matrices of their positive linear parts.

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