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Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME 23, NUMBER 5 (2020) (Print) ISSN 1311-0454
(Electronic) ISSN 1314-2224

RESEARCH PAPER

FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS IN WIENER SPACES

Vu Kim Tuan

Abstract

In this paper we study the global solvability of several ordinary and partial fractional integro-differential equations in the Wiener space of functions with bounded square averages.

MSC 2010: Primary: 44A10, 26A33, 45K05; Secondary: 35D35, 35R30

Key Words and Phrases: functions with bounded square averages; Laplace transform; Caputo fractional derivative; Riemann-Liouville fractional derivative; time fractional partial integro-differential equation

1. **Introduction**

Consider the time fractional integro-differential equation

$$\begin{cases} \partial_t^\alpha u(x,t) = kAu(x,t) - \int_0^t g(t-\tau)u(x,\tau)d\tau + h(x), \ (x,t) \in \Omega \times \mathbb{R}_+, \\ u(x,t) = 0, \qquad (x,t) \in \partial\Omega \times \mathbb{R}_+, \\ u(x,0) = f(x), \qquad x \in \Omega, \end{cases} \tag{1.1}$$

where ∂_t^α is either the Caputo or the Riemann-Liouville fractional derivatives of order $\alpha \in (0,\ 1]$ with respect to time variable t [9], and A is a self-adjoint differential operator acting in $L^2(\Omega)$, $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$. Equation (1.1) is a well known model for heat distribution of a visco-elastic material with memory and has important applications in material science [1, 3, 4]. When $\alpha = 1$, the local and global existence of solutions is usually handled by semi-group theory, [7, 13]. However, when $0 < \alpha < 1$, the proof for the local existence of solutions by semi-group theory is not possible, because

pp. 1300–1328 , DOI: 10.1515/fca-2020-0065 DE GRUYTER

∂_t^α is a non-local operator. Due to the presence of the convolution operator, Laplace transform becomes the tool of choice as it takes care of both the nonlocal operation ∂_t^α and the Laplace convolution $\int_0^t g(t-\tau)u(x,\tau)d\tau$. Until now it is not clear which function space (especially in time variable) does the solution $u(x,t)$ belong to. Clearly L^p-spaces do not serve the purpose. The main contribution of this paper is to show that the Wiener space of functions with bounded square averages [17] is the right function space for (1.1).

In this paper we are interested in two objectives:

A: Characterize the Laplace transform of Wiener functions with bounded square averages and prove the global existence of solutions in direct problems in the Wiener space.

B: Given $A = \Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2}$, the Laplace operator, solve the inverse problem of reconstructing the fractional order α, the parameter k, and the memory function g from a single observation of the solution at one point.

The outline of the paper is as follows: In Section 2 we introduce the Wiener space of functions with bounded square averages and characterize the Laplace transform of such functions. In Section 3 we solve Caputo and Riemann-Liouville fractional ordinary integro-differential equations, i.e. when A is the identity operator, while in Sections 4 and 5, $A = \Delta$, and we deal with time fractional partial integro-differential equations where we prove the global existence of solutions in the Wiener space and then solve inverse problems for such equations in Section 6.

2. **Functions with bounded square averages**

Denote by $\mathcal{L}$ and $\mathcal{L}^{-1}$ the Laplace transform and its inverse ([16])

$$\begin{aligned} F(s) &= (\mathcal{L}f)(s) = \int_0^\infty e^{-st} f(t)\,dt, \qquad (2.1)\\ f(t) &= \left(\mathcal{L}^{-1}F\right)(t) = \frac{1}{2\pi i}\int_{\operatorname{Re} s = d} F(s)\,e^{st}\,ds. \end{aligned}$$

In practice given a function F, the crucial issue is to find conditions such that it is the Laplace transform of a certain function f? Characterizing the range of the Laplace transform has been investigated extensively and the first result in this direction is the celebrated Paley-Wiener theorem for the Fourier transform [11], which takes the following form for the Laplace transform [16].

PROPOSITION 2.1 (Paley-Wiener). *$f \in L^2(\mathbb{R}_+)$ if and only if $F \in H^2(\mathbb{R}_+^2)$.*

Here $H^p(\mathbb{R}^2_+)$ is the Hardy space of functions $F(s) = F(x+iy)$, analytic in the *right* half-plane with

$$\sup_{x>0} \int_{-\infty}^{\infty} |F(x+iy)|^p dy < \infty.$$

For $f \in L^\infty(\mathbb{R}_+)$ Widder [16] proved the following result.

Proposition 2.2 (Widder). *A function F is the Laplace transform of $f \in L^\infty(\mathbb{R}_+)$ if and only if F is infinitely differentiable on $\mathbb{R}_+$, and*

$$\sup \left\{ \left| \frac{x^{n+1}}{n!} F^{(n)}(x) \right| : x > 0, n = 0, 1, \cdots \right\} < \infty.$$

In [14] it was shown that

$$\sup_{x>0} \; x \int_{-\infty}^{\infty} |F(x+iy)|^2 \; dy \qquad \Longleftrightarrow \qquad \sup_{T>0} \frac{1}{T} \int_0^T |f(t)|^2 \, dt < \infty.$$

It turns out that functions with bounded square averages on $\mathbb{R}_+$, first defined on $\mathbb{R}$ by N. Wiener in the celebrated paper [17], can play very important role in studying fractional integro-differential equations. Let us start with the definition of such functions.

Definition 2.1. By $BSA(\mathbb{R}_+)$, the Wiener linear space of functions with bounded square averages on $\mathbb{R}_+$, we denote the set of locally integrable functions f on $\mathbb{R}_+$ such that

$$\sup_{T>0} \frac{1}{T+1} \int_0^T |f(t)|^2 \, dt < \infty. \tag{2.2}$$

We say $f \in BSA^m(\mathbb{R}_+)$ if $f, f', \cdots, f^{(m)} \in BSA(\mathbb{R}_+)$.

It is readily seen that $L^2(\mathbb{R}_+) \cup L^\infty(\mathbb{R}_+) \subset BSA(\mathbb{R}_+)$ and by Hölder's inequality $L^p(\mathbb{R}_+) \subset BSA(\mathbb{R}_+)$ for $2 \le p \le \infty$. However, note that, for $-\frac{1}{2} \le \beta \le 0$, we have $f(t) = t^\beta \in BSA(\mathbb{R}_+)$, and yet $f(t) \notin L^p(\mathbb{R}_+)$, $2 \le p \le \infty$.

Now we characterize the Laplace transform of functions from $BSA(\mathbb{R}_+)$.

Theorem 2.1. *A function $F(s)$ is the Laplace transform of $f \in BSA(\mathbb{R}_+)$ if and only if $F(s)$ is analytic in the right-half plane $\operatorname{Re} s > 0$, and*

$$\sup_{x>0} \frac{x}{x+1} \int_{-\infty}^{\infty} |F(x+iy)|^2 \; dy < \infty. \tag{2.3}$$

P r o o f. The proof follows [14]. Let $f \in BSA(\mathbb{R}_+)$. Denote $\tilde{f}(T) = \int_0^T f(t)\,dt$. Integration by parts gives

$$F(s) := \int_0^\infty e^{-st} f(t)\,dt = e^{-sT}\tilde{f}(T)\Big|_{T=0}^{T=\infty} + s\int_0^\infty e^{-st}\tilde{f}(t)\,dt, \quad \operatorname{Re} s > 0.$$

By Hölder's inequality we have, for $T > 0$,

$$\begin{aligned} |\tilde{f}(T)| &\le \int_0^T 1.|f(t)|dt \le \sqrt{\int_0^T dt \int_0^T |f(t)|^2 dt} \\ &= \sqrt{T}\sqrt{\int_0^T |f(t)|^2 dt} \le M\sqrt{T}\sqrt{T+1}. \end{aligned}$$

Here and throughout the paper M denotes a universal constant that can be distinct in different places. Hence

$$e^{-sT}\tilde{f}(T)\Big|_{T=0}^{T=\infty} = 0, \quad \operatorname{Re} s > 0,$$

and

$$F(s) = s\int_0^\infty e^{-st}\tilde{f}(t)\,dt, \quad \operatorname{Re} s > 0.$$

Since $|\tilde{f}(t)| \le M\sqrt{t(t+1)}$, the Laplace transform of $\tilde{f}(t)$, i.e. $\frac{F(s)}{s}$, exists and is analytic in the right half plane $\operatorname{Re} s > 0$.

Integration by parts yields

$$\begin{aligned} &\int_0^\infty e^{-2xt}|f(t)|^2 dt = e^{-2xT}\int_0^T |f(t)|^2\,dt\Big|_{T=0}^{T=\infty} \\ &+2x\int_0^\infty e^{-2xT}\int_0^T |f(t)|^2\,dt\,dT \le Mx\int_0^\infty (T+1)e^{-2xT}\,dT \le \frac{M(x+1)}{x}. \end{aligned} \tag{2.4}$$

Hence, $e^{-xt}f(t) \in L^2(\mathbb{R}_+)$ for any $x > 0$. Consequently, $F(s)$ with $\operatorname{Re} s > x_0 > 0$ is the Laplace transform of $e^{-x_0 t}f(t) \in L^2(\mathbb{R}_+)$ at the point $s - x_0$. The Parseval formula for the Laplace transform in $L^2(\mathbb{R}_+)$, see [16], gives

$$\int_0^\infty e^{-2xt}|f(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^\infty |F(x+iy)|^2\,dy, \quad x > x_0 > 0. \tag{2.5}$$

Since x_0 is an arbitrary positive constant, equality (2.5) holds for any $x > 0$. Combining formulas (2.4) and (2.5) we obtain

$$\int_{-\infty}^\infty |F(x+iy)|^2\,dy \le \frac{M(x+1)}{x},$$

that yields (2.3).

Conversely, assume that $F(s)$ is analytic in the right-half plane $\operatorname{Re} s > 0$ and formula (2.3) holds. Then

$$\sup_{x>x_0}\int_{-\infty}^{\infty}|F(x+iy)|^2dy<\infty,\quad x_0>0.$$

Hence $F(s+x_0)$ is a Hardy function $H^2(\mathbb{R}^2_+)$ in the right-half plane $\operatorname{Re} s>0$. Therefore, by the Paley-Wiener Proposition 2.1 function $F(x_0+s)$ is the Laplace transform of a function, say, $f_{x_0}(t)\in L^2(\mathbb{R}_+)$

$$F(x_0+s)=\int_0^{\infty}e^{-st}f_{x_0}(t)\,dt,\quad \operatorname{Re} s>0.$$

Thus

$$\begin{aligned}F(x_0+x_1+s)&=\int_0^{\infty}e^{(-x_1-s)t}f_{x_0}(t)dt\\&=\int_0^{\infty}e^{(-x_0-s)t}f_{x_1}(t)dt,\quad \operatorname{Re} s,x_0,x_1>0.\end{aligned}$$

Consequently, $e^{-x_1t}f_{x_0}(t)=e^{-x_0t}f_{x_1}(t)$. Denote $f(t)=e^{x_0t}f_{x_0}(t)$. It is clear that $f(t)$ is independent of $x_0>0$ and F is the Laplace transform of f,

$$F(s)=\int_0^{\infty}e^{-st}f(t)\,dt,\quad \operatorname{Re} s>x_0+x_1.$$

As $e^{-x_0t}f(t)=f_{x_0}(t)\in L^2(\mathbb{R}_+)$, the Parseval formula for the Laplace transform [16] yields

$$\int_0^{\infty}e^{-2x_0t}|f(t)|^2\,dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}|F(x_0+iy)|^2\,dy\le\frac{M(x_0+1)}{x_0},\quad x_0>0.$$

Let g be a bounded function on $\mathbb{R}_+$. Then

$$\int_0^{\infty}e^{-2xt}g(e^{-2xt})|f(t)|^2\,dt\le\|g\|_\infty\int_0^{\infty}e^{-2xt}|f(t)|^2\,dt\le\frac{M(x+1)}{x}\|g\|_\infty,\tag{2.6}$$

for $x>0$. Take

$$g(t)=\begin{cases}\frac{1}{t}, & t>e^{-2}\\ 0, & 0<t\le e^{-2}\end{cases}.$$

Then $\|g\|_\infty=e^2$, and (2.6) becomes

$$\int_0^{1/x}|f(t)|^2\,dt\le\frac{M(x+1)}{x},\quad x>0.$$

Replacing x by $\frac{1}{T}$ we arrive at

$$\frac{1}{T+1}\int_0^{T}|f(t)|^2\,dt\le M,\quad T>0.$$

Thus $f\in BSA(\mathbb{R}_+)$ and Theorem 2.1 is proved. □

COROLLARY 2.1. *Let $F(s)$ be analytic in the right half plane* $\mathrm{Re}\, s > 0$ *and* $|F(s)| \le C|s|^{-\alpha}$, $\frac{1}{2} < \alpha \le 1$. *Then F is the Laplace transform of a function* $f \in BSA(\mathbb{R}_+)$.

P r o o f. Because $\alpha > \frac{1}{2}$, $F(x + i\bullet) \in L^2(\mathbb{R})$, and for $\frac{1}{2} < \alpha \le 1$,

$$\begin{aligned} \frac{x}{x+1}\int_{-\infty}^{\infty} |F(x+iy)|^2 dy &\le \frac{Mx}{x+1}\int_{-\infty}^{\infty} (x^2+y^2)^{-\alpha} dy \\ &= \frac{M\sqrt{\pi}\,\Gamma\left(\alpha-\frac{1}{2}\right)}{\Gamma(\alpha)} \frac{x^{2-2\alpha}}{x+1} < \infty, \end{aligned}$$

hence, formula (2.3) holds, i.e., $f \in BSA(\mathbb{R}_+)$. □

The following result is very important in solving fractional integro-differential equations in Sections 3-5.

THEOREM 2.2. *Let* $\|g\|_1 < k$, $0 < \alpha \le 1$, *then for the inverse Laplace transform* $\mathcal{L}^{-1}$

$$f := \mathcal{L}^{-1}\left(\frac{s^{\alpha-1}}{s^\alpha + k + G(s)}\right) \in BSA(\mathbb{R}_+). \tag{2.7}$$

P r o o f. Since $g \in L^1(\mathbb{R}_+)$, its Laplace transform $G(s)$ is analytic in the right half-plane, and from

$$|G(s)| \le \int_0^\infty e^{-(\mathrm{Re}\, s)\, t}\, |g(t)|\, dt \le \|g\|_1 < k, \qquad \text{for} \quad \mathrm{Re}\, s \ge 0,$$

we deduce that $\dfrac{s^{\alpha-1}}{s^\alpha + k + G(s)}$ is also analytic in the right half-plane. Let us denote by

$$h(s) = k + G(s),$$

then $h(s)$ is clearly analytic in the right half-plane, and for $\mathrm{Re}\, s > 0$,

$$0 < \varepsilon := k - \|g\|_1 \le \mathrm{Re}\, h(s) \quad \text{and also} \quad \varepsilon \le |h(s)| < 2k. \tag{2.8}$$

It is enough to show that $\dfrac{s^{\alpha-1}}{s^\alpha + h(s)}$ satisfies (2.3). Put $s = x + iy,\ x > 0$,

$$\begin{aligned} \frac{x}{x+1}\int_{-\infty}^{\infty} \frac{|s|^{2\alpha-2}}{|s^\alpha + h(s)|^2} dy &= I_1(x) + I_2(x) \\ &:= \frac{x}{x+1}\int_{y\in\mathbb{R},\ \mathrm{Re}\left(s^\alpha \overline{h(s)}\right)\ge 0} \frac{|s|^{2\alpha-2}}{|s^\alpha + h(s)|^2}\, dy \\ &+ \frac{x}{x+1}\int_{y\in\mathbb{R},\ \mathrm{Re}\left(s^\alpha \overline{h(s)}\right)< 0} \frac{|s|^{2\alpha-2}}{|s^\alpha + h(s)|^2}\, dy. \end{aligned}$$

We need to show that $I_1(x)$ and $I_2(x)$ are uniformly bounded.

For $I_1(x)$ we see first that

$$|s^\alpha + h(s)|^2 = (s^\alpha + h(s))\left(\overline{s}^\alpha + \overline{h(s)}\right) = |s|^{2\alpha} + 2\operatorname{Re}\left(s^\alpha \overline{h(s)}\right) + |h(s)|^2,$$

and thus it follows that

$$\operatorname{Re}\left(s^\alpha \overline{h(s)}\right) \geq 0 \implies |s^\alpha + h(s)|^2 \geq |s|^{2\alpha}. \tag{2.9}$$

Using (2.9) when $\operatorname{Re}\left(s^\alpha\ \overline{h(s)}\right) \geq 0$, we have

$$I_1(x) \leq \frac{x}{x+1}\int_{y\in\mathbb{R},\ \operatorname{Re}\left(s^\alpha \overline{h(s)}\right)\geq 0} \frac{|s|^{2\alpha-2}}{|s|^{2\alpha}}\,dy \leq \frac{x}{x+1}\int_{-\infty}^{\infty}\frac{1}{|s|^2}\,dy$$
$$= \frac{x}{x+1}\int_{-\infty}^{\infty}\frac{dy}{x^2+y^2} = \frac{\pi}{x+1} \leq \pi.$$

For $\operatorname{Re}\left(s^\alpha\ \overline{h(s)}\right) < 0$ write

$$h(s) = h_1(s) + ih_2(s),$$

where $h_1 = \operatorname{Re} h$ and $h_2 = \operatorname{Im} h$. We then have for $s = re^{i\varphi}$, $-\frac{\pi}{2} < \varphi < \frac{\pi}{2}$,

$$0 > \operatorname{Re}\left(s^\alpha\ \overline{h(s)}\right) = r^\alpha\left(h_1(s)\cos\alpha\varphi + h_2(s)\sin\alpha\varphi\right). \tag{2.10}$$

Since $\alpha\varphi \in \left(\frac{-\pi}{2},\ \frac{\pi}{2}\right)$ then $\cos\alpha\varphi > 0$ and by (2.8)

$$h_1(s) = \operatorname{Re} h(s) \geq \varepsilon > 0, \tag{2.11}$$

which means that

$$h_2(s)\sin\alpha\varphi < 0 \qquad \text{when} \quad \operatorname{Re}\left(s^\alpha\ \overline{h(s)}\right) < 0$$

by (2.10). In other words $h_2(s)$ and $\sin(\alpha\varphi)$ have different signs, and furthermore

$$0 < h_1(s)\cos\alpha\varphi < |h_2(s)\sin\alpha\varphi|. \tag{2.12}$$

When $\operatorname{Re}\left(s^\alpha\ \overline{h(s)}\right) < 0$ we can write

$$|s^\alpha + h(s)|^2 = |s|^{2\alpha} + 2\operatorname{Re}\left(s^\alpha \overline{h(s)}\right) + |h(s)|^2 \tag{2.13}$$
$$= \left(|h(s)| + \frac{\operatorname{Re}\left(s^\alpha \overline{h(s)}\right)}{|h(s)|}\right)^2 + \left(|s|^{2\alpha} - \frac{\left[\operatorname{Re}\left(s^\alpha \overline{h(s)}\right)\right]^2}{|h(s)|^2}\right)$$
$$\geq \frac{\left|s^\alpha\ \overline{h(s)}\right|^2 - \left[\operatorname{Re}\left(s^\alpha\ \overline{h(s)}\right)\right]^2}{|h(s)|^2} = \frac{\left[\operatorname{Im}\left(s^\alpha\ \overline{h(s)}\right)\right]^2}{|h(s)|^2}.$$

Thus we have from (2.13)

$$\begin{aligned} I_2(x) &\leq \frac{x}{x+1} \int_{y\in\mathbb{R},\ \mathrm{Re}\left(s^\alpha \overline{h(s)}\right)<0} \frac{|s|^{2\alpha-2}}{|s^\alpha + h(s)|^2}\, dy \\ &\leq \frac{x}{x+1} \int_{y\in\mathbb{R},\ \mathrm{Re}\left(s^\alpha \overline{h(s)}\right)<0} \frac{|h(s)|^2\ |s|^{2\alpha}}{\left[\mathrm{Im}\left(s^\alpha\ \overline{h(s)}\right)\right]^2 |s|^2}\, dy. \end{aligned} \tag{2.14}$$

To proceed further we shall need the following estimate:
If $\mathrm{Re}\left(s^\alpha\ \overline{h(s)}\right) < 0$ then

$$\left|s^\alpha\ \overline{h(s)}\right|^2 \leq \left(\frac{4k^2}{\varepsilon^2} + 1\right) \left[\mathrm{Im}\left(s^\alpha\ \overline{h(s)}\right)\right]^2 \qquad \text{for} \quad \mathrm{Re}\, s > 0. \tag{2.15}$$

In fact, from (2.12) and (2.8) we have

$$|\mathrm{Re}\,(s^\alpha\ h(s))| < r^\alpha\, |h_2\,(s) \sin\alpha\varphi| < r^\alpha 2k\, |\sin\alpha\varphi| \leq r^\alpha \frac{2k}{\varepsilon}\ \varepsilon\, |\sin\alpha\varphi|\,,$$

and by (2.11) we have

$$\begin{aligned} &|\mathrm{Re}\,(s^\alpha\ h(s))| \leq r^\alpha \frac{2k}{\varepsilon}\ \ |h_1(s) \sin\alpha\varphi| \\ &\leq \frac{2k}{\varepsilon} \left(r^\alpha\ \ |h_1(s)\sin\alpha\varphi| + r^\alpha\, |h_2(s)\cos\alpha\varphi|\right) \\ &\leq \frac{2k}{\varepsilon} r^\alpha\, |\ h_1(s)\sin\alpha\varphi - h_2(s)\cos\alpha\varphi| \end{aligned}$$

as $h_1(s)\sin\alpha\varphi$ and $-h_2(s)\cos\alpha\varphi$ have the same sign. Thus we deduce that

$$|\mathrm{Re}\,(s^\alpha\ h(s))| \leq \frac{2k}{\varepsilon}\ \left|\mathrm{Im}\left(s^\alpha\ \overline{h(s)}\right)\right|,$$

and it follows that

$$\begin{aligned} \left|s^\alpha\ \overline{h(s)}\right|^2 &= \left[\mathrm{Re}\left(s^a \overline{h(s)}\right)\right]^2 + \left[\mathrm{Im}\left(s^a \overline{h(s)}\right)\right]^2 \\ &\leq \left(\frac{4k^2}{\varepsilon^2} + 1\right) \left[\mathrm{Im}\left(s^a \overline{h(s)}\right)\right]^2, \end{aligned}$$

which proves (2.15).

Going back to (2.14) we get

$$
\begin{aligned}
I_2(x) &\le \frac{x}{x+1} \int_{-\infty}^{\infty} \frac{|h(s)|^2 \ |s|^{2\alpha}}{\left[\operatorname{Im}\left(s^\alpha \ \overline{h(s)}\right)\right]^2 |s|^2} \, dy \\
&\le \frac{x}{x+1} \left(\frac{4k^2}{\varepsilon^2} + 1 \right) \int_{-\infty}^{\infty} \frac{1}{|s|^2} \, dy \\
&\le \frac{x}{x+1} \left(\frac{4k^2}{\varepsilon^2} + 1 \right) \int_{-\infty}^{\infty} \frac{dy}{x^2+y^2} \\
&\le \pi \left(\frac{4k^2}{\varepsilon^2} + 1 \right).
\end{aligned}
$$

Consequently, $f \in BSA(\mathbb{R}_+)$. □

REMARK 2.1. From the proof of Theorem 2.2 we have the following estimate

$$
\left| \frac{s^{\alpha-1}}{s^\alpha + k + G(s)} \right| \le \sqrt{\left(\left(\frac{2k}{k - \|g\|_1} \right)^2 + 1 \right) \frac{1}{|s|}}, \qquad \operatorname{Re} s > 0. \tag{2.16}
$$

Combining Corollary 2.1 and Remark 2.1, we arrive at

COROLLARY 2.2. *Let $\|g\|_1 < k$, $\frac{1}{2} < \alpha \le 1$, then the inverse Laplace transform*

$$
f := \mathcal{L}^{-1} \left(\frac{1}{s^\alpha + k + G(s)} \right) \tag{2.17}
$$

is from $BSA(\mathbb{R}_+)$.

LEMMA 2.1. *Let $f \in BSA(\mathbb{R}_+)$ and $g \in L^1(\mathbb{R}_+)$. Then their Laplace convolution*

$$
h(t) = (f * g)(t) := \int_0^t f(t-\tau) \, g(\tau) \, d\tau \tag{2.18}
$$

belongs to $BSA(\mathbb{R}_+)$.

P r o o f. In fact, by applying the Laplace transform to (2.18), we obtain $H(s) = F(s)G(s)$, therefore, $|H(s)| \le |F(s)| \, \|g\|_1$, and thus

$$
\sup_{x>0} \frac{x}{x+1} \int_{-\infty}^{\infty} |H(x+iy)|^2 \, dy \le \|g\|_1^2 \sup_{x>0} \frac{x}{x+1} \int_{-\infty}^{\infty} |F(x+iy)|^2 \, dy < \infty.
$$

□

3. Fractional integro-differential equations

3.1. Caputo fractional integro-differential equation. Now consider the following Caputo fractional integro-differential equation

$$ {}^{\mathcal{C}}\partial_t^\alpha f(t) + kf(t) + \int_0^t g(t-\tau)f(\tau)d\tau = h(t), \qquad f(0) = f_0, \qquad \frac{1}{2} < \alpha \le 1, \tag{3.1} $$

where $g, h \in L^1(\mathbb{R}_+)$ are given, and f is an unknown. Here ${}^{\mathcal{C}}\partial_t^\alpha$ is the Caputo fractional derivative defined by ([9])

$$ {}^{\mathcal{C}}\partial_t^\alpha f(t) = \int_0^t \frac{(t-\tau)^{n-\alpha-1}}{\Gamma(n-\alpha)} f^{(n)}(\tau)\, d\tau,\ n-1<\alpha<n;\ {}^{\mathcal{C}}\partial_t^n f(t) = f^{(n)}(t). \tag{3.2} $$

It is well known [9] that

$$ \mathcal{L}\left({}^{\mathcal{C}}\partial_t^\alpha f\right)(s) = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0), \quad n-1<\alpha\le n. \tag{3.3} $$

THEOREM 3.1. *Let $k > 0$, $f_0 \in \mathbb{R}$, $g, h \in L^1(\mathbb{R}_+)$, be given, and $\|g\|_1 < k$. Then the Caputo fractional integro-differential equation* (3.1) *has a unique solution f from $BSA(\mathbb{R}_+)$.*

P r o o f. Applying the Laplace transform to equation (3.1) and taking into account (3.3) we obtain

$$ s^\alpha F(s) - s^{\alpha-1} f_0 + kF(s) + G(s)F(s) = H(s). \tag{3.4} $$

Solving for $F(s)$ yields

$$ F(s) = \frac{s^{\alpha-1} f_0 + H(s)}{s^\alpha + k + G(s)}. \tag{3.5} $$

Denote

$$ L(s) = \frac{s^{\alpha-1}}{s^\alpha + k + G(s)}, \quad M(s) = \frac{1}{s^\alpha + k + G(s)}, \tag{3.6} $$

then according to Theorem 2.2 and Corollary 2.2, their inverse Laplace transforms, namely $l(t), m(t)$, belong to $BSA(\mathbb{R}_+)$, and

$$ f(t) = f_0\, l(t) + \int_0^t m(t-\tau)h(\tau)\, d\tau. \tag{3.7} $$

Since $m \in BSA(\mathbb{R}_+)$ and $h \in L^1(\mathbb{R}_+)$, by Lemma 2.1, their Laplace convolution $m * h$ belongs to $BSA(\mathbb{R}_+)$. Hence, f, defined by (3.7), is from $BSA(\mathbb{R}_+)$. Using the Tauberian theorem for the Laplace transform ([16])

$$ F(s) \sim \frac{A}{s^\alpha}, \quad s \to \infty \quad \Longrightarrow \quad f(t) \sim \frac{At^{\alpha-1}}{\Gamma(\alpha)}, \quad t \to 0+, \tag{3.8} $$

we have

$$L(s) \sim \frac{1}{s}, \quad s \to \infty \quad \Longrightarrow \quad l(t) \sim 1, \quad t \to 0+.$$

Consequently, $f(0) = f_0$.

Conversely, let f be given by (3.7), where l, m are defined by (3.6). Then $f \in BSA(\mathbb{R}_+)$ and $f(0) = f_0$. Applying the Laplace transform to (3.7) and taking into account (3.6) we arrive at (3.5). Hence, (3.4) holds. The Laplace inverse of (3.4) yields (3.1). □

3.2. **Riemann-Liouville fractional integro-differential equation.** Consider now the following Riemann-Liouville fractional integro-differential equation

$$D_{0+}^{\alpha} f(t) + k f(t) + \int_0^t g(t-\tau) f(\tau) d\tau = h(t), \quad I_{0+}^{1-\alpha} f(0+) = f_0, \quad \frac{1}{2} < \alpha \leq 1, \tag{3.9}$$

where $g, h \in L^1(\mathbb{R}_+)$ are given, and f is an unknown. Here D_{0+}^{α} is the Riemann-Liouville fractional derivative ([9])

$$D_{0+}^{\alpha} f(t) = \frac{d^n}{dt^n} I_{0+}^{n-\alpha} f(t), \quad I_{0+}^{n-\alpha} f(t) = \int_0^t \frac{(t-\tau)^{n-\alpha-1}}{\Gamma(n-\alpha)} f(\tau)\, d\tau, \quad \alpha < n. \tag{3.10}$$

It is well known [9] that

$$\mathcal{L}\left(D_{0+}^{\alpha} f\right)(s) = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{n-k-1} D_{0+}^{\alpha+k-n} f(0+), \quad n-1 < \alpha \leq n. \tag{3.11}$$

THEOREM 3.2. *Let $k > 0$, $f_0 \in \mathbb{R}$, $g, h \in L^1(\mathbb{R}_+)$, be given, and $\|g\|_1 < k$. Then the Riemann-Liouville fractional integro-differential equation (3.9) has a unique solution f from $BSA(\mathbb{R}_+)$.*

P r o o f. Applying the Laplace transform to equation (3.9) and taking into account (3.11) we obtain

$$s^{\alpha} F(s) - f_0 + k F(s) + G(s) F(s) = H(s). \tag{3.12}$$

Solving for $F(s)$ yields

$$F(s) = \frac{f_0 + H(s)}{s^{\alpha} + k + G(s)}. \tag{3.13}$$

Define again $M(s)$ by (3.6), then according to Corollary 2.2, its inverse Laplace transform $m(t)$ belongs to $BSA(\mathbb{R}_+)$, and

$$f(t) = f_0 m(t) + \int_0^t m(t-\tau) h(\tau)\, d\tau. \tag{3.14}$$

Since $m \in BSA(\mathbb{R}_+)$ and $h \in L^1(\mathbb{R}_+)$, by Lemma 2.1, their Laplace convolution $m * h$ belongs to $BSA(\mathbb{R}_+)$. Hence, f, defined by (3.14), is from $BSA(\mathbb{R}_+)$. Using the Tauberian theorem for the Laplace transform (3.8) we have

$$M(s) \sim \frac{1}{s^\alpha}, \quad s \to \infty \quad \Longrightarrow \quad m(t) \sim \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad t \to 0+.$$

Consequently, $I_{0+}^{1-\alpha} m(t) \sim 1, \quad t \to 0+$. Together with (3.14) it yields $I_{0+}^{1-\alpha} f(0+) = f_0$.

Conversely, let f be given by (3.14), where m is defined by (3.6). Then $f \in BSA(\mathbb{R}_+)$ and $I_{0+}^{1-\alpha} f(0+) = f_0$. Applying the Laplace transform to (3.14) and taking into account (3.6) we arrive at (3.13). Hence, (3.12) holds. The Laplace inverse of (3.12) yields (3.9). □

4. **Partial Caputo fractional integro-differential equation**

In this section we study the following partial Caputo fractional integro-differential equation

$$\begin{cases} {}^C\partial_t^\alpha u(x,t) = k\Delta u(x,t) - \int_0^t g(t-\tau) u(x,\tau) d\tau, & (x,t) \in Q := \Omega \times \mathbb{R}^+, \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = f(x), & x \in \Omega, \end{cases} \tag{4.1}$$

with $\frac{1}{2} < \alpha \leq 1$, where $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega \in C^{[\frac{d}{2}]+1}$. Here $[a]$ denotes the integer part of a. The model in (4.1) appears in many modeling situations of new viscoelastic materials such as polymers [1, 3, 4, 12].

We will show the observability of the solution for large time, which means its global existence in the Wiener space $BSA(\mathbb{R}_+)$. Local existence results in the case of Dirichlet boundary conditions are known, however the global existence results presented here are new and do not rely on semi-group techniques, [7].

As we shall use spectral methods associated with the Dirichlet Laplacian, denote its eigenvalues indexed in the ascending order and counting their multiplicity, by λ_j and associate eigenfunctions by φ_j, i.e.

$$\begin{cases} \Delta\varphi_j(x) = -\lambda_j \varphi_j(x), & \text{in } \Omega, \\ \varphi_j(x) = 0, & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

It is known [10] that $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_j \leq \cdots$, with $\lim_{j\to\infty} \lambda_j = \infty$, and the set $\{\varphi_j\}_{j\geq 1}$, normalized by $||\varphi_j||_{L^2(\Omega)} = 1$, is an orthonormal basis for $L^2(\Omega)$. Moreover, $\varphi_j \in C^\infty(\Omega)$, and the smoothness condition on the boundary guarantees that $\varphi_j \in C(\overline{\Omega})$ (see [10, Theorem 7, Section 2, Chapter IV]).

First we can look for a particular solution of (4.1) in the form that can be written as

$$u(x,t) = c_j(t)\varphi_j(x),$$

where $u(x,0) = \varphi_j(x)$, so c_j satisfies the Caputo fractional integro-differential equation, thanks to (4.1) and (4.2),

$$ {}^C\partial_t^\alpha c_j(t) = -k\lambda_j c_j(t) - \int_0^t g(t-s)c_j(s)\,ds, \qquad \text{with} \qquad c_j(0) = 1. \tag{4.3}$$

Equation (4.3) is a special case of (3.1), so from (3.5) we have its solution

$$C_j(s) = \frac{s^{\alpha-1}}{s^\alpha + \lambda_j k + G(s)}, \quad c_j(t) = \left(\mathcal{L}^{-1}C_j\right)(s). \tag{4.4}$$

THEOREM 4.1. *Let* $\frac{1}{2} < \alpha \le 1$, $\|g\|_1 < \lambda_1 k$. *Then* $c_j(t)$, *defined by* (4.4), *belongs to* $BSA^1(\mathbb{R}_+)$.

P r o o f. Since $\lambda_j k \ge \lambda_1 k > \|g\|_1$, Theorem 2.2 shows that $c_j(t) \in BSA(\mathbb{R}_+)$ for $j = 1, 2, \cdots$.

We have

$$\left(\mathcal{L}c_j'\right)(s) = s\,C_j(s) - c_j(0) = \frac{s^\alpha}{s^\alpha + \lambda_j k + G(s)} - 1 = \frac{\lambda_j k + G(s)}{s^\alpha + \lambda_j k + G(s)}. \tag{4.5}$$

In (2.16) we have shown that

$$\left|\frac{s^{\alpha-1}}{s^\alpha + \lambda_j k + G(s)}\right| \le \frac{M}{|s|}, \quad \operatorname{Re} s > 0.$$

Since $\lambda_j k + |G(s)| < (\lambda_1 + \lambda_j)k$, we have

$$\left|\frac{\lambda_j k + G(s)}{s^\alpha + \lambda_j k + G(s)}\right| < \frac{M\lambda_j}{|s|^\alpha}, \quad \operatorname{Re} s > 0.$$

Consequently, by Corollary 2.1 the Laplace inverse of $s\,C_j(s) - c_j(0)$, or $c_j'(t)$ exists and belongs to $BSA(\mathbb{R}_+)$. Thus, $c_j \in BSA^1(\mathbb{R}_+)$. □

If we take $f(x) = \varphi_j(x)$, then $u(x,t)$ defined by (4) with $c_j(0) = 1$ satisfies (4.1). Thus we have proved

THEOREM 4.2. *Let* $\frac{1}{2} < \alpha \le 1$, $\|g\|_1 < \lambda_1 k$, *and* $f(x) = \sum_{j=1}^m a_j\varphi_j(x)$, *then the classical solution to the problem* (4.1), *exists for all* $t > 0$, *i.e. is global.*

Now we go to the general case. The Weyl law for the asymptotics of the eigenvalues λ_j has the form [5, 6]

$$\lambda_j \simeq \delta j^{\frac{2}{d}}, \qquad j \to \infty, \quad \text{where} \quad \delta = \left[\frac{(2\sqrt{\pi})^{-d}}{\Gamma\left(\frac{d}{2}+1\right)} \mathrm{Vol}\,(\Omega) \right]^{-\frac{2}{d}}. \tag{4.6}$$

For the eigenfunctions $\varphi_j(x)$ the following asymptotics formula holds uniformly on any compact subset K of Ω, see [2],

$$\sum_{|\sqrt{\lambda_j}-\lambda|\le 1} \varphi_j^2(x) = O\left(\lambda^{d-1}\right), \quad \lambda \to \infty.$$

In particular,

$$\varphi_j(x) = O\left(\lambda_j^{\frac{d-1}{4}}\right) = O\left(j^{\frac{d-1}{2d}}\right), \qquad j \to \infty, \quad x \in K \Subset \Omega. \tag{4.7}$$

By f_j we denote the j^{th} Fourier coefficient of $f \in L^2(\Omega)$ in the basis $\{\varphi_j\}_{j\ge 1}$, namely,

$$f_j = \int_\Omega f(x)\varphi_j(x)\; dx.$$

Recall that if $f \in H_0^m(\Omega)$, the Sobolev space of functions with compact supports in Ω with generalized derivatives up to order $m \ge 0$, [10], then its Fourier coefficient f_j has the asymptotics ([2])

$$f_j = O\left(\lambda_j^{-\frac{m}{2}}\right) = O\left(j^{-\frac{m}{d}}\right), \quad j \to \infty, \tag{4.8}$$

and the following convergence result will be essential for studying solutions of (4.1).

LEMMA 4.1. *Let* $f \in H_0^m(\Omega)$.
a) [2] *If* $m > \frac{d}{2}$, *then the series*

$$\sum_{j=1}^{\infty} f_j \varphi_j(x) \tag{4.9}$$

converges absolutely and uniformly to $f(x)$ *on any compact subset of* Ω.
b) [10, Theorem 8, Chapter IV] *If* $\partial\Omega \in C^m$, *then*

$$\sum_{j=1}^{\infty} f_j^2 \lambda_j^m \le C\|f\|^2_{H^m(\Omega)}, \tag{4.10}$$

and the series (4.9) *converges to* $f(x)$ *in* $H^m(\Omega)$.
c) [10, Theorem 9, Chapter IV] *If* $\partial\Omega \in C^m$ *and* $m \ge \left[\frac{d}{2}\right]+1$, *then the sum* (4.9) *belongs to* $C^{m-\left[\frac{d}{2}\right]-1}(\overline{\Omega})$.

The absolute convergence of (4.9) should be understood in the following unconventional way. With the presence of multiple eigenvalues, let us regroup all eigenvalues into a strictly increasing sequence $\mu_1 < \mu_2 < \ldots$ such that the sets $\{\lambda_1, \lambda_2, \cdots, \lambda_j, \cdots\}$ and $\{\mu_1, \mu_2, \cdots, \mu_l, \cdots\}$ coincide. Then the absolute convergence of (4.9) means the convergence of the series

$$\sum_{l=1}^{\infty} \left| \sum_{\lambda_j=\mu_l} f_j \varphi_j(x) \right|. \tag{4.11}$$

Theorem 4.3. *Let $g \in L^1(\mathbb{R}_+) \cup L^\infty(\mathbb{R}_+), f \in H_0^m(\Omega), \|g\|_1 < k\lambda_1, \frac{1}{2} < \alpha \leq 1$, and c_j be defined by (4.4).*
a) If $\partial\Omega \in C^m$, then the series

$$u(x,t) := \sum_{j=1}^{\infty} f_j c_j(t) \varphi_j(x) \tag{4.12}$$

converges in $H^m(\Omega)$ norm for each $t \geq 0$. If, moreover, $m \geq \left[\frac{d}{2}\right] + 1$, then $u(.,t) \in C^{m-\left[\frac{d}{2}\right]-1}(\overline{\Omega})$.
b) If $m > \frac{d}{2}$, then the series (4.12) converges absolutely on $Q := \Omega \times \mathbb{R}_+$.
c) If $m > \frac{3d-1}{2}$, then the series (4.12) converges uniformly on any compact subset of Q. Moreover, if $\partial\Omega \in C^{\left[\frac{d}{2}\right]+1}$, then $u \in C(\overline{\Omega}) \times BSA(\mathbb{R}_+)$.

P r o o f. Consider the equation

$$^C\partial_t^\alpha y(t) = -\lambda y(t) + f(t), \quad y(0) = 1. \tag{4.13}$$

Its solution has the form [9]

$$y(t) = E_\alpha(-\lambda t^\alpha) + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-\tau)^\alpha) f(\tau)\, d\tau, \tag{4.14}$$

where $E_\alpha(z), E_{\alpha,\beta}(z)$ are the classical and two parametric Mittag-Leffler functions ([8])

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad E_\alpha(z) = E_{\alpha,1}(z). \tag{4.15}$$

Applying (4.13) and (4.14) to (4.3) with $f(t) = -(g * c_j)(t)$, and λ being replaced by $k\lambda_j$, we obtain

$$\begin{aligned} c_j(t) &= E_\alpha(-k\lambda_j t^\alpha) \\ &\quad - \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-k\lambda_j(t-\tau)^\alpha) \int_0^\tau g(\tau-\eta) c_j(\eta)\, d\eta\, d\tau \\ &= E_\alpha(-k\lambda_j t^\alpha) - \int_0^t \beta(t,\eta) c_j(\eta)\, d\eta, \end{aligned}$$

where

$$\beta(t,\eta) = \int_\eta^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-k\lambda_j (t-\tau)^\alpha) g(\tau-\eta)\, d\tau. \tag{4.16}$$

Since $E_{\alpha,\alpha}(-x)$ is monotone decreasing and $E_{\alpha,\alpha}(0) = \frac{1}{\Gamma(\alpha)}$ (see [8]), we have

$$|\beta(t,\eta)| \le E_{\alpha,\alpha}(0)\|g\|_\infty \int_\eta^t (t-\tau)^{\alpha-1} d\tau = \frac{(t-\eta)^\alpha}{\Gamma(\alpha+1)}\|g\|_\infty. \tag{4.17}$$

Consequently, the monotone decay of $E_\alpha(-x)$ and $E_\alpha(0) = 1$ (see [8]) yields

$$\begin{aligned} |c_j(t)| &\le E_\alpha(-k\lambda_j t^\alpha) + \frac{\|g\|_\infty}{\Gamma(\alpha+1)} \int_0^t (t-\eta)^\alpha |c_j(\eta)| d\eta \\ &\le 1 + \frac{\|g\|_\infty}{\Gamma(\alpha+1)} \int_0^t (t-\eta)^\alpha |c_j(\eta)| d\eta. \end{aligned}$$

Applying the Gronwall inequality for fractional integral [18, Corollary 2] and recalling that $E_\alpha(x)$ is monotone increasing [8], we obtain

$$|c_j(t)| \le E_\alpha\left(\frac{\|g\|_\infty t^\alpha}{\alpha}\right) \le E_\alpha\left(\frac{\|g\|_\infty T^\alpha}{\alpha}\right) =: M_T, \quad t \in [0,T]. \tag{4.18}$$

Thus, $\{c_j(t)\}_{j\ge1}$ are uniformly bounded on any interval $[0,T]$.

a) Since $f \in H_0^m(\Omega)$ and $\partial\Omega \in C^m$, then by Lemma 4.1 (b) the inequality (4.10) holds. Together with the uniform boundedness of $c_j(t)$ on $[0,T]$ it yields

$$\sum_{j=1}^\infty f_j^2 c_j^2(t) \lambda_j^m < \infty.$$

In other words, the series (4.12) converges in $H^m(\Omega)$ norm, and $u(.,t) \in H^m(\Omega)$ for any $t \ge 0$. On the other hand, when $m \ge \left[\frac{d}{2}\right] + 1$, we have [10] $H^m(\Omega) \Subset C^{m-[\frac{d}{2}]-1}(\overline{\Omega})$, therefore, $u(.,t) \subset C^{m-[\frac{d}{2}]-1}(\overline{\Omega})$.

b) Combining Lemma 4.1 (a), formula (4.11), and noticing that $c_j(t) = c_{j'}(t)$ if $\lambda_j = \lambda_{j'}$ we arrive at

$$\sum_{l=1}^\infty \left| \sum_{\lambda_j=\mu_l} f_j c_j(t)\varphi_j(x) \right| \le M_T \sum_{l=1}^\infty \left| \sum_{\lambda_j=\mu_l} f_j \varphi_j(x) \right| < \infty,$$

i.e., the absolute convergence of (4.12).

c) From (4.7), (4.8), and (4.18) we have

$$f_j c_j(t) \varphi_j(x) = O\left(j^{\frac{d-1-2m}{2d}}\right),$$

uniformly on $K \times [0,T]$, where K is any compact subset of Ω. Since $m > \frac{3d-1}{2}$, then $\frac{d-1-2m}{2d} < -1$, and therefore, the series (4.12) converges uniformly on $K \times [0,T]$.

From (2.16) and (4.4) we have

$$|C_j(s)| \leq \frac{M}{|s|}, \quad \operatorname{Re} s > 0,$$

where M is independent of j. Hence, Hölder's inequality and formula (2.5) give

$$\left[\int_0^\infty e^{-xt}|c_j(t)|dt\right]^2 \leq \int_0^\infty e^{-xt}dt \int_0^\infty e^{-xt}|c_j(t)|^2 dt$$
$$= \frac{1}{2\pi x}\int_{-\infty}^\infty \left|C_j\left(\frac{x}{2}+iy\right)\right|^2 dy \leq \frac{M}{2\pi x}\int_{-\infty}^\infty \frac{1}{\left|\frac{x}{2}+iy\right|^2}\,dy = \frac{M}{x^2}, \quad x > 0.$$

Consequently,

$$\sum_{j=1}^\infty |f_j\varphi_j(x)| \int_0^\infty e^{-xt}|c_j(t)|dt \leq \frac{\sqrt{M}}{x}\sum_{j=1}^\infty O\left(j^{\frac{d-1-2m}{2d}}\right) < \infty, \quad x > 0.$$

Thus, we can apply Lebesgue's dominated convergence theorem to obtain

$$\left(\mathcal{L}u(x,.)\right)(s) = \sum_{j=1}^\infty f_j\varphi_j(x)(\mathcal{L}c_j)(s), \quad \operatorname{Re} s > 0.$$

In other words,

$$U(x,s) = \sum_{j=1}^\infty f_jC_j(s)\varphi_j(x) = \sum_{j=1}^\infty O\left(j^{\frac{d-1-2m}{2d}}\right)O\left(\frac{1}{s}\right) = O\left(\frac{1}{s}\right). \quad (4.19)$$

By Corollary 2.1 we have $u(x,.) \in BSA(\mathbb{R}_+)$.

Now, $m > \frac{3d-1}{2} > \left[\frac{d}{2}\right]+1$, therefore, combining with Part (a) we arrive at $u \in C(\overline{\Omega}) \times BSA(\mathbb{R}_+)$.

Theorem 4.3 is proved. □

Now we are ready to prove the main theorem of this section about the global existence of classical solutions of (4.1).

THEOREM 4.4. *Let* $g \in L^1(\mathbb{R}_+) \cup L^\infty(\mathbb{R}_+)$, $f \in H_0^m(\Omega)$, $\partial\Omega \in C^m$ *with* $m > \frac{3d+3}{2}$, $\frac{1}{2} < \alpha \leq 1$, *and* $\|g\|_1 < k\lambda_1$. *Then* $u(x,t)$, *defined by* (4.12), *is the unique classical solution of* (4.1) *in* $C^2(\overline{\Omega}) \times BSA^1(\mathbb{R}_+)$.

P r o o f. Since $m - \left[\frac{d}{2}\right] - 1 > \frac{3d+3}{2} - \left[\frac{d}{2}\right] - 1 \geq 2$, by Theorem 4.3 (a) we have $u(.,t) \in C^2(\overline{\Omega})$. Moreover, from (4) and $\frac{d+3-2m}{2d} < -1$,

$$\sum_{j=1}^{\infty} |f_j c_j(t) \Delta\varphi_j(x)| = \sum_{j=1}^{\infty} |\lambda_j f_j c_j(t) \varphi_j(x)| = \sum_{j=1}^{\infty} O\left(j^{\frac{d+3-2m}{2d}}\right) < \infty,$$

uniformly on any compact subset of Q. Hence,

$$\Delta u(x,t) = \sum_{j=1}^{\infty} f_j c_j(t) \Delta\varphi_j(x) = -\sum_{j=1}^{\infty} \lambda_j f_j c_j(t) \varphi_j(x). \tag{4.20}$$

From (4.3) and (4.18) we get

$$\begin{aligned} \left|{}^{\mathcal{C}}\partial_t^{\alpha}\, c_j(t)\right| &\leq k\lambda_j M_T + M_T \int_0^t |g(t-s)| ds \leq M_T(k\lambda_j + \|g\|_1) \\ &= O(\lambda_j) = O\left(j^{\frac{2}{d}}\right), \quad t \in [0,T]. \end{aligned}$$

Consequently,

$$\sum_{j=1}^{\infty} \left| f_j\, {}^{\mathcal{C}}\partial_t^{\alpha} c_j(t) \varphi_j(x)\right| = \sum_{j=1}^{\infty} O\left(j^{\frac{d+3-2m}{2d}}\right) < \infty,$$

uniformly on $[0,T]$ for any $T > 0$, and it yields

$${}^{\mathcal{C}}\partial_t^{\alpha} u(x,t) = \sum_{j=1}^{\infty} f_j\ {}^{\mathcal{C}}\partial_t^{\alpha} c_j(t)\, \varphi_j(x). \tag{4.21}$$

It is obvious that

$$\int_0^t g(t-\tau) u(x,\tau)\, d\tau = \sum_{j=1}^{\infty} f_j \varphi_j(x) \int_0^t g(t-\tau)\, c_j(\tau)\, d\tau. \tag{4.22}$$

Combining (4.20), (4.21), (4.22), and (4.3), we arrive at

$$\begin{aligned} &{}^{\mathcal{C}}\partial_t^{\alpha} u(x,t) - k\Delta u(x,t) + \int_0^t g(t-\tau) u(x,\tau) d\tau \\ &= \sum_{j=1}^{\infty} f_j \varphi_j(x) \left[{}^{\mathcal{C}}\partial_t^{\alpha} c_j(t) + k\lambda_j c_j(t) + \int_0^t g(t-\tau)\, c_j(\tau)\, d\tau\right] = 0. \end{aligned}$$

Since $\varphi_j(x) = 0$ on $\partial\Omega$, then

$$u(x,t) = \sum_{j=1}^{\infty} f_j\, c_j(t) \varphi_j(x) = 0, \quad x \in \partial\Omega.$$

Because $c_j(0) = 1$, by Lemma 4.1

$$u(x,0) = \sum_{j=1}^{\infty} f_j\, c_j(0) \varphi_j(x) = \sum_{j=1}^{\infty} f_j \varphi_j(x) = f(x), \quad x \in \Omega.$$

Thus, $u(x,t)$, defined by (4.12), is a classical solution of (4.1).

Taking into account (4.19) and (4.5) we obtain

$$\begin{aligned}(\mathcal{L}u_t(x,t))\,(s) &= sU(x,s)-u(x,0)=s\sum_{j=1}^{\infty}f_j\varphi_j(x)C_j(s)-\sum_{j=1}^{\infty}f_j\varphi_j(x)\\ &= \sum_{j=1}^{\infty}f_j\varphi_j(x)\,(sC_j(s)-1)=\sum_{j=1}^{\infty}f_j\varphi_j(x)\frac{\lambda_j k+G(s)}{s^\alpha+\lambda_j k+G(s)}.\end{aligned}$$

Using (2.16) and (4.6) we get

$$\left|\frac{\lambda_j k+G(s)}{s^\alpha+\lambda_j k+G(s)}\right|\le\frac{M\lambda_j}{|s|^\alpha}\le\frac{M\,j^{\frac{2}{d}}}{|s|^\alpha}.$$

Together with (4.7), (4.8), it yields

$$\sum_{j=1}^{\infty}\left|f_j\varphi_j(x)\frac{\lambda_j k+G(s)}{s^\alpha+\lambda_j k+G(s)}\right|\le\frac{M}{|s|^\alpha}\sum_{j=1}^{\infty}j^{\frac{d+3-2m}{2d}}\le\frac{M}{|s|^\alpha},$$

because $\frac{d+3-2m}{2d}<-1$. By Corollary 2.1 $u_t(x,t)\in BSA(\mathbb{R}_+)$. Together with $u(x,t)\in BSA(\mathbb{R}_+)$ by Theorem 4.3 (c) it yields $u(x,t)\in BSA^1(\mathbb{R}_+)$ for any $x\in\Omega$. Thus, $u\in C^2(\overline{\Omega})\times BSA^1(\mathbb{R}_+)$.

Let $u,\tilde{u}\in C^2(\overline{\Omega})\times BSA^1(\mathbb{R}_+)$ be two solutions of (4.1). Then $w=u-\tilde{u}\in C^2(\overline{\Omega})\times BSA^1(\mathbb{R}_+)$ is a solution of

$$\begin{cases} {}^{C}\partial_t^\alpha w(x,t)=k\Delta w(x,t)-\int_0^t g(t-\tau)w(x,\tau)d\tau, & (x,t)\in\Omega\times\mathbb{R}^+,\\ w(x,t)=0, & (x,t)\in\partial\Omega\times\mathbb{R}^+,\\ w(x,0)=0, & x\in\Omega.\end{cases}\tag{4.23}$$

Taking the Laplace transform of (4.23) we get

$$\begin{cases}\Delta W(x,s)=\frac{G(s)+s^\alpha}{k}W(x,s), & x\in\Omega\\ W(x,s)=0, & x\in\partial\Omega\end{cases},\ W(x,s)\in C^2(\overline{\Omega}),\ \mathrm{Re}\,s>0.\tag{4.24}$$

If $s\in\left(\|g\|_1^{\frac{1}{\alpha}},\infty\right)$, then $-\frac{G(s)+s^\alpha}{k}<0$ cannot be an eigenvalue of the Dirichlet Laplacian (4.1), therefore the Schrödinger equation with Dirichlet's boundary condition (4.24) has only trivial solution $W(x,s)=0$, $x\in\Omega$, [10], for such s. But for a fixed parameter $x\in\Omega$, $W(x,s)$, as a function of s, is analytic in $\mathrm{Re}\,s>0$. As $W(x,s)=0$ on $s\in\left(\|g\|_1^{\frac{1}{\alpha}},\infty\right)$, the interior uniqueness theorem for holomorphic functions yields $W(x,s)=0$, $\mathrm{Re}\,s>0$. Hence, $w(x,t)=0$, and we obtain the uniqueness of u. The theorem is proved. □

5. Partial Riemann-Liouville fractional integro-differential equation

In this section we will study the global solvability of the following partial Riemann-Liouville fractional integro-differential equation

$$\begin{cases} D^{\alpha}_{0+}u(x,t) = k\Delta u(x,t) - \int_0^t g(t-\tau)u(x,\tau)d\tau, & (x,t) \in Q = \Omega \times \mathbb{R}^+, \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times \mathbb{R}^+, \\ I^{1-\alpha}_{0+}u(x,0) = f(x), & x \in \Omega, \end{cases} \tag{5.1}$$

with $\frac{1}{2} < \alpha \leq 1$, where $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega \in C^{[\frac{d}{2}]+1}$.

First we look for a particular solution of (5.1) in the form

$$u(x,t) = c_j(t)\varphi_j(x), \tag{5.2}$$

where $I^{1-\alpha}_{0+}u(x,0) = \varphi_j(x)$, so $c_j(t)$ satisfies the fractional integro-differential equation, thanks to (5.1) and (4.2),

$$D^{\alpha}_{0+}c_j(t) = -k\lambda_j c_j(t) - \int_0^t g(t-s)c_j(s)\,ds, \qquad \text{with} \qquad I^{1-\alpha}_{0+}c_j(0) = 1. \tag{5.3}$$

Equation (5.3) is a special case of (3.9), so from (3.13) we have its solution

$$C_j(s) = \frac{1}{s^{\alpha} + \lambda_j k + G(s)}, \quad c_j(t) = \left(\mathcal{L}^{-1}C_j\right)(s). \tag{5.4}$$

According to Corollary 2.2 we have the following theorem.

THEOREM 5.1. *Let $\frac{1}{2} < \alpha \leq 1$ and $\|g\|_1 < \lambda_1 k$. Then $c_j(t)$, defined by (5.4), belongs to $BSA(\mathbb{R}_+)$.*

If we take $f(x) = \varphi_j(x)$, then $u(x,t)$, defined by (5.2) with $I^{1-\alpha}_{0+}c_j(0) = 1$, satisfies (5.1). Thus we have proved the following theorems.

THEOREM 5.2. *Let $\frac{1}{2} < \alpha \leq 1$, $\|g\|_1 < \lambda_1 k$, and $f(x) = \sum_{j=1}^m a_j\varphi_j(x)$, then the classical solution to the problem (5.1), exists for all $t > 0$, i.e. is global.*

THEOREM 5.3. *Let $g \in L^1(\mathbb{R}_+) \cup L^\infty(\mathbb{R}_+), f \in H^m_0(\Omega), \|g\|_1 < k\lambda_1$, $\frac{1}{2} < \alpha \leq 1$, and c_j be defined by (5.4).*
a) If $\partial\Omega \in C^m$, then the series

$$u(x,t) := \sum_{j=1}^{\infty} f_j c_j(t)\varphi_j(x) \tag{5.5}$$

converges in $H^m(\Omega)$ *norm for each* $t > 0$. *If, moreover,* $m \geq \left[\frac{d}{2}\right] + 1$, *then* $u(.,t) \in C^{m-\left[\frac{d}{2}\right]-1}(\overline{\Omega})$.
b) If $m > \frac{d}{2}$, *then the series* (5.5) *converges absolutely on* Q.
c) If $m > \frac{3d-1}{2}$, *then the series* (5.5) *converges uniformly on any compact subset of* Q. *Moreover, if* $\partial\Omega \in C^{\left[\frac{d}{2}\right]+1}$, *then* $u \in C(\overline{\Omega}) \times BSA(\mathbb{R}_+)$.

P r o o f. Consider the equation

$$D_{0+}^{\alpha} y(t) = -\lambda y(t) + f(t), \quad I_{0+}^{1-\alpha} y(0) = 1. \tag{5.6}$$

Its solution has the form [9]

$$y(t) = t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^{\alpha}) + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-\tau)^{\alpha}) f(\tau)\, d\tau. \tag{5.7}$$

Applying (5.6) and (5.7) to (5.3) with $f(t) = -(g * c_j)(t)$, and λ being replaced by $k\lambda_j$, we obtain

$$\begin{aligned} c_j(t) &= t^{\alpha-1} E_{\alpha,\alpha}(-k\lambda_j t^{\alpha}) \\ &\quad - \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-k\lambda_j (t-\tau)^{\alpha}) \int_0^{\tau} g(\tau-\eta) c_j(\eta)\, d\eta\, d\tau \\ &= t^{\alpha-1} E_{\alpha,\alpha}(-k\lambda_j t^{\alpha}) - \int_0^t \beta(t,\eta) c_j(\eta)\, d\eta, \end{aligned}$$

where $\beta(t,\eta)$ is defined by (4.16). Using (4.17) we get

$$|c_j(t)| \leq t^{\alpha-1} \left| E_{\alpha,\alpha}(-k\lambda_j t^{\alpha}) \right| + \frac{\|g\|_\infty}{\Gamma(\alpha+1)} \int_0^t (t-\eta)^{\alpha} |c_j(\eta)| d\eta.$$

The complete monotonicity property of $E_{\alpha,\alpha}(-t)$, $0 < \alpha \leq 1$, [8], yields the monotone decay and positivity of $E_{\alpha,\alpha}(-t)$, $0 < \alpha \leq 1$,

$$E_{\alpha,\alpha}(-k\lambda_1 t^{\alpha}) \geq E_{\alpha,\alpha}(-k\lambda_j t^{\alpha}) > 0.$$

Consequently,

$$|c_j(t)| \leq t^{\alpha-1} E_{\alpha,\alpha}(-k\lambda_1 t^{\alpha}) + \frac{\|g\|_\infty}{\Gamma(\alpha+1)} \int_0^t (t-\eta)^{\alpha} |c_j(\eta)| d\eta. \tag{5.8}$$

Since $t^{\alpha-1} E_{\alpha,\alpha}(-t^{\alpha}), 0 < \alpha \leq 1$, is complete monotone [8], then

$$t^{\alpha-1} E_{\alpha,\alpha}(-k\lambda_1 t^{\alpha}), \qquad 0 < \alpha \leq 1,$$

is monotone decreasing. Applying the Gronwall inequality for fractional integral [18, Corollary 2] and monotone decreasing of $t^{\alpha-1} E_{\alpha,\alpha}(-k\lambda_1 t^{\alpha}), 0 < \alpha \leq 1$, to (5.8), we obtain

$$|c_j(t)| \leq t^{\alpha-1} E_{\alpha,\alpha}(-k\lambda_1 t^{\alpha}) E_{\alpha}\left(\frac{\|g\|_\infty t^{\alpha}}{\alpha}\right) =: M(t), \quad t > 0. \tag{5.9}$$

Thus, $\{c_j(t)\}_{j\geq 1}$ are uniformly bounded for any $t>0$.

a) Since $f\in H_0^m(\Omega)$ and $\partial\Omega\in C^m$, then by Lemma 4.1 (b) the inequality (4.10) holds. Together with the uniform boundedness of $\{c_j(t)\}_{j\geq 1}$ for $t>0$ it yields

$$\sum_{j=1}^{\infty} f_j^2 c_j^2(t)\lambda_j^m < \infty.$$

In other words, the series (5.5) converges in $H^m(\Omega)$ norm, and $u(.,t)\in H^m(\Omega)$ for any $t>0$. On the other hand, when $m\geq\left[\frac{d}{2}\right]+1$, we have [10] $H^m(\Omega)\Subset C^{m-\left[\frac{d}{2}\right]-1}(\overline{\Omega})$, therefore, $u(.,t)\subset C^{m-\left[\frac{d}{2}\right]-1}(\overline{\Omega})$.

b) Combining Lemma 4.1 (a), formula (4.11), and noticing that $c_j(t)=c_{j'}(t)$ if $\lambda_j=\lambda_{j'}$, we arrive at

$$\sum_{l=1}^{\infty}\left|\sum_{\lambda_j=\mu_l} f_j c_j(t)\varphi_j(x)\right| \leq M(t)\sum_{l=1}^{\infty}\left|\sum_{\lambda_j=\mu_l} f_j\varphi_j(x)\right|<\infty,$$

i.e., the absolute convergence of (5.5).

c) From (4.7), (4.8), and (5.9) we have

$$f_j c_j(t)\varphi_j(x)=O\left(j^{\frac{d-1-2m}{2d}}\right), \tag{5.10}$$

uniformly on $K\times[T_1,T]$, where K is any compact subset of Ω, and $0<T_1<T<\infty$. Since $m>\frac{3d-1}{2}$, then $\frac{d-1-2m}{2d}<-1$, and therefore, the series (5.5) converges uniformly on $K\times[T_1,T]$.

From (2.16) and (5.4) we have

$$|C_j(s)|\leq\frac{M}{|s|^{\alpha}},\quad \operatorname{Re}s>0,$$

where M is independent of j. Hence, Hölder's inequality and formula (2.5) give

$$\begin{aligned}\left[\int_0^\infty e^{-xt}|c_j(t)|dt\right]^2 &\leq \int_0^\infty e^{-xt}dt\int_0^\infty e^{-xt}|c_j(t)|^2dt\\ &= \frac{1}{2\pi x}\int_{-\infty}^{\infty}\left|C_j\left(\frac{x}{2}+iy\right)\right|^2dy\leq\frac{M}{2\pi x}\int_{-\infty}^{\infty}\frac{1}{\left|\frac{x}{2}+iy\right|^{2\alpha}}dy\\ &= \frac{M2^{2\alpha-2}\Gamma\left(\alpha-\frac{1}{2}\right)}{x^{2\alpha}\sqrt{\pi}\Gamma(\alpha)},\quad x>0.\end{aligned}$$

Consequently,

$$\sum_{j=1}^{\infty}|f_j\varphi_j(x)|\int_0^\infty e^{-xt}|c_j(t)|dt\leq\frac{1}{x^{\alpha}}\sum_{j=1}^{\infty}O\left(j^{\frac{d-1-2m}{2d}}\right)<\infty,\quad x>0.$$

Thus, we can apply Lebesgue's dominated convergence theorem to obtain

$$(\mathcal{L}u(x,.))\,(s)=\sum_{j=1}^{\infty} f_j\varphi_j(x)(\mathcal{L}c_j)(s), \quad \operatorname{Re} s>0.$$

In other words,

$$U(x,s)=\sum_{j=1}^{\infty} f_j C_j(s)\varphi_j(x)=\sum_{j=1}^{\infty} O\left(j^{\frac{d-1-2m}{2d}}\right) O\left(\frac{1}{s^\alpha}\right)=O\left(\frac{1}{s^\alpha}\right). \tag{5.11}$$

By Corollary 2.1 we have $u(x,.)\in BSA(\mathbb{R}_+)$.

Now, $m>\frac{3d-1}{2}>\left[\frac{d}{2}\right]+1$, therefore, combining with Part (a) we arrive at $u\in C(\overline{\Omega})\times BSA(\mathbb{R}_+)$.

Theorem 5.3 is proved. □

Now we prove the main theorem of this section about the global existence of classical solutions of (5.1).

THEOREM 5.4. *Let $g\in L^1(\mathbb{R}_+)\cup L^\infty(\mathbb{R}_+)$, $f\in H_0^m(\Omega)$, $\partial\Omega\in C^m$ with $m>\frac{3d+3}{2}$, $\frac{1}{2}<\alpha\le 1$, and $\|g\|_1<k\lambda_1$. Then $u(x,t)$, defined by (5.5), is the unique classical solution of (5.1) in $C^2(\overline{\Omega})\times BSA^\alpha(\mathbb{R}_+)$.*

By $f(t)\in BSA^\alpha(\mathbb{R}_+)$ we mean both $f(t), D_{0+}^\alpha f(t)\in BSA(\mathbb{R}_+)$.

P r o o f. Since $m-\left[\frac{d}{2}\right]-1>\frac{3d+3}{2}-\left[\frac{d}{2}\right]-1\ge 2$, by Theorem 5.3 (a) we have $u(.,t)\in C^2(\overline{\Omega})$. Moreover, from (5.10) and $\frac{d+3-2m}{2d}<-1$,

$$\sum_{j=1}^{\infty}|f_jc_j(t)\Delta\varphi_j(x)|=\sum_{j=1}^{\infty}|\lambda_j f_jc_j(t)\varphi_j(x)|=\sum_{j=1}^{\infty} O\left(j^{\frac{d+3-2m}{2d}}\right)<\infty,$$

uniformly on any compact subset $K\times[T_1,T]$. Hence,

$$\Delta u(x,t)=\sum_{j=1}^{\infty} f_jc_j(t)\Delta\varphi_j(x)=-\sum_{j=1}^{\infty}\lambda_j f_jc_j(t)\varphi_j(x). \tag{5.12}$$

From (5.3) and (5.9) we get

$$\begin{aligned}\left|D_{0+}^\alpha c_j(t)\right| &\le k\lambda_j M(t)+M(t)\int_0^t|g(t-s)|ds\le M(t)(k\lambda_j+\|g\|_1)\\ &= O(\lambda_j)=O\left(j^{\frac{2}{d}}\right), \qquad t\in[T_1,T].\end{aligned}$$

Consequently,

$$\sum_{j=1}^{\infty}\left|f_j\,D_{0+}^\alpha c_j(t)\varphi_j(x)\right|=\sum_{j=1}^{\infty} O\left(j^{\frac{d+3-2m}{2d}}\right)<\infty,$$

uniformly on $[T_1,T]$ for any $0<T_1<T<\infty$, and it yields

$$D^{\alpha}_{0+}u(x,t) = \sum_{j=1}^{\infty} f_j \; D^{\alpha}_{0+}c_j(t)\,\varphi_j(x). \tag{5.13}$$

It is obvious that

$$\int_0^t g(t-\tau)u(x,\tau)\,d\tau = \sum_{j=1}^{\infty} f_j\varphi_j(x) \int_0^t g(t-\tau)\,c_j(\tau)\,d\tau. \tag{5.14}$$

Combining (5.12), (5.13), (5.14), and (5.3), we arrive at

$$\begin{aligned} & D^{\alpha}_{0+}u(x,t) - k\Delta u(x,t) + \int_0^t g(t-\tau)u(x,\tau)d\tau \\ = \; & \sum_{j=1}^{\infty} f_j\varphi_j(x) \left[D^{\alpha}_{0+}c_j(t) + k\lambda_j c_j(t) + \int_0^t g(t-\tau)\,c_j(\tau)\,d\tau\right] = 0. \end{aligned}$$

Since $\varphi_j(x) = 0$ on $\partial\Omega$, then

$$u(x,t) = \sum_{j=1}^{\infty} f_j\,c_j(t)\varphi_j(x) = 0, \quad x \in \partial\Omega.$$

Because $I^{1-\alpha}_{0+}c_j(0) = 1$, by Lemma 4.1 (a)

$$I^{1-\alpha}_{0+}u(x,0) = \sum_{j=1}^{\infty} f_j\, I^{1-\alpha}_{0+}c_j(0)\varphi_j(x) = \sum_{j=1}^{\infty} f_j\varphi_j(x) = f(x), \quad x \in \Omega. \tag{5.15}$$

Thus, $u(x,t)$, defined by (5.5), is a classical solution of (5.1).

Taking into account (5.11), (3.11), and (5.15) we obtain

$$\begin{aligned} & \left(\mathcal{L}D^{\alpha}_{0+}u(x,t)\right)(s) = s^{\alpha}\,U(x,s) - I^{1-\alpha}_{0+}u(x,0) \\ = \; & s^{\alpha}\sum_{j=1}^{\infty} f_j\varphi_j(x)C_j(s) - \sum_{j=1}^{\infty} f_j\varphi_j(x) \\ = \; & \sum_{j=1}^{\infty} f_j\varphi_j(x)\left(s^{\alpha}C_j(s) - 1\right) = \sum_{j=1}^{\infty} f_j\varphi_j(x)\frac{\lambda_j k + G(s)}{s^{\alpha} + \lambda_j k + G(s)}. \end{aligned}$$

Using (2.16) and (4.6) we get

$$\left|\frac{\lambda_j k + G(s)}{s^{\alpha} + \lambda_j k + G(s)}\right| \le \frac{M\lambda_j}{|s|^{\alpha}} \le \frac{M\,j^{\frac{2}{d}}}{|s|^{\alpha}}.$$

Together with (4.7), (4.8), it yields

$$\sum_{j=1}^{\infty}\left|f_j\varphi_j(x)\frac{\lambda_j k + G(s)}{s^{\alpha} + \lambda_j k + G(s)}\right| \le \frac{M}{|s|^{\alpha}}\sum_{j=1}^{\infty} j^{\frac{d+3-2m}{2d}} \le \frac{M}{|s|^{\alpha}},$$

because $\frac{d+3-2m}{2d} < -1$. By Corollary 2.1 $D^{\alpha}_{0+}u(x,t) \in BSA(\mathbb{R}_+)$. Together with $u(x,t) \in BSA(\mathbb{R}_+)$ by Theorem 5.3 (c) it yields $u(x,t) \in BSA^{\alpha}(\mathbb{R}_+)$ for any $x \in \Omega$. Thus, $u \in C^2(\overline{\Omega}) \times BSA^{\alpha}(\mathbb{R}_+)$.

Let $u, \tilde{u} \in C^2(\overline{\Omega}) \times BSA^\alpha(\mathbb{R}_+)$ be two solutions of (5.1). Then $w = u - \tilde{u} \in C^2(\overline{\Omega}) \times BSA^\alpha(\mathbb{R}_+)$ is a solution of

$$\begin{cases} D_{0+}^\alpha w(x,t) = k\Delta w(x,t) - \int_0^t g(t-\tau)w(x,\tau)d\tau, & (x,t) \in \Omega \times \mathbb{R}^+, \\ w(x,t) = 0, & (x,t) \in \partial\Omega \times \mathbb{R}^+, \\ I_{0+}^{1-\alpha} w(x,0) = 0, & x \in \Omega. \end{cases} \tag{5.16}$$

Taking the Laplace transform of (5.16) we get the Dirichlet Schrödinger problem (4.24), and the uniqueness of u follows. □

6. Inverse problems

We consider now an inverse problem of finding an initial function $u(x,0) = f(x)$, so that we can reconstruct the order of fractional derivative α, the constant k, and the memory function g uniquely from a single observation of the solution $\{u(x,t)\}_{t>0}$ of (4.1) at one arbitrary point $x = b \in \Omega$. For an one-dimensional case see [15].

The initial condition we choose is $f(x) = \varphi_1(x)$. Then the observation $u(b,t)$ is given by

$$u(b,t) = c_1(t)\varphi_1(b), \quad c_1(0) = 1, \text{ where } b \in \Omega.$$

Recall that $\varphi_1(b) \neq 0$, as the principal eigenfunction of the Dirichlet Laplacian never vanishes inside Ω, [10], and so the observation is not trivial.

Taking the Laplace transform of the observation $u(b,t)$ with respect to t, and recalling (4.4), we have

$$U(b,s) = \frac{s^{\alpha-1}}{s^\alpha + \lambda_1 k + G(s)}\varphi_1(b).$$

Consequently,

$$\frac{\varphi_1(b)}{s\,U(b,s)} - 1 = s^{-\alpha}(\lambda_1 k + G(s)),$$

and

$$\alpha = -\frac{\ln\left(\dfrac{\varphi_1(b)}{s\,U(b,s)} - 1\right)}{\ln s} + \frac{\ln(\lambda_1 k + G(s))}{\ln s}.$$

Using the fact that $G(s) \to 0$ as $s \to \infty$, it yields

$$\alpha = -\lim_{s\to\infty} \frac{\ln\left(\dfrac{\varphi_1(b)}{s\,U(b,s)} - 1\right)}{\ln s}. \tag{6.1}$$

For k we have

$$k = \frac{s^\alpha}{\lambda_1}\left[\frac{\varphi_1(b)}{s\,U(b,s)} - 1\right] - \frac{G(s)}{\lambda_1}.$$

Therefore, once α is known, k can be obtained as

$$k = \lim_{s\to\infty} \frac{s^\alpha}{\lambda_1} \left[\frac{\varphi_1(b)}{s\,U(b,s)} - 1 \right], \tag{6.2}$$

and $G(s)$ as

$$G(s) = s^\alpha \left[\frac{\varphi_1(b)}{s\,U(b,s)} - 1 \right] - k\lambda_1, \quad \operatorname{Re} s > 0. \tag{6.3}$$

The memory kernel $g(t)$ can be recovered by taking the Laplace inverse transform of $G(s)$. Thus we have proved

THEOREM 6.1. *Let $\frac{1}{2} < \alpha \le 1$, $g \in L^1(\mathbb{R}_+)$ with $\|g\|_1 < \lambda_1 k$. Taking $f(x) = \varphi_1(x)$ then using one observation $u(b,t)$ of* (4.1) *at a single point $b \in \Omega$ we can reconstruct uniquely the fractional order α by* (6.1)*, the parameter k by* (6.2)*, and the function g by taking the Laplace inverse of $G(s)$ from* (6.3)*.*

Assume now that the observation point b is on the boundary $\partial\Omega$. Since $u(b,t) = 0$ when $b \in \partial\Omega$, so instead of $u(b,t)$ we should observe $\frac{\partial u(b,t)}{\partial \nu}$, the outer normal derivative of the solution u at the boundary point b. With the initial condition $u(x,0) = \varphi_1(x)$ the solution $u(x,t) = c_1(t)\varphi_1(x) \in C^1(\overline{\Omega})$ for each $t \ge 0$ when $\partial\Omega \in C^{[\frac{d}{2}]+2}$, [10]. Since $\frac{\partial \varphi_1(b)}{\partial \nu} \neq 0$, [10], the observation $\frac{\partial u(b,t)}{\partial \nu}$ is meaningful.

Taking the Laplace transform of the observation $\frac{\partial u(b,t)}{\partial \nu}$ with respect to t, and recalling (4.4), we have

$$\frac{\partial U(b,s)}{\partial \nu} = \frac{s^{\alpha-1}}{s^\alpha + \lambda_1 k + G(s)} \frac{\partial \varphi_1(b)}{\partial \nu}.$$

Consequently,

$$\alpha = -\lim_{s\to\infty} \frac{\ln\left(\dfrac{\frac{\partial \varphi_1(b)}{\partial \nu}}{s\,\frac{\partial U(b,s)}{\partial \nu}} - 1 \right)}{\ln s}, \tag{6.4}$$

$$k = \lim_{s\to\infty} \frac{s^\alpha}{\lambda_1} \left[\frac{\frac{\partial \varphi_1(b)}{\partial \nu}}{s\,\frac{\partial U(b,s)}{\partial \nu}} - 1 \right], \tag{6.5}$$

and

$$G(s) = s^\alpha \left[\frac{\frac{\partial \varphi_1(b)}{\partial \nu}}{s\,\frac{\partial U(b,s)}{\partial \nu}} - 1 \right] - k\lambda_1, \quad \operatorname{Re} s > 0. \tag{6.6}$$

THEOREM 6.2. *Let $\frac{1}{2} < \alpha \le 1$, $\partial\Omega \in C^{[\frac{d}{2}]+2}$, $g \in L^1(\mathbb{R}_+)$ with $\|g\|_1 < \lambda_1 k$. Taking $f(x) = \varphi_1(x)$, then using one observation $\frac{\partial u(b,t)}{\partial \nu}$ of* (4.1) *at a single point $b \in \partial\Omega$ we can reconstruct uniquely the fractional* *order α by* (6.4)*, the parameter k by* (6.5)*, and the function g by taking the Laplace inverse of $G(s)$ from* (6.6)*.*

Similarly, consider now an inverse problem of reconstructing the order of fractional derivative α, the constant k, and the memory function g uniquely from a single observation of the solution $\{u(x,t)\}_{t>0}$ of (5.1) at one point $x = b \in \Omega$.

Choose the initial condition $f(x) = \varphi_1(x)$. Then the observation $u(b,t)$ is given by

$$u(b,t) = c_1(t)\varphi_1(b), \quad I_{0+}^{1-\alpha} c_1(0) = 1, \text{ where } b \in \Omega.$$

Taking the Laplace transform of the observation $u(b,t)$ with respect to t, and recalling (5.4), we have

$$U(b,s) = \frac{1}{s^\alpha + \lambda_1 k + G(s)}\varphi_1(b).$$

Consequently,

$$\frac{\varphi_1(b)}{U(b,s)} = s^\alpha + \lambda_1 k + G(s) \sim s^\alpha, \quad s \to \infty,$$

and therefore

$$\alpha = \lim_{s\to\infty} \frac{\ln\left(\dfrac{\varphi_1(b)}{s\,U(b,s)}\right)}{\ln s}. \tag{6.7}$$

Once α is known, k can be obtained as

$$k = \lim_{s\to\infty} \frac{1}{\lambda_1}\left[\frac{\varphi_1(b)}{U(b,s)} - s^\alpha\right], \tag{6.8}$$

and $G(s)$ as

$$G(s) = \frac{\varphi_1(b)}{U(b,s)} - s^\alpha - k\lambda_1, \quad \operatorname{Re} s > 0. \tag{6.9}$$

The memory kernel $g(t)$ can be recovered by taking the Laplace inverse transform of $G(s)$. Thus we have proved the following thorem.

THEOREM 6.3. *Let $\frac{1}{2} < \alpha \le 1$, $g \in L^1(\mathbb{R}_+)$ with $\|g\|_1 < \lambda_1 k$. Taking $f(x) = \varphi_1(x)$, then using one observation $u(b,t)$ of* (5.1) *at a single point $b \in \Omega$ we can reconstruct uniquely the fractional order α by* (6.7)*, the parameter k by* (6.8)*, and the function g by taking the Laplace inverse of $G(s)$ from* (6.9)*.*

If, moreover, $\partial\Omega \in C^{[\frac{d}{2}]+2}$, and $b \in \partial\Omega$, then from the observation $\frac{\partial u(b,t)}{\partial \nu}$ of (5.1) one can find

$$\alpha = \lim_{s\to\infty} \frac{\ln\left(\dfrac{\frac{\partial\varphi_1(b)}{\partial\nu}}{s\,\frac{\partial U(b,s)}{\partial\nu}}\right)}{\ln s},$$
$$k = \lim_{s\to\infty} \frac{1}{\lambda_1}\left[\frac{\frac{\partial\varphi_1(b)}{\partial\nu}}{\frac{\partial U(b,s)}{\partial\nu}} - s^\alpha\right],$$

and

$$G(s) = \frac{\frac{\partial\varphi_1(b)}{\partial\nu}}{\frac{\partial U(b,s)}{\partial\nu}} - s^\alpha - k\lambda_1, \quad \operatorname{Re} s > 0, \qquad g(t) = (\mathcal{L}^{-1}G)(t).$$

Acknowledgement

The author would like to thank Dr. Amin Boumenir for fruitful discussions.

References

[1] A. Akilandeeswari, K. Balachandran, and N. Annapoorani, On fractional partial differential equations of diffusion type with integral kernel. In: *Mathematical Modelling, Optimization, Analytic and Numerical Solutions*, Eds: P. Manchanda, R.P. Lozi, and A.H. Siddiqi, Springer, Singapore, 2020, 333–349.

[2] Sh.A. Alimov, V.A. Ilyin, and E.M. Nikishin, Convergence problems of multiple Fourier series and spectral decompositions, II (In Russian). *Uspekhi Mat. Nauk* **32** (1977), 107–130; Engl. transl. in: *Russian Math. Surveys* **32** (1977), 115–139.

[3] R.L. Bagley and P.J. Torvik, Fractional calculus, a different approach to the analysis of viscoelastically damped structures. *AIAA J.* 21(1983), 741–748.

[4] R.L. Bagley and P.J. Torvik, On the fractional calculus model of viscoelastic behavior, *J. Rheol.* **30** (1986), 133–155.

[5] M.Sh. Birman and M.Z. Solomyak, The principal term of the spectral asymptotics for non-smooth elliptic problems (In Russian). *Funktsional. Anal. i Prilozhen.* **4**, No 4 (1970), 1–13; Engl. transl. in *Functional Analysis Appl.* **4** (1971), 265–275.

[6] M.Sh. Birman and M.Z. Solomyak, Spectral asymptotics of nonsmooth elliptic operators, I (In Russian). *Trudy Moskov. Mat. Obshch.* **27** (1972), 3–52; Engl. transl. in *Trans. Moscow Math. Soc.* **27** (1972/ 1975), 1–52.

[7] M.M. El-Borai and A. Debbouche, On some fractional integro-differential equations with analytic semigroups. *Intern. J. of Contemp. Math. Sci.* **4**, No 25-28 (2009), 1361–1371.

[8] R. Gorenflo, A.A. Kilbas, F. Mainardi, and S.V. Rogosin, *Mittag-Leffler Functions, Related Topics and Applications*. Springer, Berlin, 2014.

[9] A.A. Kilbas, H.M. Srivastava, and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*. North-Holland Math. Studies, Vol. 204, Elsevier, New York, 2006.

[10] V.P. Mikhailov, *Partial Differential Equations*. Mir, Moscow, 1978.

[11] R.E.A.C. Paley and N. Wiener, *Fourier Transforms in the Complex Domain*. Amer. Math. Soc. Coll. Publ. 19, 1934.

[12] Y.A. Rossikhin and M.V. Shitikova, Applications of fractional calculus to dynamic problems of linear and nonlinear hereditary mechanics of solids. *Appl. Mech. Rev.* **50** (1997), 15–67.

[13] F. Saedpanah, *Existence and Uniqueness of the Solution of an Integro-Differential Equation with Weakly Singular Kernel*. Dept. of Math. Sciences, Chalmers University of Technology, 2009.

[14] Vu Kim Tuan, Laplace transform of functions with bounded averages. *Intern. J. of Evolution Equations* **1**, No 4 (2005), 429–433.

[15] Vu Kim Tuan, Inverse problem for fractional diffusion equation. *Fract. Calc. Appl. Anal.* **14**, No 1 (2011), 31–55; DOI:10.2478/s13540-011-0004-x; https://www.degruyter.com/view/journals/fca/14/1/fca.14.issue-1.xml.

[16] D. V. Widder, *The Laplace Transform*. Princeton Univ. Press, Princeton, 1946.

[17] N. Wiener, Generalized harmonic analysis. *Acta Math.* **55** (1930), 117–258.

[18] H. Ye, J. Gao, and Y. Ding, A generalized Gronwall inequality and its application to a fractional differential equation. *J. Math. Anal. Appl.* **328** (2007), 1075–1081.

Department of Mathematics, University of West Georgia
Carrollton, GA 30118 – USA

e-mail: vu@westga.edu

Received: March 22, 2020, Revised: May 6, 2020

Please cite to this paper as published in:

Fract. Calc. Appl. Anal., Vol. **23**, No 5 (2020), pp. 1300–1328,
DOI: 10.1515/fca-2020-0065