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RESEARCH PAPER

ANALYSIS OF FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH NONLOCAL ERDÉLYI-KOBER TYPE INTEGRAL BOUNDARY CONDITIONS

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Abstract

In this article, we study the existence and uniqueness of solutions for nonlinear fractional integro-differential equations with nonlocal Erdélyi-Kober type integral boundary conditions. The existence results are based on Krasnoselskii's and Schaefer's fixed point theorems, whereas the uniqueness result is based on the contraction mapping principle. Examples illustrating the applicability of our main results are also constructed.

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Key Words and Phrases: fractional differential equation; Erdélyi-Kober type fractional integral; integro-differential equation; existence; fixed point

1. Introduction

Through the development of mathematics, fractional derivatives are gradually occurring in various research areas, such as viscoelasticity, electromagnetic, material science, aerodynamics, etc. Most works concentrated on the solvability of initial value problems for differential equations of fractional-order. Recently, attention has given to the theory of boundary value problems for nonlinear fractional differential equations, and many aspects of this theory need to be explored. Many scholars have studied the differential and integrodifferential fractional equations, supplemented with

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various types of initial and boundary conditions, see, [1, 2, 4, 5, 17, 18, 20] and the references cited therein.

Several researchers have involved in their studies either on fractional derivatives of the Riemann-Liouville or Caputo. Since 1940, Arthur Erdélyi and Hermann Kober [6] introduced new more general form of fractional calculus operators called as Erdélyi-Kober fractional integrals and derivatives.

Several researchers have recently studied fractional differential equations with integral boundary conditions of the Erdélyi-Kober type, and the application aspects of the topic can be found, see [3, 7, 8, 14, 13, 15]. For example, in [19], results of existence and uniqueness for the Riemann-Liouville fractional differential equations with non-local Erdélyi-Kober fractional integral conditions of the form:

$$D^q x(t) = f(t, x(t)), \qquad t \in (0, T),$$
$$x(0) = 0, \quad \alpha x(T) = \sum_{i=1}^{m} \beta_i I_{\eta_i}^{\gamma_i,\delta_i} x(\xi_i),$$

are discussed, where $1 < q \leq 2$, D^q is the Riemann-Liouville fractional derivative of order q, $I_{\eta_i}^{\gamma_i,\delta_i}$ are Erdélyi-Kober fractional integrals of order $\delta_i > 0$, with $\eta_i > 0$ and $\gamma_i \in R$, $i = 1, 2, ..., m$.

Motivated by the aforementioned papers, in this paper we study the existence and uniqueness of the results for the fractional integro-differential equation with non-local Erdélyi-Kober type integral boundary conditions under Krasnoselskii's fixed point theorem, Schaefer's fixed point theorem and contraction mapping principle. Precisely, we investigate the following problem:

$$ {}^cD^v x(t) = g(t, x(t), \int_0^t h(t, s, x(s))ds), \ 1 < v \leq 2, \ t \in J = [0, 1], \tag{1.1}$$

$$\begin{aligned} x(0) &= \alpha \frac{\sigma\zeta^{-\sigma(\theta+\epsilon)}}{\Gamma(\theta)} \int_0^\zeta \frac{s^{\sigma\epsilon+\sigma-1}}{(\zeta^\sigma - s^\sigma)^{1-\theta}} x(s)ds = \alpha I_\sigma^{\epsilon,\theta} x(\zeta), \\ x(1) &= \beta \frac{\eta\xi^{-\eta(\delta+\gamma)}}{\Gamma(\delta)} \int_0^\xi \frac{s^{\eta\gamma+\eta-1}}{(\xi^\eta - s^\eta)^{1-\delta}} x(s)ds = \beta I_\eta^{\gamma,\delta} x(\xi), \\ &\theta, \sigma > 0, \ 0 < \zeta < 1, \ \epsilon \in \mathbb{R}, \ \delta, \eta > 0, \ 0 < \xi < 1, \ \gamma \in \mathbb{R}, \end{aligned} \tag{1.2}$$

where ${}^cD^v$ is the Caputo fractional derivative of order $1 < v \leq 2$, $g : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $I_\sigma^{\epsilon,\theta}$ and $I_\eta^{\gamma,\delta}$ denote Erdélyi-Kober fractional integrals and $X = C(J, \mathbb{R})$ denotes the Banach space of all continuous functions from $[0, 1] \times \mathbb{R}$ endowed with a topology of uniform convergence with the norm denoted by $\|x\| = \sup\{|x(t)| : t \in J\}$.

Here we use the notation $Hx(s) = \int_0^t h(t, s, x(s))ds$.

The paper is organized as follows: In Section 2 we present some necessary preliminaries and lemmas. Section 3 deals with the existence and uniqueness results by using fixed point theorems of Krasnoselskii, Schaefer, and Banach for the boundary value problem (1.1)-(1.2). Finally, several examples are provided in Section 4 to illustrate the applicability of our main results.

2. Preliminary results

In this section, we recall some basic definitions on the fractional calculus [9, 11, 12] and lemmas, which are used to our main results.

Definition 2.1. The Riemann-Liouville fractional integral of order v for a function g is defined as

$$J^v g(t) = \int_0^t \frac{(t-s)^{v-1}}{\Gamma(v)} g(s)ds, \quad t>0, v>0,$$

provided the right hand-side is point-wise defined on $[0,\infty)$, where Γ is the gamma function.

Definition 2.2. For a function $g:[0,\infty)\to\mathbb{R}$, the Caputo derivative of order v is defined by

$$^cD^v g(t) = \frac{1}{\Gamma(n-v)} \int_0^t (t-s)^{v-1} g(s)ds, \quad t>0,$$

$n-1<v<n, n=[v]+1$, where $[v]$ denotes the integer part of the real number v.

Definition 2.3. The Erdélyi-Kober fractional integral of order $\delta>0$ with $\eta>0$ and $\gamma\in\mathbb{R}$ of a continuous function $g:(0,\infty)\to\mathbb{R}$ is defined by

$$I_\eta^{\gamma,\delta} g(t) = \frac{\eta t^{-\eta(\delta+\gamma)}}{\Gamma(\delta)} \int_0^t \frac{s^{\eta\gamma+\eta-1} g(s)}{(t^\eta - s^\eta)^{1-\delta}} ds,$$

provided the right hand side is point wise defined on $\mathbb{R}_+$.

Remark 2.1. For $\eta=1$, the above operator is reduced to the Kober operator

$$I_1^{\gamma,\delta} g(t) = \frac{t^{-(\delta+\gamma)}}{\Gamma(\delta)} \int_0^t \frac{s^\gamma g(s)}{(t-s)^{1-\delta}} ds, \quad \gamma,\delta>0,$$

that was introduced for the first time by Kober in [10]. For $\gamma=0$, the Kober operator is reduced to the Riemann-Liouville fractional integral with a power weight:

$$I_1^{0,\delta} g(t) = \frac{t^{-\delta}}{\Gamma(\delta)} \int_0^t \frac{g(s)}{(t-s)^{1-\delta}} ds, \quad \delta>0.$$

LEMMA 2.1. *If $\delta, \eta > 0$ and $\gamma, v \in \mathbb{R}$, then we have*

$$I_\eta^{\gamma,\delta} t^v = \frac{t^v \Gamma(\gamma + \frac{v}{\eta} + 1)}{\Gamma(\gamma + \frac{v}{\eta} + \delta + 1)}.$$

LEMMA 2.2. *For any $k \in C(J, \mathbb{R}), x \in C^2(J, \mathbb{R})$ is a solution for the fractional differential equation*

$$^cD^v x(t) = k(t), \tag{2.1}$$

with the boundary conditions (1.2) if and only if

$$\begin{aligned} x(t) &= J^v k(t) + \frac{\alpha}{\Lambda}(u_4 - tu_3) I_\sigma^{\epsilon,\theta} J^v k(\zeta) \\ &+ \frac{1}{\Lambda}(u_2 + tu_1)[\beta I_\eta^{\gamma,\delta} J^v k(\xi) - J^v k(1)], \end{aligned} \tag{2.2}$$

where

$$\Lambda = u_1 u_4 + u_2 u_3 \neq 0, \tag{2.3}$$

$$\begin{aligned} u_1 &= 1 - \alpha \tfrac{\Gamma(\epsilon+1)}{\Gamma(\epsilon+\theta+1)}, \ u_2 = \alpha\zeta \tfrac{\Gamma(\epsilon+\frac{1}{\sigma}+1)}{\Gamma(\epsilon+\frac{1}{\sigma}+\theta+1)}, \\ u_3 &= 1 - \beta \tfrac{\Gamma(\gamma+1)}{\Gamma(\gamma+\delta+1)}, \ u_4 = 1 - \beta\xi \tfrac{\Gamma(\gamma+\frac{1}{\eta}+1)}{\Gamma(\gamma+\frac{1}{\eta}+\delta+1)}. \end{aligned} \tag{2.4}$$

P r o o f. We know that, the general solution of (2.1) is of the form,

$$x(t) = c_0 + c_1 t + J^v k(t), \tag{2.5}$$

where $c_0, c_1 \in \mathbb{R}$ are arbitrary constants.

Applying the boundary conditions (1.2) into (2.5) together with Lemma 2.1, we get

$$\Big(1 - \alpha \frac{\Gamma(\epsilon + 1)}{\Gamma(\epsilon + \theta + 1)}\Big) c_0 - \alpha\zeta \frac{\Gamma(\epsilon + \frac{1}{\sigma} + 1)}{\Gamma(\epsilon + \frac{1}{\sigma} + \theta + 1)} c_1 = \alpha I_\sigma^{\epsilon,\theta} J^v k(\zeta), \tag{2.6}$$

$$\begin{aligned} \Big(1 - \beta \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + \delta + 1)}\Big) c_0 + \Big(1 - \beta\xi \frac{\Gamma(\gamma + \frac{1}{\eta} + 1)}{\Gamma(\gamma + \frac{1}{\eta} + \delta + 1)}\Big) c_1 \\ = \beta I_\eta^{\gamma,\delta} J^v k(\xi) - J^v k(1). \end{aligned} \tag{2.7}$$

Solving (2.6) and (2.7) and using (2.3) and (2.4), we get

$$c_0 = \frac{1}{\Lambda}\Big\{u_4 \alpha I_\sigma^{\epsilon,\theta} J^v k(\zeta) + u_2 \beta I_\eta^{\gamma,\delta} J^v k(\xi) - u_2 J^v k(1)\Big\},$$

$$c_1 = \frac{1}{\Lambda}\Big\{u_1 \beta I_\eta^{\gamma,\delta} J^v k(\xi) - u_1 J^v k(1) - u_3 \alpha I_\sigma^{\epsilon,\theta} J^v k(\zeta)\Big\}.$$

Substituting the values of c_0 and c_1 into (2.5), we obtain (2.2).

Conversely, it can be easily verify that (2.2) satisfies by the direct computation and the boundary conditions (1.2). This completes the proof. □

3. Main results

From Lemma 2.2, we define an operator $F : X \to X$ by

$$\begin{aligned}(Fx)(t) &= J^v g(s,x(s),Hx(s))(t)\\ &+\frac{\alpha}{\Lambda}(u_4-tu_3)I_\sigma^{\epsilon,\theta}J^v g(s,x(s),Hx(s))(\zeta)\\ &+\frac{1}{\Lambda}(u_2+tu_1)\Big[|\beta|I_\eta^{\gamma,\delta}J^v g(s,x(s),Hx(s))(\xi)\\ &+J^v g(s,x(x),Hx(s))(1)\Big],\quad t\in J.\end{aligned}$$

Here, we use the following expressions, for $t \in J,\ \zeta,\xi \in (0,1)$:

$$\begin{aligned}J^v g(s,x(s),Hx(s))(t) &= \frac{1}{\Gamma(v)}\int_0^t (t-s)^{v-1} g(s,x(s),Hx(s))ds,\\ I_\sigma^{\epsilon,\theta}J^v g(s,x(s),Hx(s))(\zeta) &= \frac{\sigma\zeta^{-\sigma(\epsilon+\theta)}}{\Gamma(v)\Gamma(\theta)}\int_0^\zeta\int_0^\tau \frac{\tau^{\sigma\epsilon+\sigma-1}(\tau-s)^{(v-1)}}{(\zeta^\sigma-\tau^\sigma)^{1-\theta}}\\ &\quad\times g(s,x(s),Hx(s))dsd\tau,\\ I_\eta^{\gamma,\delta}J^v g(s,x(s),Hx(s))(\xi) &= \frac{\eta\xi^{-\eta(\gamma+\delta)}}{\Gamma(v)\Gamma(\delta)}\int_0^\xi\int_0^z \frac{z^{\eta\gamma+\eta-1}(z-s)^{(v-1)}}{(\xi^\eta-z^\eta)^{1-\delta}}\\ &\quad\times g(s,x(s),Hx(s))dsdz.\end{aligned}$$

For the forthcoming analysis, we need the following assumptions:

(A1) There exists positive constants L_g and L_h such that
(i) $|g(t,x_1,y_1)-g(t,x_2,y_2)| \le L_g(\|x_1-x_2\|+\|y_1-y_2\|),\ t\in J,\ x_1,x_2,y_1,y_2\in X$.
(ii) $|h(t,s,x_1)-h(t,s,x_2)| \le L_h(\|x_1-x_2\|),\ t,s\in J,\ x_1,x_2\in X$.

(A2) $|g(t,x,y)| \le l(t)\phi(\|x\|), (t,x,y)\in J\times\mathbb{R}^2$, where $l\in L^1(J,\mathbb{R}^+)$ and $\phi:[0,\infty)\to(0,\infty)$ is a continuous nondecreasing function.

For our convenience, we can take

$$\begin{aligned}\chi = \tfrac{L_g}{\Gamma(v+1)}\Bigg\{&1+\tfrac{L_h}{(v+1)}+\tfrac{|\alpha|}{|\Lambda|}(|u_3|+|u_4|)\tfrac{\zeta^v\Gamma(\epsilon+\frac{v}{\sigma}+1)}{\Gamma(\epsilon+\frac{v}{\sigma}+\theta+1)}(1+\tfrac{\zeta L_h}{(v+1)})\\ &+\tfrac{1}{|\Lambda|}(|u_1|+|u_2|)\Big[|\beta|\tfrac{\xi^v\Gamma(\gamma+\frac{v}{\eta}+1)}{\Gamma(\gamma+\frac{v}{\eta}+\delta+1)}(1+\tfrac{\xi L_h}{(v+1)})+1+\tfrac{L_h}{(v+1)}\Big]\Bigg\},\end{aligned}\tag{3.1}$$

and

$$\chi_1 = \frac{1}{\Gamma(v+1)}\Bigg\{1+\frac{|\alpha|}{|\Lambda|}(|u_3|+|u_4|)\frac{\zeta^v\Gamma(\epsilon+\frac{v}{\sigma}+1)}{\Gamma(\epsilon+\frac{v}{\sigma}+\theta+1)} + \frac{1}{|\Lambda|}(|u_1|+|u_2|)\Big[|\beta|\frac{\xi^v\Gamma(\gamma+\frac{v}{\eta}+1)}{\Gamma(\gamma+\frac{v}{\eta}+\delta+1)}+1\Big]\Bigg\}. \tag{3.2}$$

Existence result via Krasnoselskii's fixed point theorem:

Lemma 3.1. ([16]) *(Krasnoselskii's fixed point theorem) Let S be a closed, convex, nonempty subset of a Banach space X. Let A, B be two operators such that:*
(i). $Ax + By \in S$, whenever $x, y \in S$;
(ii). A is compact and continuous;
(iii). B is a contraction mapping.
Then there exists $z \in S$ such that $z = Az + Bz$.

Theorem 3.1. *If the assumptions (A1) - (A2) hold, then there exists a solution for the boundary value problem (1.1)-(1.2) on J, provided*

$$\frac{L_g}{\Gamma(v+1)}\Bigg\{\frac{|\alpha|}{|\Lambda|}(|u_3|+|u_4|)\frac{\zeta^v\Gamma(\epsilon+\frac{v}{\sigma}+1)}{\Gamma(\epsilon+\frac{v}{\sigma}+\theta+1)}\Big(1+\frac{\zeta L_h}{(v+1)}\Big) + \frac{1}{|\Lambda|}(|u_1|+|u_2|)\Big[|\beta|\frac{\xi^v\Gamma(\gamma+\frac{v}{\eta}+1)}{\Gamma(\gamma+\frac{v}{\eta}+\delta+1)}\Big(1+\frac{\xi L_h}{(v+1)}\Big)+1+\frac{L_h}{(v+1)}\Big]\Bigg\} < 1. \tag{3.3}$$

P r o o f. Consider $B_r = \{x \in X : \|x\| \le r\}$.
We define the operators F_1 and F_2 by

$$\begin{aligned} F_1x(t) &= J^v g(s, x(s), Hx(s))(t),\ t \in J, \\ F_2x(t) &= \frac{\alpha}{\Lambda}(u_4 - tu_3)I_\sigma^{\epsilon,\theta}J^v g(s, x(s), Hx(s))(\zeta) \\ &\quad + \frac{1}{\Lambda}(u_2 + tu_1)\Big[|\beta|I_\eta^{\gamma,\delta}J^v g(s, x(s), Hx(s))(\xi) \\ &\quad + J^v g(s, x(x), Hx(s))(1)\Big],\quad t \in J. \end{aligned}$$

Choosing $r \ge \|l\|\phi(r)\chi_1$, where χ_1 is defined by (3.2). For any $x, y \in B_r$, we have

$$|F_1x(t) + F_2y(t)| \le \sup_{t\in J}\Bigg\{J^v|g(s, x(s), Hx(s))|(t)$$
$$+\frac{|\alpha|}{|\Lambda|}|u_4 - tu_3|I_\sigma^{\epsilon,\theta}J^v|g(s, y(s), Hy(s))|(\zeta)$$

$$
\begin{aligned}
&+\frac{1}{|\Lambda|}|u_2+tu_1|\Big[|\beta|I_\eta^{\gamma,\delta}J^v|g(s,y(s),Hy(s))|(\xi)\\
&+J^v|g(s,y(s),Hy(s))|(1)\Big]\Bigg\}\\
\le\ & l(s)\phi(\|x\|)J^v(1)+\frac{|\alpha|}{|\Lambda|}(|u_3|+|u_4|)l(s)\phi(\|y\|)I_\sigma^{\epsilon,\theta}J^v(\zeta)\\
&+\frac{1}{|\Lambda|}(|u_1|+|u_2|)\Big[|\beta|l(s)\phi(\|y\|)I_\eta^{\gamma,\delta}J^v(\xi)+l(s)\phi(\|y\|)J^v(1)\Big]\\
\le\ & \|l\|_{L^1}\phi(r)\frac{1}{\Gamma(v+1)}\Bigg\{1+\frac{|\alpha|}{|\Lambda|}(|u_3|+|u_4|)\frac{\zeta^v\Gamma(\epsilon+\frac{v}{\sigma}+1)}{\Gamma(\epsilon+\frac{v}{\sigma}+\theta+1)}\\
&+\frac{1}{|\Lambda|}(|u_1|+|u_2|)\Big[|\beta|\frac{\xi^v\Gamma(\gamma+\frac{v}{\eta}+1)}{\Gamma(\gamma+\frac{v}{\eta}+\delta+1)}+1\Big]\Bigg\}\\
=\ & \|l\|\phi(r)\chi_1\le r.
\end{aligned}
$$

Thus $F_1x+F_2y\in B_r$.

To show that F_2 is a contraction mapping,

$$
\begin{aligned}
&|F_2x(t)-F_2y(t)|\\
&\le \sup_{t\in J}\Bigg\{\frac{|\alpha|}{|\Lambda|}|u_4-tu_3|I_\sigma^{\epsilon,\theta}J^v|g(s,x(s),Hx(s))-g(s,y(s),Hy(s))|(\zeta)\\
&\quad+\frac{1}{|\Lambda|}|u_2+tu_1|\Big[|\beta|I_\eta^{\gamma,\delta}J^v|g(s,x(s),Hx(s))-g(s,y(s),Hy(s))|(\xi)\\
&\quad+J^v|g(s,x(s),Hx(s))-g(s,y(s),Hy(s))|(1)\Big]\Bigg\}\\
&\le L_g\|x-y\|\Bigg\{\Bigg[\frac{|\alpha|}{|\Lambda|}(|u_3|+|u_4|)\frac{\zeta^v\Gamma(\epsilon+\frac{v}{\sigma}+1)}{\Gamma(v+1)\Gamma(\epsilon+\frac{v}{\sigma}+\theta+1)}\\
&\quad+\frac{1}{|\Lambda|}(|u_1|+|u_2|)\left(|\beta|\frac{\xi^v\Gamma(\gamma+\frac{v}{\eta}+1)}{\Gamma(v+1)\Gamma(\gamma+\frac{v}{\eta}+\delta+1)}+\frac{1}{\Gamma(v+1)}\right)\Bigg]\\
&\quad+L_h\Bigg[\frac{|\alpha|}{|\Lambda|}(|u_3|+|u_4|)\frac{\zeta^{v+1}\Gamma(\epsilon+\frac{v}{\sigma}+1)}{\Gamma(v+2)\Gamma(\epsilon+\frac{v}{\sigma}+\theta+1)}\\
&\quad+\frac{1}{|\Lambda|}(|u_1|+|u_2|)\left(|\beta|\frac{\xi^{v+1}\Gamma(\gamma+\frac{v}{\eta}+1)}{\Gamma(v+2)\Gamma(\gamma+\frac{v}{\eta}+\delta+1)}+\frac{1}{\Gamma(v+2)}\right)\Bigg]\Bigg\}
\end{aligned}
$$

$$\leq \frac{L_g\|x-y\|}{\Gamma(v+1)}\Bigg\{\frac{|\alpha|}{|\Lambda|}(|u_3|+|u_4|)\frac{\zeta^v\Gamma(\epsilon+\frac{v}{\sigma}+1)}{\Gamma(\epsilon+\frac{v}{\sigma}+\theta+1)}\Big(1+\frac{\zeta L_h}{(v+1)}\Big)$$
$$+\frac{1}{|\Lambda|}(|u_1|+|u_2|)\Big[|\beta|\frac{\xi^v\Gamma(\gamma+\frac{v}{\eta}+1)}{\Gamma(\gamma+\frac{v}{\eta}+\delta+1)}\Big(1+\frac{\xi L_h}{(v+1)}\Big)+1+\frac{L_h}{(v+1)}\Big]\Bigg\}.$$

Since by (3.3), we have F_2 is a contraction.

The continuity of g and h implies that the operator F_1 is continuous. Also, F_1 is uniformly bounded on B_r as

$$|F_1x(t)| \leq \sup_{t\in J} J^v|g(s,x(s),Hx(s))|(t) \leq \frac{\|l\|_{L^1}\phi(r)}{\Gamma(v+1)}.$$

This is to prove the compactness of F_1.

Now, for $t_1, t_2 \in J$ with $t_1 < t_2$ and $x \in B_r$, we have

$$\begin{aligned}
&|F_1x(t_2)-F_1x(t_1)|\\
&= |J^v g(s,x(s),Hx(s))(t_2) - J^v g(s,x(s),Hx(s))(t_1)|\\
&\leq \left|\int_0^{t_2}\frac{(t_2-s)^{v-1}}{\Gamma(v)}l(s)\phi(\|x\|)ds - \int_0^{t_1}\frac{(t_1-s)^{v-1}}{\Gamma(v)}l(s)\phi(\|x\|)ds\right|\\
&= \frac{1}{\Gamma(v)}\Bigg|\int_0^{t_1}[(t_2-s)^{v-1}-(t_1-s)^{v-1}]\phi(r)l(s)ds\\
&\quad + \int_{t_1}^{t_2}(t_2-s)^{v-1}\phi(r)l(s)ds\Bigg|\\
&\leq \frac{\|l\|_{L^1}\phi(r)}{\Gamma(v+1)}[t_2^v - t_1^v + 2(t_2-t_1)^v],
\end{aligned}$$

which is independent of x and tends to zero as $t_2 \to t_1$. Thus F_1 is equicontinuous. By Arzela-Ascoli's theorem, F_1 is compact on B_r. Hence by Lemma 3.1, there exists a fixed point $x \in X$ such that $Fx = x$ which is a solution to the problem (1.1)-(1.2) on J. This completes the proof. □

Existence result via Schaefer's fixed point theorem:

LEMMA 3.2. ([16]) *(Schaefer's fixed point theorem) Let X be a Banach Space. Assume that $T : X \to X$ is a completely continuous operator and the set $\Omega = \{u \in X : u = \mu Tu, 0 < \mu < 1\}$ is bounded. Then T has a fixed point in X.*

THEOREM 3.2. *Assume that (A_2) hold, then there exists a solution for the problem (1.1)-(1.2) on J.*

P r o o f. Since f is continuous on J, therefore the operator F is continuous.

Next, we are to show that the operator F is completely continuous.

Consider $B_r = \{x \in X : \|x\| \leq r\}$. Then for $t \in J$, we have

$$\begin{aligned}
|Fx(t)| &\leq J^v|g(s,x(s),Hx(s))|(t) \\
&\quad + \frac{|\alpha|}{|\Lambda|}|u_4 - tu_3| I_\sigma^{\epsilon,\theta} J^v |g(s,x(s),Hx(s))|(\zeta) \\
&\quad + \frac{1}{|\Lambda|}|u_4 + tu_1|\Big[|\beta| I_\eta^{\gamma,\delta}|g(s,x(s),Hx(s))|(\xi) \\
&\quad + J^v|g(s,x(s),Hx(s))|(1)\Big] \\
&\leq \frac{\|l\|_{L^1}\phi(r)}{\Gamma(v+1)}\Bigg\{1 + \frac{|\alpha|}{|\Lambda|}(|u_3|+|u_4|)\frac{\zeta^v\Gamma(\epsilon+\frac{v}{\sigma}+1)}{\Gamma(\epsilon+\frac{v}{\sigma}+\theta+1)} \\
&\quad + \frac{1}{|\Lambda|}(|u_1|+|u_2|)\Big[|\beta|\frac{\xi^v\Gamma(\gamma+\frac{v}{\eta}+1)}{\Gamma(\gamma+\frac{v}{\eta}+\delta+1)} + 1\Big]\Bigg\} \\
&\leq \|l\|_{L^1}\phi(r)\chi_1.
\end{aligned}$$

Next, we prove that F maps bounded sets into equicontinuous sets of X.

Let $\tau_1, \tau_2 \in J$ with $\tau_1 < \tau_2$ and $x \in B_r$, we have

$$\begin{aligned}
&|(Fx)(\tau_2) - (Fx)(\tau_1)| \\
&\quad \leq |J^v g(s,x(s),Hx(s))(\tau_2) - J^v g(s,x(s),Hx(s))(\tau_1)| \\
&\qquad + \frac{|\alpha|}{|\Lambda|}|u_3||\tau_2 - \tau_1| I_\sigma^{\epsilon,\theta} J^v|g(s,x(s),Hx(s))|(\zeta) \\
&\qquad + \frac{1}{|\Lambda|}|u_1||\tau_2-\tau_1|\Big[|\beta| I_\eta^{\gamma,\delta} J^v |g(s,x(s),Hx(s))|(\xi) \\
&\qquad + J^v|g(s,x(s),Hx(s))|(1)\Big] \\
&\quad \leq \|l\|_{L^1}\phi(r)\Bigg\{\frac{1}{\Gamma(v)}\Bigg|\int_0^{\tau_1}[(\tau_2-s)^{v-1} - (\tau_1-s)^{v-1}]ds \\
&\qquad + \int_{\tau_1}^{\tau_2}(\tau_2-s)^{v-1}ds\Bigg| + \frac{|\alpha|}{|\Lambda|}|u_3||\tau_2-\tau_1| I_\sigma^{\epsilon,\theta} J^v(\zeta) \\
&\qquad + \frac{1}{|\Lambda|}|u_1||\tau_2-\tau_1|\Big[|\beta| I_\eta^{\gamma,\delta} J^v(\xi) + J^v(1)\Big]\Bigg\},
\end{aligned}$$

which is independent of x and tends to zero as $\tau_2 \to \tau_1$.

Hence, by Arzela-Ascoli theorem, the operator $F : X \to X$ is completely continuous.

Now it remains to show that the set $\Omega = \{x \in X : x = \lambda F(x), 0 < \lambda < 1\}$ is bounded. For $x \in \Omega$ and $t \in J$, we have

$$\begin{aligned} \|x\| &\leq \|l\|_{L^1}\phi(r)\frac{1}{\Gamma(v+1)}\left\{1 + \frac{|\alpha|}{|\Lambda|}(|u_3| + |u_4|)\frac{\zeta^v\Gamma(\epsilon + \frac{v}{\sigma} + 1)}{\Gamma(\epsilon + \frac{v}{\sigma} + \theta + 1)}\right. \\ &\quad \left. + \frac{1}{|\Lambda|}(|u_1| + |u_2|)\left[|\beta|\frac{\xi^v\Gamma(\gamma + \frac{v}{\eta} + 1)}{\Gamma(\gamma + \frac{v}{\eta} + \delta + 1)} + 1\right]\right\} \\ &= \|l\|_{L^1}\phi(r)\chi_1. \end{aligned}$$

This shows that Ω is bounded. Hence, by Lemma 3.2, the boundary value problem (1.1)-(1.2) has at least one solution on J. This completes the proof. □

Uniqueness result via Banach's fixed point theorem:

THEOREM 3.3. *If the assumptions (A1), (A2) are satisfied and $\chi < 1$, then there exists a unique solution for the boundary value problem (1.1)-(1.2) on J.*

P r o o f. Let $M_1 = \sup_{t \in J}|g(t, 0, 0)|$, $M_2 = \sup_{t \in J}|h(t, s, 0)|$ and choose $r \geq \triangle_1 + \triangle_2$.

Take

$$\begin{aligned} \triangle_1 = \quad & \frac{rL_g + M_1}{\Gamma(v+1)}\left\{1 + \frac{|\alpha|}{|\Lambda|}(|u_3| + |u_4|)\frac{\zeta^v\Gamma(\epsilon + \frac{v}{\sigma} + 1)}{\Gamma(\epsilon + \frac{v}{\sigma} + \theta + 1)}\right. \\ & \left. + \frac{1}{|\Lambda|}(|u_1| + |u_2|)\left[|\beta|\frac{\xi^v\Gamma(\gamma + \frac{v}{\eta} + 1)}{\Gamma(\gamma + \frac{v}{\eta} + \delta + 1)} + 1\right]\right\}, \\ \triangle_2 = \quad & \frac{rL_gL_h + L_gM_2}{\Gamma(v+2)}\left\{1 + \frac{|\alpha|}{|\Lambda|}(|u_3| + |u_4|)\frac{\zeta^{v+1}\Gamma(\epsilon + \frac{v}{\sigma} + 1)}{\Gamma(\epsilon + \frac{v}{\sigma} + \theta + 1)}\right. \\ & \left. + \frac{1}{|\Lambda|}(|u_1| + |u_2|)\left[|\beta|\frac{\xi^{v+1}\Gamma(\gamma + \frac{v}{\eta} + 1)}{\Gamma(\gamma + \frac{v}{\eta} + \delta + 1)} + 1\right]\right\}. \end{aligned}$$

Now we show that $FB_r \subset B_r$. Now for each $x \in B_r$ and $t \in J$, we have

$$\begin{aligned} &|(Fx)(t)| \\ &\quad \leq J^v(|g(s, x(s), Hx(s)) - g(s, 0, 0)| + |g(s, 0, 0)|)(1) \\ &\quad\quad + \frac{|\alpha|}{|\Lambda|}(|u_3| + |u_4|)I_\sigma^{\epsilon,\theta}J^v(|g(s, x(s), Hx(s)) - g(s, 0, 0)| \\ &\quad\quad + |g(s, 0, 0)|)(\zeta) + \frac{1}{|\Lambda|}(|u_1| + |u_2|)\Big[|\beta|I_\eta^{\gamma,\delta}J^v(|g(s, x(s), Hx(s)) \end{aligned}$$

$$
\begin{aligned}
& -g(s,0,0)| + |g(s,0,0)|)(\xi) \\
& +J^{v}(|g(s,x(s),Hx(s)) - g(s,0,0)| + |g(s,0,0)|)(1)\Big] \\
\leq\; & J^{v}[Lg(\|x(s)\| + \|Hx(s)\|) + M_1](1) \\
& +\frac{|\alpha|}{|\Lambda|}(|u_3|+|u_4|)I_{\sigma}^{\epsilon,\theta}J^{v}[L_g(\|x\| + Hx(s)\|) + M_1](\zeta) \\
& \frac{1}{\Lambda}(|u_1|+|u_2|)\Big[|\beta|I_{\eta}^{\gamma,\delta}J^{v}[L_g(\|x(s)\| + \|Hx(s)\|) + M_1](\xi) \\
& +J^{v}[L_g(\|x(s)\| + \|Hx(s)\|) + M_1](1)\Big] \\
\leq\; & \frac{rL_g + M_1}{\Gamma(v+1)}\Bigg\{1 + \frac{|\alpha|}{|\Lambda|}(|u_3|+|u_4|)\frac{\zeta^{v}\Gamma(\epsilon+\frac{v}{\sigma}+1)}{\Gamma(\epsilon+\frac{v}{\sigma}+\theta+1)} \\
& +\frac{1}{|\Lambda|}(|u_1|+|u_2|)\Big[|\beta|\frac{\xi^{v}\Gamma(\gamma+\frac{v}{\eta}+1)}{\Gamma(\gamma+\frac{v}{\eta}+\delta+1)} + 1\Big]\Bigg\} \\
& +\frac{rL_gL_h + L_gM_2}{\Gamma(v+2)}\Bigg\{1 + \frac{|\alpha|}{|\Lambda|}(|u_3|+|u_4|)\frac{\zeta^{v+1}\Gamma(\epsilon+\frac{v}{\sigma}+1)}{\Gamma(\epsilon+\frac{v}{\sigma}+\theta+1)} \\
& +\frac{1}{|\Lambda|}(|u_1|+|u_2|)\Big[|\beta|\frac{\xi^{v+1}\Gamma(\gamma+\frac{v}{\eta}+1)}{\Gamma(\gamma+\frac{v}{\eta}+\delta+1)} + 1\Big]\Bigg\} \\
=\; & \triangle_1 + \triangle_2 \leq r.
\end{aligned}
$$

This shows that $FB_r \subset B_r$. Now, for $x, y \in X$ and $t \in J$, we have

$$
\begin{aligned}
& |Fx(t) - Fy(t)| \\
\leq\; & \sup_{t\in J}\Big\{J^{v}|g(s,x(s),Hx(s)) - g(s,y(s),Hy(s))|(t) \\
& +\frac{|\alpha|}{|\Lambda|}(|u_3|+|u_4|)I_{\sigma}^{\epsilon,\theta}J^{v}|g(s,x(s),Hx(s)) - g(s,y(s),Hy(s))|(\zeta) \\
& +\frac{1}{|\Lambda|}(|u_1|+|u_2|)\Big[|\beta|I_{\eta}^{\gamma,\delta}J^{v}|g(s,x(s),Hx(s)) - g(s,y(s),Hy(s))|(\xi) \\
& +J^{v}|g(s,x(s),Hx(s)) - g(s,y(s),Hy(s))|(1)\Big]\Big\} \\
\leq\; & L_g\Bigg[\Bigg\{\frac{1}{\Gamma(v+1)} + \frac{|\alpha|}{|\Lambda|}(|u_3|+|u_4|)\frac{\zeta^{v}\Gamma(\epsilon+\frac{v}{\sigma}+1)}{\Gamma(v+1)\Gamma(\epsilon+\frac{v}{\sigma}+\theta+1)} \\
& +\frac{1}{|\Lambda|}(|u_1|+|u_2|)\Bigg[|\beta|\frac{\xi^{v}\Gamma(\gamma+\frac{v}{\eta}+1)}{\Gamma(v+1)\Gamma(\gamma+\frac{v}{\eta}+\delta+1)} + \frac{1}{\Gamma(v+1)}\Bigg]\Bigg\}\|x-y\|
\end{aligned}
$$

$$
\begin{aligned}
&+L_h\Bigg\{\frac{1}{\Gamma(v+2)}+\frac{|\alpha|}{|\Lambda|}(|u_3|+|u_4|)\frac{\zeta^{v+1}\Gamma(\epsilon+\frac{v}{\sigma}+1)}{\Gamma(v+2)\Gamma(\epsilon+\frac{v}{\sigma}+\theta+1)}\\
&+\frac{1}{|\Lambda|}(|u_1|+|u_2|)\bigg[|\beta|\frac{\xi^{v+1}\Gamma(\gamma+\frac{v}{\eta}+1)}{\Gamma(v+2)\Gamma(\gamma+\frac{v}{\eta}+\delta+1)}+\frac{1}{\Gamma(v+2)}\bigg]\Bigg\}\|x-y\|\Bigg]\\
\le\ & \frac{L_g\|x-y\|}{\Gamma(v+1)}\Bigg\{1+\frac{L_h}{(v+1)}+\frac{|\alpha|}{|\Lambda|}(|u_3|+|u_4|)\frac{\zeta^{v}\Gamma(\epsilon+\frac{v}{\sigma}+1)}{\Gamma(\epsilon+\frac{v}{\sigma}+\theta+1)}\Big(1+\frac{\zeta L_h}{(v+1)}\Big)\\
&+\frac{1}{|\Lambda|}(|u_1|+|u_2|)\Big[|\beta|\frac{\xi^{v}\Gamma(\gamma+\frac{v}{\eta}+1)}{\Gamma(\gamma+\frac{v}{\eta}+\delta+1)}\Big(1+\frac{\xi L_h}{(v+1)}\Big)+1+\frac{L_h}{(v+1)}\Big]\Bigg\}\\
=\ & \chi\|x-y\|.
\end{aligned}
$$

Since $\chi<1$, we have F is a contraction. Hence, by the Banach's fixed point theorem, F has a fixed point which is the unique solution for the problem (1.1)-(1.2) on J. This completes the proof. □

4. **Examples**

EXAMPLE 4.1. Consider the following nonlinear fractional integro-differential equation

$$
{}^cD^{\frac{3}{2}}x(t)=\frac{1}{2}\frac{e^{-3t}}{(t+4)^2}\frac{|x(t)|}{1+|x(t)|}+\frac{1}{32}\int_0^t\frac{e^{-2s}}{15}\frac{|x(s)|}{1+|x(s)|}ds, \tag{4.1}
$$

with the boundary conditions

$$
x(0)=\frac{\sqrt{3}}{2}I_{\frac{1}{2}}^{\frac{3}{7},\frac{\sqrt{5}}{9}}x\Big(\frac{1}{3}\Big),\text{and } x(1)=\frac{3}{5}I_{\frac{1}{3}}^{\frac{\sqrt{3}}{7},\frac{2}{5}}x\Big(\frac{2}{3}\Big). \tag{4.2}
$$

Here $v=\frac{3}{2},\alpha=\frac{\sqrt{3}}{2},\beta=\frac{3}{5},\sigma=\frac{1}{2},\epsilon=\frac{3}{7},\theta=\frac{\sqrt{5}}{9},\zeta=\frac{1}{3},\eta=\frac{1}{3},\gamma=\frac{\sqrt{3}}{4},\delta=\frac{2}{5},\xi=\frac{2}{3}$. Also $L_g=\frac{1}{32},L_h=\frac{1}{15}$.

Using the given data, we found that $u_1=0.15160103681942672, u_2=0.7814552314958013, u_3=0.4348335918159857, u_4=0.7734309135282774$ and $\Lambda=0.4570559135537951\neq 0$.

Thus,

$$
\begin{aligned}
&\frac{L_g}{\Gamma(v+1)}\Bigg\{\frac{|\alpha|}{|\Lambda|}(|u_3|+|u_4|)\frac{\zeta^{v}\Gamma(\epsilon+\frac{v}{\sigma}+1)}{\Gamma(\epsilon+\frac{v}{\sigma}+\theta+1)}\Big(1+\frac{\zeta L_h}{(v+1)}\Big)\\
&+\frac{1}{|\Lambda|}(|u_1|+|u_2|)\Big[|\beta|\frac{\xi^{v}\Gamma(\gamma+\frac{v}{\eta}+1)}{\Gamma(\gamma+\frac{v}{\eta}+\delta+1)}\Big(1+\frac{\xi L_h}{(v+1)}\Big)+1+\frac{L_h}{(v+1)}\Big]\Bigg\}\\
\cong\ & 0.040229397218974706<1.
\end{aligned}
$$

Clearly, all conditions of Theorem 3.1 are satisfied. Hence the problem (4.1) with (4.2) has at least one solution on J.

Example 4.2. Consider the nonlinear fractional integro-differential equation:

$$ {}^cD^{\frac{7}{4}}x(t) = \frac{1}{6}\Big(\frac{x^2(t)+2|x(t)|}{1+|x(t)|}\Big)\sin^2 t + \frac{1}{3}\int_0^t \frac{e^{-s}}{7}\cos s ds, \qquad (4.3) $$

with the boundary conditions (4.2).

Here $v = \frac{7}{4}, \alpha = \frac{\sqrt{3}}{2}, \beta = \frac{3}{5}, \sigma = \frac{1}{2}, \epsilon = \frac{3}{7}, \theta = \frac{\sqrt{5}}{9}, \zeta = \frac{1}{3}, \eta = \frac{1}{3}, \gamma = \frac{\sqrt{3}}{4}, \delta = \frac{2}{5}, \xi = \frac{2}{3}$. Also $L_g = \frac{1}{3}, L_h = \frac{1}{7}$.

Using the given data, we found that $u_1 = 0.15160103681942672, u_2 = 0.7814552314958013, u_3 = 0.4348335918159857, u_4 = 0.7734309135282774$ and $\Lambda = 0.4570559135537951 \neq 0$.

Thus

$$ \frac{L_g}{\Gamma(v+1)}\Bigg\{1 + \frac{L_h}{v+1} + \frac{|\alpha|}{|\Lambda|}(|u_3|+|u_4|)\frac{\zeta^v\Gamma(\epsilon+\frac{v}{\sigma}+1)}{\Gamma(\epsilon+\frac{v}{\sigma}+\theta+1)}\Big(1+\frac{\zeta L_h}{(v+1)}\Big) $$

$$ +\frac{1}{|\Lambda|}(|u_1|+|u_2|)\Big[|\beta|\frac{\xi^v\Gamma(\gamma+\frac{v}{\eta}+1)}{\Gamma(\gamma+\frac{v}{\eta}+\delta+1)}\Big(1+\frac{\xi L_h}{(v+1)}\Big)+1+\frac{L_h}{(v+1)}\Big]\Bigg\} $$

$$ \cong 0.5633023999907423 < 1. $$

Clearly, all assumptions of Theorem 3.3 are satisfied. Therefore, the problem (4.3) with (4.2) has a unique solution on J.

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