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RESEARCH PAPER

ON A QUANTITATIVE THEORY OF LIMITS: ESTIMATING THE SPEED OF CONVERGENCE

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Abstract

The classical "ε-δ" definition of limits is of little use to quantitative purposes, as is needed, for instance, for computational and applied mathematics. Things change whenever a realistic and computable estimate of the function $\delta(\varepsilon)$ is available. This may be the case for Lipschitz continuous and Hölder continuous functions, or more generally for functions admitting of a modulus of continuity. This, provided that estimates for first derivatives, fractional derivatives, or the modulus of continuity might be obtained. Some examples are given.

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Key Words and Phrases: limits; Lipschitz; Hölder; modulus of continuity; fractional derivatives; Riemann-Liouville derivative; Caputo derivative; convex functions

1. Introduction

Stan Ulam [12, 19] was used to say that being able to discriminate between the smaller and the larger is more important than being able to solve differential equations. This is definitely true for the applied mathematician and for all scientists who use mathematics quantitatively. The existing classical theory of limits provides an example of the opposite case.

The basic definition of limits given in any textbook, is in fact the so-called ε-δ definition. Confining our attention, for simplicity, to finite limits of real-valued functions of one real variable, say $f(x)$, one says that

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$\lim_{x\to x_0} f(x) = \lambda$, with λ and x_0 finite, when, for every given $\varepsilon > 0$ a $\delta > 0$ exists such that $|f(x) - \lambda| < \varepsilon$ for all x such that $|x - x_0| < \delta$. A special case occurs when $f(x)$ is continuous at x_0, since then $\lambda = f(x_0)$. In higher dimensions, the absolute values can be replaced by norms.

From the definition follows that δ depends on ε, but no hints are given, however, about the function $\delta(\varepsilon)$, in particular, no estimate of it are provided. This definition is clearly *qualitative* rather than *quantitative* in nature.

From Numerical Analysis point of view, this occurrence is rather disappointing. In fact, if one wants to compute approximately the limiting value, λ, of $f(x)$, and would like to know how close to x_0 the values of x should be to infer that $f(x)$ falls near λ within a prescribed (small) tolerance ε, no answer exists. An estimate, but a *realistic* and *computable* one, for the function $\delta(\varepsilon)$, if not the explicit form of it, is what is needed.

Similar observations could be made concerning the asymptotic (Landau) symbols, o and $\sim$, used in relations like $f(x) = o(g(x))$ and $f(x) \sim g(x)$, since they are formulated in terms of limits, and also about the $\mathcal{O}$-symbol (which does not involve a limit). One says that $f(x) = \mathcal{O}(g(x))$ in some neighborhood of a certain given x_0 if an estimate like $|f(x)| \leq K\,|g(x)|$ holds in that neighborhood, but the value of K is not specified. This does not imply the existence of a limit, but, clearly, if one intends to approximate the value of $f(x)$, the value of K definitely matters. In many cases, one would like to know the smallest constant K implied by the $\mathcal{O}$-symbol, [15, 18].

2. Modulus of continuity, Lipschitz, Hölder

In this section, we consider some possible answers to the issue raised in Introduction.

2.1. **The modulus of continuity.** The so called "modulus of continuity" [4] might be sufficiently general to the purpose of assessing the relation between increments of functions and of their argument. The local modulus of continuity of the function $f(x)$ at the given point t, is defined as

$$\omega_f(\delta; t) := \sup_{x:|x-t|\leq\delta} |f(x) - f(t)|. \tag{2.1}$$

The (global) modulus of continuity of $f(x)$, continuous in a set $\mathcal{K}$, is then defined as

$$\omega_f(\delta) := \sup_t \omega_f(\delta; t) = \sup_{x,t:|x-t|\leq\delta} |f(x) - f(t)|. \tag{2.2}$$

Clearly, $|f(x) - f(t)| \leq \omega_f(x - t)$ in this case.

Some useful properties of the (global) modulus of continuity, well known in the literature and that can be easily checked, are the following:

- $\omega_f(\delta)$ is a nondecreasing function for $\delta > 0$; (hence if $0 < \delta_1 < \delta_2$, then $\omega_f(\delta_1) \leq \omega_f(\delta_2)$; and more, $\omega_f(\delta_1 + \delta_2) \leq \omega_f(\delta_1) + \omega_f(\delta_2)$.
- $\lim_{\delta \to 0^+} \omega_f(\delta) = 0$ if and only if f is uniformly continuous;
- for every integer $n \geq 1$, we have $\omega_f(n\delta) \leq n\,\omega_f(\delta)$;
- for every $\lambda > 0$, we have $\omega_f(\lambda\delta) \leq (1 + \lambda)\,\omega_f(\delta)$.

More generally, a monotone nondecreasing function $\omega(\delta)$, defined and continuous for $\delta \geq 0$, is [called] *a* modulus of continuity if $\omega(0) = 0$ and $\omega(\delta_1 + \delta_2) \leq \omega(\delta_1) + \omega(\delta_2)$ for every $\delta_1, \delta_2 \geq 0$ (note that $\omega(\delta) \geq 0$ for every $\delta \geq 0$).

Moduli of continuity may be also measured in other norms, leading, e.g., to the notion of "integral modulus of continuity", but we will not discuss these cases here.

It is advisable to confine to closed (or compact) sets for the functions for which a modulus of continuity is wanted. In fact, on unbounded sets, e.g., the function $f(x) := \sin(x^2)$ has a "modulus of continuity" $\omega_f(\delta) = 2$, for every value of δ.

2.2. **The special case of Lipschitz continuity.** In order to gain some insight on the issue we are considering, one needs *to know more* about the function $f(x)$. Suppose for instance that, instead of being merely continuous at the point x_0, the function $f(x)$ is *Lipschitz* continuous in a neighborhood of x_0, with known or well-estimated constants. This means that an estimate like

$$|f(x) - f(x_0)| < L\,|x - x_0| \tag{2.3}$$

holds, for all x in some neighborhood of x_0. Clearly, such a relation implies continuity at x_0, while differentiability implies the Lipschitz continuity. Whenever $f(x)$ is differentiable in some compact set, say $\mathcal{K}$, e.g., a closed interval, containing x_0, $f(x)$ turns out to be *uniformly* Lipschitz for all $x \in \mathcal{K}$.

In such occurrences, from $|x - x_0| < \delta$ the precise estimate $|f(x) - f(x_0)| < L\,\delta$ follows, and hence the explicit function $\delta(\varepsilon) := \varepsilon/L$.

It is noteworthy that, also on compact sets, nice continuous functions may be not Lipschitz continuous; see for instance $f(x) := \sqrt{x}$ on $[0,1]$.

The function $f(x) := x\,\sin(1/x)$ for $x \in [0,1] \setminus \{0\}$, $f(0) := 0$, is Lipschitz but not differentiable at $x = 0$.

If $f(x)$ is Lipschitz continuous, then it has the modulus of continuity $\omega_f(\delta) = C\,\delta$, for some $C > 0$.

2.3. **The special case of Hölder continuity.** If two constants A and α, with $A > 0$ and $0 < \alpha < 1$, exist, such that the function $f(x)$ defined, e.g., in some interval $[a, b]$, obeys

$$|f(x) - f(x_0)| < A\,|x - x_0|^\alpha, \tag{2.4}$$

being $x_0 \in [a, b]$, for all x in some neighborhood of x_0, the function is termed locally Hölder continuous at x_0. It a relation like (2.4) holds for all x and x_0 in $[a, b]$, then $f(x)$ is termed uniformly Hölder continuous in $[a, b]$.

What is the relation between Hölder continuity, Lipschitz continuity, and differentiability? Differentiability implies Lipschitz that implies Hölder, but not conversely: for instance, the function $f(x) := x^\alpha$ for $x > 0$, with $0 < \alpha < 1$, is Hölder but not Lipschitz.

For f Hölder continuous, sometimes denoted by Lip_α with $0 < \alpha < 1$, $\omega_f(\delta) = C\,\delta^\alpha$, for some $C > 0$.

What is needed now, in practice, is a (realistic and computable) bound for the Caputo (or the Riemann-Liouville) fractional derivative.

2.4. **The special case of convex functions.** Confining to real valued functions of one real variable defined on an interval, we recall that $f(x)$ is convex if and only if the ratio

$$R(x_1, x_2) := \frac{f(x_1) - f(x_2)}{x_1 - x_2}$$

is a *monotone nondecreasing* function of x_1 for every fixed x_2 (x_1 and x_2 can be interchanged). From this it follows that every convex function defined on the open interval I is continuous on I and is Lipschitz-continuous on every closed subinterval of I. We have the following results:

• If $f(x)$ is convex in some domain $\mathcal{D}$ with nonempty interior, and $m \leq f(x) \leq M$ in the ball $B(x; r_0)$, then it is locally Lipschitz there, and

$$|f(x_1) - f(x_2)| \leq \frac{M - m}{r - r_0}\,|x_1 - x_2| \tag{2.5}$$

in the closed ball $B(x; r_0)$ included in $\mathcal{D}^{\mathrm{o}}$, with any $r < r_0$, [5].

• If $f(x)$ is convex in some domain, then a uniform Lipschitz estimate like $f(x_1) - f(x_2)| \leq L\,|x_1 - x_2|$ holds with some L, for all x belonging to any closed and bounded set, K (i.e., compact set, since we are in $\mathbf{R}$), provided that K is contained in the relative interior of the domain of f. For instance, the previous result fails for:

(1) $f(x) := 1/x$ in the domain $D := (0, +\infty)$, with $K := (0, 1]$, since K is not closed;
(2) $f(x) := x^2$ in the domain $D := \mathbf{R}$, with $K := \mathbf{R}$, since K is not bounded;
(3) $f(x) := -\sqrt{x}$ in the domain $D := [0, +\infty)$, with $K := [0, 1]$, since K is not contained in the interior of D,

and in fact these functions are not Lipschitz, see [11].

3. Making useful the previous relations

In practice, it is important to be able to estimate the constants inherent to the relations above, and moreover, to be practically useful, such estimates should be both, *realistic* and easily *computable*.

3.1. **Estimating the modulus of continuity.** In [6], the following result, may be one of the the most useful one, was established.

A) If $f \in C^0([a,b])$ is *nonconcave* and *monotone* on $[a,b]$, then, for any $\delta \in (0, b-a)$, we have that:

- $\omega(f,\delta) = f(b) - f(b-\delta)$ if f is nondecreasing on $[a,b]$;
- $\omega(f,\delta) = f(a) - f(a+\delta)$ if f is nonincreasing on $[a,b]$.

B) If $f \in C^0([a,b])$ is *nonconcave* on $[a,b]$, then

$$\omega(f,\delta) = \max\{|f(a+\delta) - f(a)|, |f(b-\delta) - f(b)|\},$$

for δ sufficiently small.

The functions in the following examples are clearly *not* Hölder continuous at $x=0$, but have a modulus of continuity.

Example 3.1. Consider for instance the continuous, nonincreasing, nonconcave (actually decreasing and convex) function

$$f(x) := \begin{cases} \frac{1}{\log x}, & 0 < x \le e^{-2}, \\ 0, & x = 0, \end{cases} \tag{3.1}$$

where log denotes the natural logarithm, hence, by (B), we infer that it has the modulus of continuity

$$\omega_f(\delta) = f(0) - f(\delta) = -\frac{1}{\log \delta}, \quad 0 < \delta < e^{-2}. \tag{3.2}$$

Note that this function is *neither* Lipschitz *nor* Hölder continuous in the interval we considered. In [6, p. 200] the moduli of continuity for a few nonconcave or nonconvex elementary functions are also given.

Example 3.2. The function

$$f(x) := \begin{cases} \frac{1}{\log^{1/2}(\frac{1}{x})}, & 0 < x \le e^{-2}, \\ 0, & x = 0, \end{cases} \tag{3.3}$$

is continuous, nondecreasing, nonconcave (actually increasing and convex), and hence, by (A), it has the modulus of continuity

$$\omega_f(\delta) = 2^{-1/2} - \log^{-1/2}((e^{-2} - \delta)^{-1}). \tag{3.4}$$

Another result established in [6] reads:

C) Any function, $f(x)$, *continuous* and *nonconcave* on $[a,b]$ has the modulus of continuity

$$\omega_f(\delta) = \max\{f(a) - f(a+\delta), f(b) - f(b-\delta)\}, \quad \text{for } 0 < \delta < \varphi(m), \quad (3.5)$$

where $\varphi(x) := \min\{b-x, x-a\}$, for $x \in (a,b)$ and m is the point of global minimum of $f(x)$ in $[a,b]$.

Note that not only the definition (3.5) holds for "δ sufficiently small", but its smallness is estimated precisely by the quantity $\varphi(m)$.

Thus we can take $\varepsilon = \omega_f(\delta)$, and hence, being $\omega(\delta)$ monotone for all $\delta \geq 0$ (by definition), we obtain the function, of ε, $\delta = \omega_f^{-1}(\varepsilon)$. Unfortunately, generally speaking, estimates for the moduli of continuity are not always available, as it would be desirable.

3.2. **Estimating the Lipschitz constant.** Recall that $\omega_f(\delta) = C\,\delta$. For differentiable functions, one can use (first) derivatives to estimate Lipschitz constants, L (also called modulus of uniform continuity). In fact, if $f(x)$ is defined in some neighborhood of x_0 and differentiable at x_0, a relation like

$$|f(x) - f(x_0)| < L\,|x - x_0| \quad (3.6)$$

holds with $L = |f'(x_0)| + \delta$, for some $\delta > 0$. This is called *local* Lipschitz continuity. If $f(x)$ is differentiable in some interval, (a,b), then the estimate (3.6) holds for all x, x_0 belonging to any compact subset K of (a,b), and $L = \max_{x \in K} |f'(x)|$ (uniform Lipschitz continuity).

3.3. **Estimating the Hölder constants.** Similarly, fractional derivatives can be used to estimate Hölder constants.

Let us give here the definitions of the Riemann-Liouville, Caputo, and Grünwald-Letnikov fractional derivatives [8]:

• ([8, Definition 2]). If $0 < \alpha < 1$ and $x_0 \in \mathbf{R}$, the *Riemann-Liouville* fractional derivative of $f(x) \in L^1([x_0, x_1])$, of order α and with starting point x_0, at the point x, is defined as

$${}^{RL}\mathrm{D}^{\alpha}_{x_0} f(x) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{x_0}^{x} \frac{f(t)}{(x-t)^{\alpha}}\, dt, \quad x_0 < x_0 < x - 1. \quad (3.7)$$

• ([8, Definition 3]). If $0 < \alpha < 1$ and $x_0 \in \mathbf{R}$, the *Caputo* fractional derivative of $f(x) \in AC([x_0, x_1])$, of order α and with starting point x_0, at the point x, is defined as

$${}^{C}\mathrm{D}^{\alpha}_{x_0} f(x) := \frac{1}{\Gamma(1-\alpha)} \int_{x_0}^{x} \frac{f'(t)}{(x-t)^{\alpha}}\, dt, \quad x > x_0. \quad (3.8)$$

Here $AC(I)$ denotes the set of the functions absolute continuous in I.

• ([8, Definition 5]). If $\alpha > 0$ and $x_0 \in \mathbf{R}$, the (truncated) *Grünwald-Letnikov* fractional derivative of $f(x)$, of order α and with starting point x_0, at the point x, is defined as

$$ {}^{GL}\mathrm{D}^{\alpha}_{x_0} f(x) := \lim_{h\to 0^+} \frac{1}{h^\alpha} \sum_{j=0}^{N} (-1)^j \binom{\alpha}{j} f(x-jh), \quad x > x_0, \quad N := \left\lceil \frac{x-x_0}{h} \right\rceil . \tag{3.9} $$

Sometimes the point x_0 above is taken at $-\infty$. It can be shown that letting $x_0 \to -\infty$, the Grünwald-Letnikov derivative becomes [8, Definition 4]

$$ {}^{GL}\mathrm{D}^{\alpha}_{-\infty} f(x) :={}^{GL} \mathrm{D}^{\alpha} f(x) := \lim_{h\to 0^+} \frac{1}{h^\alpha} \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} f(x-jh), \quad x \in \mathbf{R}. \tag{3.10} $$

One can show that for smooth functions, more precisely for $f \in C^1([x_0, x_1])$, it turns out that

$$ {}^{GL}\mathrm{D}^{\alpha}_{x_0} f(x) ={}^{RL} \mathrm{D}^{\alpha}_{x_0} f(x) \tag{3.11} $$

for any $x \in (x_0, x_1]$, [8, eq. 17], [2, Theorem 2.25].

All these derivatives are linear but *nonlocal* operators. The GL derivatives may be convenient for numerical work.

It is also useful to observe that truncating $f(x)$ at $x = x_0$, so to define the truncated function $f^R(x) := f(x)$ for $x < x_0$ and $f^R(x) = 0$ for $x \geq x_0$ leads to the result

$$ {}^{GL}\mathrm{D}^{\alpha} f^R(x) ={}^{GL} \mathrm{D}^{\alpha}_{x_0} f(x) ={}^{RL} \mathrm{D}^{\alpha}_{x_0} f(x) \tag{3.12} $$

for all $\alpha > 0$ and $f \in C^1([x_0, x_1])$, and for each $x \in (x_0, x_1]$, [8, Proposition 4], while for the function $f^C(x)$ defined as equal to $f(x)$ for $x < x_0$ and as its Taylor polynomial, centered at $x = x_0$ and such that $f^C(x)$ will be a C^1 smooth function over all $\mathbf{R}$, we obtain [8, Proposition 5]

$$ {}^{GL}\mathrm{D}^{\alpha} f^C(x) ={}^{C} \mathrm{D}^{\alpha}_{x_0} f(x). \tag{3.13} $$

Roughly speaking, RL, C, and GL fractional derivative of $f(x)$ coincide for every smooth function, $f(x)$.

Then, recall the following results:

- ([10, § 5.7, p. 591, Theorem 20], [16]) If $0 < \alpha < \beta \leq 1$ and $f \in \mathrm{Lip}_\beta$, then the Riemann-Liouville derivative ${}^{RL}\mathrm{D}^{\alpha} f(x)$ exists and is continuous, actually belongs to $\mathrm{Lip}_{\beta-\alpha}$.

  Conversely, any function, f, possessing, e.g., a fractional Caputo derivative of order α, with $0 < \alpha < 1$, is α-Hölder continuous [7].

Actually, if f is in $L^1(a,b)$ and its Caputo derivative of order α is bounded, then f is α-Hölder continuous and the estimate

$$|f(x) - f(y)| \leq M\,|x-y|^\alpha, \quad \forall x, y \in (a,b), \tag{3.14}$$

$$M := \frac{1}{\Gamma(\alpha+1)} \sup_x |{}^C\mathrm{D}^\alpha f(x)| \tag{3.15}$$

holds. This result can be derived, e.g., from [7, Proposition 1], where however there is an extra factor 2.

- ([3, Corollary 2.4, p. 308], [14, Theorem 1, p. 288]) If $0 < \alpha \leq 1$ and $f(x) \in C^0([a,b])$ is such that ${}^C\mathrm{D}^\gamma f(x) \in C^0([a,b])$, the there exists some $\xi \in (a,b)$ such that

$$f(b) - f(a) = \frac{1}{\Gamma(\alpha)}\, {}^C\mathrm{D}_a^\alpha f(\xi)\,(b-a), \tag{3.16}$$

where $\Gamma(\alpha)$ is the Euler gamma function, that is, a mean-value theorem exists for the Caputo derivative. Note that, being $0 < \alpha \leq 1$, we have $1/\Gamma(\alpha) \leq 1$, hence, *a fortiori* (3.16) holds omitting the factor $1/\Gamma(\alpha)$.

Similarly to the case of Lipschitz continuous functions, in practice what is now needed is a (realistic and computable) bound for the fractional derivative derivative above. This can be obtained, apart from some elementary instances, by the so-called Grünwald-Letnikov fractional derivative, since this can approximate Riemann-Liouville and Caputo derivatives by means of finite differences [2, 8].

Explicit expressions exist for the RL, C, and GL fractional derivatives of several simple functions. For instance,

$${}^{RL}\mathrm{D}_0^\alpha x^\beta =^C \mathrm{D}_0^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\beta-\alpha+1}\, x^{\beta-\alpha}, \tag{3.17}$$

for $x > 0$ and any $0 < \beta \neq \alpha - 1$;

$${}^{GL}\mathrm{D}^\alpha e^{\omega x} = \omega^\alpha e^{\omega x}, \tag{3.18}$$

for $\omega > 0$ and $x \in \mathbf{R}$;

$${}^{GL}\mathrm{D}^\alpha \sin(\omega x) = \omega^\alpha \sin\left(\omega x + \alpha\,\frac{\pi}{2}\right), \tag{3.19}$$

for $\alpha > 0$ and $x \in \mathbf{R}$.

Other fractional derivatives (RL, C, or GL) of exponentials and sinusoidal functions can be expressed in terms of the two-parameter Mittag-Leffler function, that is the entire function

$$E_{\alpha,\beta}(x) := \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)}, \tag{3.20}$$

with $\alpha > 0$, for which estimates, asymptotic estimates, and numerical method for evaluating it do exist; see [8] or references therein.

4. **Some applications**

A case when the Lipschitz continuity is seen to be extremely useful can be found in the celebrated 1922 Banach contraction lemma (or Banach fixed point theorem). Here, the Lipschitz continuity, with a constant $0 \le L < 1$, plays a fundamental role: not only existence and uniqueness of fixed points for the equation $x = f(x)$ can be established, but the rate of convergence to the limit of a suitable approximating sequence, and even a criterion to stop the convergence process when a prescribed accuracy is attained, are provided.

This lemma states the following. Assume that $\mathcal{M}$ is a *nonempty closed* subspace of the *metric* space X, which is *complete* with respect to the distance d. Let the operator T map $\mathcal{M}$ into itself and be L-contractive, i.e., there exists a constant L, with $0 < L < 1$, such that $d(Tx, Ty) \le L\, d(x, y)$ for every $x, y \in \mathcal{M}$. Then, the map T has a unique fixed point in M, i.e., the equation

$$x = Tx$$

has a unique solution, say x^*, $x^* \in \mathcal{M}$, and the sequence $\{x_n\}_0^\infty$ of successive approximations, iteratively defined by

$$x_{n+1} = Tx_n,\ n = 0, 1, 2, \dots,$$

started at $x_0 \in \mathcal{M}$, converges to x^* as $n \to \infty$ for any choice of x_0 (in $\mathcal{M}$). In addition, the following "error estimates" can be established for $n = 0, 1, 2, \dots$:

$$d(x_n, x^*) \le L^n (1 - L)^{-1} d(x_0, x_1) \quad (\textit{a priori} \text{ estimate}), \tag{4.1}$$

$$d(x_{n+1}, x^*) \le L(1 - L)^{-1} d(x_{n+1}, x_n) \quad (\textit{a posteriori}). \tag{4.2}$$

Moreover, the estimate for the "speed of convergence"

$$d(x_{n+1}, x^*) \le L\, dx_n, x^*) \tag{4.3}$$

can also be proved, for $n = 0, 1, 2, \dots$.

In many practical cases, the aforementioned *metric* space, X, will be a *normed* space.

It is remarkable here that *a precise* estimate of the convergence rate, not just the fact that x_n converges to x^*, is provided. For instance, from (4.1) one can infer how large n should be in order to get $d(x_n, x^*) < \varepsilon$. In fact, one should choose

$$n > \frac{\log\left(\frac{1-L}{d(x_0,x_1)}\,\varepsilon\right)}{\log L} = \frac{\log\frac{1}{\varepsilon} + \log\left(\frac{d(x_0,x_1)}{1-L}\right)}{\log\frac{1}{L}}. \tag{4.4}$$

Note that here $0 < L < 1$, and $\varepsilon > 0$ can be assumed to be small (< 1), so that both $\log L$ and $\log\varepsilon$ are negative numbers.

In one dimension, in practice, what is needed is a (realistic and computable) bound for the derivative.

The Nekhoroshev theorem.

We encounter an important example where the function $\delta(\varepsilon)$ is available in the framework of the so-called KAM (Kolmogorov-Arnold-Moser) theory, and more precisely in the Nekhoroshev and Nekhoroshev-like theorems. These concern what is called "exponential stability" of dynamical systems [13, 9]. The typical formulation reads

$$|I(t) - I(0)| \leq \varepsilon \qquad \forall t,\ t \leq T_{Nekh} := \frac{T}{\varepsilon^a} \exp\left(\frac{c}{\varepsilon^b}\right), \tag{4.5}$$

for some real positive constants $a, b, c, T, \varepsilon_0$, the relation holding for every $0 < \varepsilon < \varepsilon_0$.

While the practical usefulness of this kind of results rest on the violation of the estimate for $I(t) - I_(0)$, namely in the fact that after times longer than the Nekhoroshev time T_{Nekh} there is no guarantee that $I(t)$ will remain closer to $I(0)$ (and in fact this typically will be the case), we consider (4.5) as the formulation of a finite limit, where the rather special function $\delta(\varepsilon) := (T/\varepsilon^a)\, e^{c/\varepsilon^b}$ is given.

In some practical applications, e.g., to the stability of the solar system, the parameter ε is positive but usually estimated as extremely small, thus letting the solar system to be stable for very long Nekhoroshev times, T_{Nekh}. A time interval longer than the (estimated) life of the sun itself makes it safe the life on the Earth as long as the Earth will be able to take advantage from its star. In this respect, to have a good estimate of the "threshold" parameter ε_0 is quite important.

Recently, Nekhoroshev-type theorems have been established also for some infinite-dimensional dynamical systems, namely systems governed by nonlinear wave equations and by other kinds of nonlinear partial differential equations [1].

Also in the asymptotic theory for nonlinear ODEs there are estimates valid on "expanding intervals", i.e., of the form $|y_n(t) - y(t)| < \varepsilon$ for $0 < t \leq T/\varepsilon$, that is a kind of *algebraic* (rather than *exponential*) stability; see e.g., [17].

5. Results and discussion

In this paper, we considered the possibility to formulate in a quantitative form the classical ε-δ definition of limits for a given function, $f(x)$, in order to obtain a quantitative estimate of the speed of convergence. This amounts to being able to estimate the function $\delta(\varepsilon)$. Functions $f(x)$ admitting of a modulus of continuity, for which a realistic and computable estimates exist, for instance, fit this requirement.

6. **Conclusions**

Assessing the function $\delta(\varepsilon)$ inherent to the classical definition of limits is useful in many practical applications of computational and applied mathematics. In the special cases of convex, differentiable, or fractionally differentiable functions, this can be done exploiting (often) computable estimates.

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