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RESEARCH PAPER

α-FRACTIONALLY CONVEX FUNCTIONS

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Abstract

In this article, we introduce the concept of α-fractionally convex functions. We primarily focus on characterizing some of the qualitative properties of convex functions with the assistance of fractional order operators. Also, we discuss the connection of optimality, monotonicity, and convexity of a function in the sense of fractional calculus. Several examples are provided to support the proposed formulation, and one can establish the fractional convexity of a non-convex function.

MSC 2010: Primary 26A33; Secondary 26A48, 26A51, 52A41

Key Words and Phrases: convex function; monotonic function; fractional derivative; fractional integral; fractional Taylor-Riemann series

1. Introduction

The field of fractional calculus originated with an innovative idea to extend the order of derivative (or integration) from an integer to a non-integer. Some of its extensively analyzed problems include generalized Abel's integral equation and its applications [3], generalized Taylor's series expansion for fractional derivatives introduced by Osler [21], fractional variational (see [15, 19]) and fractional optimal control problems (see [16, 25, 26]). The modernization of such problems from classical to fractional calculus motivates us to inspect the convexity of a function in the sense of fractional calculus.

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As per the author's knowledge, fractionally convex functions have not been studied in the literature. Although, Zhou [28] discussed the monotonicity and convexity of fractional differential operators. Some of the related work includes convexity for nabla and delta fractional difference operators (see [1, 8, 12, 13], and references therein), fractional mean value theorem [7, 22], monotonicity of fractional difference operators [2, 5, 6], and convexity results for fractional differences [9, 10].

In this paper, we focus on analyzing the role of fractional order derivatives to define the convexity of a real-valued function. The prime objective is to define the notion of fractional convexity, even for non-convex functions. Apart from the convexity of fractional operators, it is necessary to characterize the conditions under which a non-convex function becomes fractionally convex. We provide a few well-founded reasons in support of formulating fractionally convex functions, verified with given examples and their graphical representations. Lastly, we discuss the necessary optimality condition for optimizing a fractionally convex function.

1.1. **Motivation: Why fractional sense convexity?** The question of primary interest: What is the necessity of fractional sense convexity? If required, is it possible to define some sort of fractionally convex functions as an augmentation of classically convex functions? Usually, one come across non-convex functions. And, we are aware of the functions which are not differentiable in the classical sense but fractionally differentiable (e.g. $f(x) = |x|$). So, similar to fractional differentiability, we prefer to talk about fractionally convex functions. Apart from necessity, at a certain point, one may have any of the following questions:

(1) If a function turns out to be non-convex, what kind of fractional sense convexity can be imposed?
(2) Does the proposed fractional sense convexity meet all the basic criterion of convex functions?
(3) Will there be any sort of link between convex and fractionally convex functions?
(4) Is it possible to find some functions which are convex but not fractionally convex and vice-versa?

We shall answer these questions in our present investigation. Concerning the above points, we shall see that fractional sense convexity should possess certain properties to make the proposed work meaningful. These remarks trace the presence and possibility of making an effort to impose the fractional sense convexity on non-convex functions. Outlining the insights of fractionally convex functions in present work, we have observed:

- functions which are non-convex in the classical sense but convex in fractional sense and vice versa,

- necessary conditions under which a convex function turns out to be a fractionally convex function.

2. Preliminaries

In this section, we give some basic definitions and properties of fractional order derivatives/integrals (see e.g. [24, 23, 14, 17]), which will be needed in the sequel.

Let $f \in C[a,b]$, where $C[a,b]$ is the space of all continuous functions defined on the closed interval $[a,b]$.

DEFINITION 2.1. For all $x \in [a,b]$ and $\alpha > 0$, the left Riemann-Liouville fractional integral of order α is defined as

$$ {}_aI_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\tau)^{\alpha-1} f(\tau)\, d\tau. $$

DEFINITION 2.2. For all $x \in [a,b]$ and $n-1 \le \alpha < n$, the left Riemann-Liouville fractional derivative of order α is defined as

$$ {}_aD_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x (x-\tau)^{n-\alpha-1} f(\tau)\, d\tau. $$

Let us consider $f \in C^n[a,b]$, where $C^n[a,b]$ is the space of n times continuously differentiable functions defined on $[a,b]$.

DEFINITION 2.3. For $n-1 \le \alpha < n$, the left Caputo fractional derivative of order α is defined as

$$ {}_a^cD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\, d\tau. \tag{2.1} $$

Properties of fractional order operators: For $n-1 \le \alpha < n$, $n \in \mathbb{N}$:

1. The Riemann-Liouville and Caputo fractional derivatives are related by

$$ {}_aD_x^\alpha f(x) = {}_a^cD_x^\alpha f(x) + \sum_{k=0}^{n-1} \frac{(x-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(a)\,. \tag{2.2} $$

2. Composition rules for fractional derivatives and integrals are given by

- $$ {}_aI_x^\alpha\, {}_aD_x^\alpha f(x) = f(x) - \sum_{j=0}^{n-1} \frac{(x-a)^{\alpha-j}}{\Gamma(\alpha-j+1)} {}_aD_x^{\alpha-j} f(a)\,, $$
- $$ {}_aI_x^\alpha\, {}_a^cD_x^\alpha f(x) = f(x) - \sum_{j=0}^{n-1} \frac{(x-a)^{j}}{\Gamma(j+1)} f^{(j)}(a)\,. $$

3. **Main results**

In this section, we discuss convex functions in the sense of fractional calculus. Throughout the work, we deal with univariate real-valued functions and intend to accomplish the objectives listed below:

- *Monotonicity:* To assimilate the monotonicity of a real-valued function in terms of fractional order derivatives, and the monotonicity of Riemann-Liouville and Caputo fractional derivatives.
- *Convexity:* To introduce the notion of α-fractionally convex functions, and analyze the resemblance by deducing results analogous to classically convex functions. We illustrate some examples graphically for fractional convexity with distinct fractions $\alpha \in (0, 1)$.
- *Optimal Points:* To describe an unconstrained optimization problem for minimizing a fractionally convex function.

3.1. **Comments on monotonicity of functions with fractional derivatives.** In this section, we investigate the criteria of monotonicity of a function with fractional derivative of order $\alpha \in (0, 1)$. We start by recalling the classical property of a monotone function.

A real-valued function $f \in C^1[a, b]$ is said to be monotonically increasing on $[a, b]$, if $\forall x, y \in [a, b]$ such that $x \leq y$ one must have $f(x) \leq f(y)$. In terms of first derivative, we must have $f'(x) \geq 0$, $\forall x \in [a, b]$. Similarly, f is said to be monotonically decreasing if, whenever $x \leq y$ then $f(x) \geq f(y)$. Or, $f'(x) \leq 0$, $\forall x \in [a, b]$. Now, we will discuss the monotonicity of f with Riemann-Liouville and Caputo fractional order derivatives.

THEOREM 3.1. *Let $f \in C^1[a, b]$, and $\alpha \in (0, 1)$. Then, f is monotonically increasing on $[a, x]$ if and only if ${}_a^cD_x^\alpha f(x) \geq 0$. Or, $f'(x) \geq 0$ if and only if ${}_a^cD_x^\alpha f(x) \geq 0$.*

P r o o f. Let f be monotonically increasing function on $[a, x]$. Next, we consider the Caputo fractional derivative of order α,

$$ {}_a^cD_x^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x (x-t)^{-\alpha} f'(t)\, dt. \tag{3.1}$$

Clearly, $f'(x) \geq 0$ on $[a, x]$ as f is monotonically increasing, and also $(x - t)^{-\alpha} > 0$. So, we conclude that the integral given by Eq. (3.1) is non-negative, that is, ${}_a^cD_x^\alpha f(x) \geq 0$. Conversely, let us now assume $g(x) = {}_a^cD_x^\alpha f(x) \geq 0$, which implies that

$$\begin{aligned} {}_aI_x^\alpha g(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} g(t)\, dt \geq 0, \quad \forall\, x \geq a, \\ \text{or,} \quad {}_aI_x^\alpha\, {}_a^cD_x^\alpha f(x) &\geq 0, \quad \forall\, x \geq a, \end{aligned}$$

$$\text{or,} \quad {}_aI_x^\alpha \, {}_a^cD_x^\alpha f(x) = f(x) - f(a) \geq 0, \quad \forall\, x \geq a,$$

which proves that f is a monotonically increasing function on $[a, x]$. □

The next theorem gives the condition for monotonically decreasing functions in terms of the Caputo fractional order derivative.

THEOREM 3.2. *Let $f \in C^1[a,b]$, and $\alpha \in (0,1)$. Then, f is monotonically decreasing on $[a,x]$ if and only if ${}_a^cD_x^\alpha f(x) \leq 0$. Or, $f'(x) \leq 0$ if and only if ${}_a^cD_x^\alpha f(x) \leq 0$.*

P r o o f. The proof is omitted as f is monotonically decreasing in an interval $[a, b]$, if $-f$ is monotonically increasing in $[a, b]$. □

COROLLARY 3.1. *Let $f \in C^1[a,b]$, and $\alpha \in (0,1)$. If f is monotonically increasing and $f(a) \geq 0$, then the α^{th}-order Riemann-Liouville derivative ${}_aD_x^\alpha f(x) \geq 0$.*

P r o o f. For $\alpha \in (0, 1)$, using the relationship between Riemann-Liouville and Caputo fractional derivative given by Eq. (2.2) , we have

$${}_aD_x^\alpha f(x) = \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)} f(a) + {}_a^cD_x^\alpha f(x). \tag{3.2}$$

As f is monotonically increasing, we have ${}_a^cD_x^\alpha f(x) \geq 0$ (Theorem 3.1). Since $f(a) \geq 0$, from Eq. (3.2) we can easily observe that ${}_aD_x^\alpha f(x) \geq 0$. □

COROLLARY 3.2. *Let $f \in C^1[a,b]$, and $\alpha \in (0,1)$. If ${}_aD_x^\alpha f(x) \geq 0$ and $f(a) \leq 0$, then f is monotonically increasing.*

P r o o f. With Eq. (3.2), we can write

$${}_a^cD_x^\alpha f(x) = {}_aD_x^\alpha f(x) - \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)} f(a).$$

Let ${}_aD_x^\alpha f(x) \geq 0$ and $f(a) \leq 0$, thus ${}_a^cD_x^\alpha f(x) \geq 0$. Again Theorem 3.1 states that f is monotonically increasing if and only if ${}_a^cD_x^\alpha f(x) \geq 0$, which completes the proof. □

On combining Corollaries 3.1 and 3.2 stated above, we arrive at the following theorem.

THEOREM 3.3. *Let $f \in C^1[a,b]$, $f(a) = 0$, and $\alpha \in (0,1)$. Then, f is monotonically increasing on $[a,x]$ if and only if the Riemann-Liouville derivative ${}_aD_x^\alpha f(x) \geq 0$.*

The following example describes the relevance of Theorems 3.1- 3.3.

EXAMPLE 3.1. Let $f(x) = x^2 - 1,\ x \geq 0$. Clearly f is monotonically increasing for $x \geq 0$, as $f'(x) \geq 0$. For $\alpha \in (0,1)$, we first consider the Caputo fractional derivative of f,

$$ {}_a^cD_x^\alpha f(x) = \frac{\Gamma(3)}{\Gamma(3-\alpha)}\, x^{2-\alpha},\ \forall\, x \geq 0, $$

which shows that ${}_a^cD_x^\alpha f(x) \geq 0$. But, if we consider the Riemann-Liouville derivative of f,

$$ {}_aD_x^\alpha f(x) = \frac{2}{\Gamma(3-\alpha)}\, x^{2-\alpha} - \frac{1}{\Gamma(1-\alpha)}\, x^{-\alpha},\ \forall\, x \geq 0. $$

Clearly, ${}_aD_x^\alpha f(x) \geq 0$ for $2\,x^2 \geq (1-\alpha)(2-\alpha)$. It does guarantee the non-negativity of ${}_aD_x^\alpha f(x)$, since $\alpha \in (0,1)$ is not fixed. Here $f(0) = -1$, so monotonically increasing nature of f need not imply ${}_aD_x^\alpha f(x) \geq 0$. We can verify the same by monitoring the graphs of α^{th}-order Caputo and Riemann-Liouville derivatives of f, as shown in Figure 3.1 .

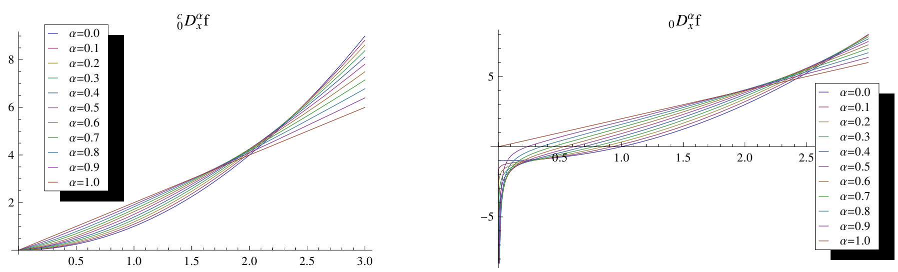

Fig. 3.1: Plot of α^{th}-order Caputo and Riemann-Liouville fractional derivatives of $f(x) = x^2 - 1$, Example 3.1

At this point, it is important to mention that the above results deal with the monotonicity of a function f in terms of fractional derivatives. But, it is not same as the monotonicity of fractional differential operators. For monotonicity of Riemann-Liouville or Caputo fractional derivative, we must emphasize on the $(\alpha+1)^{\text{th}}$-order derivative. And, we must have $\frac{d}{dx}\ [{}_a^cD_x^\alpha f(x)] \geq 0$ (or ≤ 0).

THEOREM 3.4. *Let $f \in C^2[a,b]$, and $\alpha \in (0,1)$. Then, f' is monotonically increasing on $[a,x]$ if and only if ${}_a^cD_x^{\alpha+1} f(x) \geq 0$. Or, $f''(x) \geq 0$ if and only if ${}_a^cD_x^{\alpha+1} f(x) \geq 0$.*

P r o o f. Let $f''(x) \geq 0,\ \forall\, x \geq a$, which implies that

$$ {}_{a}^{c}D_{x}^{\alpha+1} f(x) = \frac{1}{\Gamma(1-\alpha)} \int_{a}^{x} (x-t)^{-\alpha} f''(t)\, dt \geq 0. $$

Conversely, we have ${}_{a}^{c}D_{x}^{\alpha+1} f(x) \geq 0$. Let ${}_{a}^{c}D_{x}^{\alpha+1} f(x) = {}_{a}^{c}D_{x}^{\alpha} g(x) \geq 0$, where $g(x) = f'(x)$. Thus,

$$ \begin{aligned} {}_{a}I_{x}^{\alpha}\, {}_{a}^{c}D_{x}^{\alpha} g(x) &\geq 0,\ \forall\, x \geq a, \\ \text{i.e.,}\quad g(x) - g(a) &\geq 0,\ \forall\, x \geq a, \\ \text{or,}\quad f'(x) - f'(a) &\geq 0,\ \forall\, x \geq a, \end{aligned} $$

that is, $f'(x) \geq f'(a),\ \forall\, x \geq a$, which implies that $f'(x)$ is monotonically increasing on $[a, x]$. □

The next theorem states the similar kind of result for $f'(x)$ to be monotonically decreasing.

THEOREM 3.5. *Let $f \in C^2[a,b]$, and $\alpha \in (0,1)$. Then, f' is monotonically decreasing on $[a,x]$ if and only if ${}_{a}^{c}D_{x}^{\alpha+1} f(x) \leq 0$. Or, $f''(x) \leq 0$ if and only if ${}_{a}^{c}D_{x}^{\alpha+1} f(x) \leq 0$.*

THEOREM 3.6. *Let $f \in C^2[a,b]$, and $\alpha \in (0,1)$. If $f'(x)$ is monotonically increasing and $f'(a) \geq 0$, then ${}_{a}^{c}D_{x}^{\alpha} f(x)$ is monotonically increasing on $[a,x]$.*

P r o o f. We consider

$$ \begin{aligned} \frac{d}{dx}\left[{}_{a}^{c}D_{x}^{\alpha} f(x)\right] &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{a}^{x} (x-t)^{-\alpha} f'(t)\, dt, \\ &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{0}^{x-a} Y^{-\alpha} f'(x-Y)\, dY, \quad Y = x - t \\ &= \frac{1}{\Gamma(1-\alpha)} \left[\int_{0}^{x-a} Y^{-\alpha} f''(x-Y)\, dY + (x-a)^{-\alpha} f'(a) \right], \\ &= \frac{1}{\Gamma(1-\alpha)} \int_{a}^{x} (x-t)^{-\alpha} f''(t)\, dt + \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)} f'(a). \end{aligned} $$

For $f''(x) \geq 0$ and $f'(a) \geq 0$, $\frac{d}{dx}\left[{}_{a}^{c}D_{x}^{\alpha} f(x)\right] \geq 0$, i.e., the α^{th}-order Caputo's fractional derivative of f is monotonically increasing. □

The next theorem states the condition for α^{th}-order Caputo's fractional derivative to be monotonically decreasing.

THEOREM 3.7. *Let $f \in C^2[a,b]$, and $\alpha \in (0,1)$. If $f'(x)$ is monotonically decreasing and $f'(a) \le 0$, then ${}_a^cD_x^\alpha f(x)$ is monotonically decreasing on $[a,x]$.*

Note that $\frac{d}{dx}\,[{}_aD_x^\alpha f(x)] = {}_aD_x^{\alpha+1} f(x)$, so one must have ${}_aD_x^{\alpha+1} f(x) \ge 0$ (or ≤ 0) for Riemann-Liouville derivative to be monotonically increasing (or decreasing).

COROLLARY 3.3. *Let $f \in C^2[a,b]$, and $\alpha \in (0,1)$. If $f'(x)$ is monotonically increasing, $f(a) \le 0$, and $f'(a) \ge 0$. Then ${}_aD_x^{\alpha+1} f(x) \ge 0$, that is, the α^{th}-order Riemann-Liouville derivative ${}_aD_x^\alpha f(x)$ is monotonically increasing.*

P r o o f. For $1 < \alpha + 1 < 2$, using the relationship between Riemann-Liouville and Caputo fractional derivative, we get

$$\begin{aligned} {}_aD_x^{\alpha+1} f(x) &= {}_a^cD_x^{\alpha+1} f(x) + \sum_{k=0}^{1} \frac{(x-a)^{k-(\alpha+1)}}{\Gamma(k-(\alpha+1)+1)}\, f^k(a), \\ &= {}_a^cD_x^{\alpha+1} f(x) + \frac{(x-a)^{-(\alpha+1)}}{\Gamma(-\alpha)}\, f(a) + \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)}\, f'(a), \\ &= {}_a^cD_x^{\alpha+1} f(x) - \frac{\alpha(x-a)^{-(\alpha+1)}}{\Gamma(1-\alpha)} f(a) + \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)} f'(a). \end{aligned} \tag{3.3}$$

Let f' be monotonically increasing on $[a,x]$, which implies ${}_a^cD_x^{\alpha+1} f(x) \ge 0$ (by Theorem 3.5). One may observe from Eq. (3.3), ${}_aD_x^{\alpha+1} f(x) \ge 0$ for $f(a) \le 0$, and $f'(a) \ge 0$. □

COROLLARY 3.4. *Let $f \in C^2[a,b]$, and $\alpha \in (0,1)$. If ${}_aD_x^{\alpha+1} f(x) \ge 0$, $f(a) \ge 0$, and $f'(a) \le 0$. Then, $f'(x)$ is monotonically increasing function on $[a,x]$.*

P r o o f. By Eq. (3.3), we have

$${}_a^cD_x^{\alpha+1} f(x) = {}_aD_x^{\alpha+1} f(x) + \frac{\alpha\,(x-a)^{-(\alpha+1)}}{\Gamma(1-\alpha)}\, f(a) - \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)}\, f'(a).$$

Clearly ${}_a^cD_x^{\alpha+1} f(x) \ge 0$, since ${}_aD_x^{\alpha+1} f(x) \ge 0$, $f(a) \ge 0$ and $f'(a) \le 0$. Thus, Theorem 3.5 assures that f' is monotonically increasing function on the interval $[a,x]$. □

On combining Corollaries 3.3 and 3.4, we arrive at the following theorem.

THEOREM 3.8. *Let $f \in C^2[a,b]$, $f(a) = 0 = f'(a)$, and $\alpha \in (0,1)$. Then, f' is monotonically increasing (or $f''(x) \geq 0$) on $[a,x]$ if and only if the Riemann-Liouville derivative ${}_aD_x^{\alpha+1}f(x) \geq 0$.*

The above theorem asserts that the non-negativity of second derivative of a function f is equivalent to the non-negativity of $(\alpha+1)^{\text{th}}$-order Riemann-Liouville derivative of f, provided $f(a) = 0 = f'(a)$. Conventionally, the convexity of a function $f \in C^2[a,b]$ is same as the non-negativity of f''. Thus, the interconnection of the second derivative and $(\alpha+1)^{\text{th}}$-order derivative of f suggests an idea to propose the notion of fractionally convex functions.

3.2. **α-Fractionally Convex functions.** In this section, we attend convex functions with Riemann-Liouville fractional order derivative in place of classical derivatives. This work introduces the concept of fractionally convex functions, and characterizes the conditions for which a function becomes convex in fractional calculus sense.

We know that a twice differentiable function f defined on an interval I is convex if and only if $f''(x) \geq 0$, $\forall x \in I$. The monotonicity criterion of f' in terms of fractional derivatives (Theorem 3.8) assists us to define the notion of fractionally convex functions by attaching a fraction $\alpha \in (0,1)$, as described below.

DEFINITION 3.1. **α-fractionally convex function:** Let $\alpha \in (0,1)$. A real-valued univariate function $f \in C^2[a,b]$ is said to be fractionally convex of order α, or α-fractionally convex on $[a,x)$ if the $(\alpha+1)^{\text{th}}$-order Riemann-Liouville derivative of f is non-negative, i.e. ${}_aD_x^{\alpha+1}f \geq 0$. And, strictly α-fractionally convex if ${}_aD_x^{\alpha+1}f > 0$.

Note:

- One of the vital inspection is the selection of a fractional order derivative while defining α-fractionally convex functions. The existence of a variety of definitions in literature such as Riemann-Liouville, Grunwald-Letnikov, Weyl, Caputo, and Riesz fractional derivatives (see [17, 20, 24]) make the desired choice crucial. We observe that the left Riemann-Liouville derivative appears to be appropriate as ${}_aD_x^0 f(x) = f(x)$. And ${}_xD_b^0 f(x) = -f(x)$, so one can use right Riemann-Liouville derivative to define the fractional convexity of $-f$ or fractional concavity of f.
- In case of Caputo's fractional derivative ${}_a^cD_x^0 f(x) = f(a) + f(x)$ which further restricts the extension of classically convex functions (as for $\alpha = 0$ one must reach f).

- One may note that Riemann-Liouville derivative is used here to explain fractional convexity of f in some interval $[a, b]$. But, we have to use Weyl's fractional order derivative whenever the interval is extended to whole real line.

REMARK 3.1. For $\alpha = 0$, ${}_aD_x^{\alpha+1} f(x) \geq 0 \Rightarrow f'(x) \geq 0$, that is, the function f is a monotonically increasing function in $[a, x)$.

REMARK 3.2. If $\alpha = 1$, ${}_aD_x^{\alpha+1} f(x) \geq 0 \Rightarrow f''(x) \geq 0$, that is, the function f is a convex function in $[a, x)$.

The raised remarks facilitate us to consider the fractional convexity of f as the lower order convexity of f lying between 0 and 1, i.e., ${}_aD_x^{\alpha+1} f \geq 0$ gives the fractional sense convexity of f for $\alpha \in (0, 1)$.

Characterization of α-fractionally convex functions.

As fractionally convex functions are structured similar to convex functions, we must explain the extensive properties and characteristics analogous to classically convex functions. In addition, we specify the conditions under which classical convexity of a function implies fractional convexity and vice versa.

COROLLARY 3.5. *Let $f \in C^2([a, b])$ be a classically convex function on $[a, x]$. If $f(a) \leq 0$ and $f'(a) \geq 0$, then f is α-fractionally convex for $\alpha \in (0, 1)$*

P r o o f. The proof follows from Corollary 3.3. □

COROLLARY 3.6. *Let $\alpha \in (0, 1)$, and $f \in C^2([a, b])$ be α-fractionally convex. If $f(a) \geq 0$ and $f'(a) \leq 0$, then f is classically convex in $[a, x]$.*

P r o o f. The proof follows from Corollary 3.4. □

The next example enables us to understand the criteria of fractional sense convexity of a function.

EXAMPLE 3.2. Let $f(x) = x^{\frac{1}{2}}$, $x \in \mathbb{R}$. And, $f''(x) = -\frac{1}{4} x^{-\frac{3}{2}} < 0$ for all $x > 0$. Next, we compute the $(1+\alpha)^{\text{th}}$- order Riemann-Liouville derivative of f

$$ {}_0D_x^{1+\alpha} f(x) = \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma(\frac{1}{2} - \alpha)} \, x^{-\left(\frac{1}{2}+\alpha\right)}. $$

Clearly ${}_0D_x^{1+\alpha} f(x) \geq 0$, for $\alpha \leq \frac{1}{2}$. Thus, f is α-fractionally convex in $(0, x)$ for all $\alpha \in (0, \frac{1}{2}]$. But, f is not convex in classical sense for $x > 0$. We may also observe that $f(0) = 0$, and $f'(0) > 0$, that is Corollary 3.6 is not satisfied for f to be classically convex. In Figure 3.2, the non-negativity of ${}_0D_x^{\alpha+1} f$ is justified for $\alpha \in (0, \frac{1}{2}]$. We have also plotted the α^{th}-order Riemann-Liouville derivative of f to verify the monotonicity of f in terms of fractional derivatives (see Theorem 3.3).

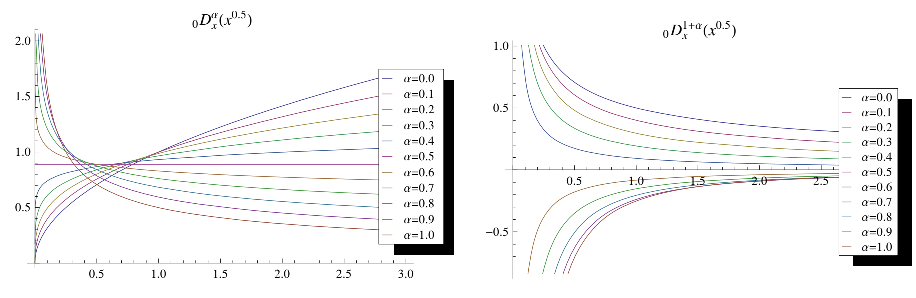

Fig. 3.2: Plot of α^{th}, $(1+\alpha)^{\text{th}}$ order Riemann-Liouville derivatives of $f(x) = x^{\frac{1}{2}}$, Example 3.2

On combining Corollaries 3.5 and 3.6, we get the condition for a convex function to be fractionally convex and vice versa.

Theorem 3.9. *Let $f \in C^2([a, b])$ and $\alpha \in (0, 1)$. If $f(a) = 0 = f'(a)$, then f is convex on $[a, x]$ if and only if f is α-fractionally convex.*

The above theorem asserts that the classical convexity of f in $[a, x)$ is same as the fractional convexity of f, provided $f(a) = 0 = f'(a)$.

Example 3.3. Let $f(x) = x^n$, $n \geq 1$, $n \in \mathbb{R}^+$. Clearly, f is convex in $(0, x)$ as $f''(x) = n(n-1)\, x^{n-1} > 0$, $\forall\, x > 0$. And,

$$ {}_0D_x^{1+\alpha} f(x) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha)}\, x^{n-\alpha-1} > 0, \ \forall\, x > 0, $$

that is, f is classically as well as α-fractionally convex in $(0, x)$. One may also observe that $f(0) = 0 = f'(0)$, as given in Theorem 3.9. In particular, for $n = 2, 4$, we have plotted the $(1+\alpha)^{\text{th}}$ derivative of f in Figure 3.3.

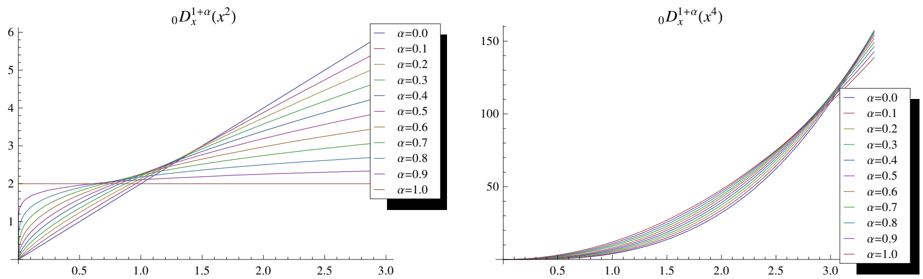

Fig. 3.3: Plot of $(1+\alpha)^{\text{th}}$-order Riemann-Liouville derivatives of $f(x) = x^n$, Example 3.3

Example 3.4. Let $f(x) = x^3 + k\,x$, where k is a positive real number. Clearly, f is classically convex (strictly) for $x > 0$. One may observe that $f(0) = 0$, and $f'(0) = 1 > 0$, thus f is α-fractionally convex for all $\alpha \in (0,1)$ (Corollary 3.5). It can also be validated by computing

$$_0D_x^{\alpha+1} f(x) = 6\frac{x^{2-\alpha}}{\Gamma(3-\alpha)} + k\frac{x^{-\alpha}}{\Gamma(1-\alpha)} > 0, \;\; \forall\, x > 0.$$

In Figure 3.4, the graphical representation is given for $(1+\alpha)^{\text{th}}$-order derivative of $f = x^3 + kx$ by taking different values of k.

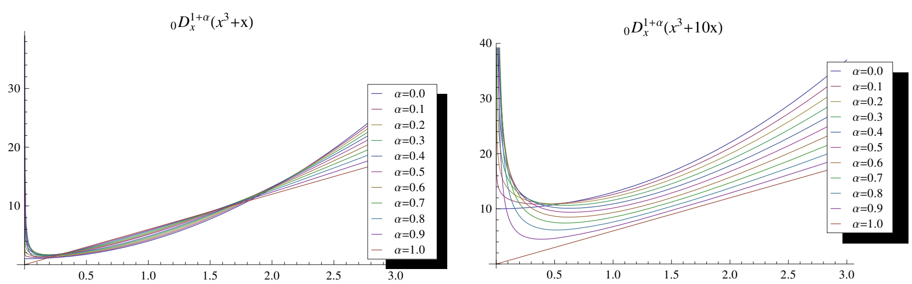

Fig. 3.4: Plot of $(1+\alpha)^{\text{th}}$-order Riemann-Liouville derivatives of $f(x) = x^3 + kx$, for $k = 1, 10$ in Example 3.4

All the above examples deal with convex and fractionally convex function over the same interval $(0, x)$. But, this is not the case in general. Next example discusses the fractional sense convexity of a function in $(0, x)$, which is classically convex in $(1, x)$.

EXAMPLE 3.5. Let $f(x) = \frac{3}{4}x^2 - 2x^{-\frac{1}{2}},\ x > 0$, classically convex for $x \geq 1$ as $f''(x) = \frac{3}{2}(1 - x^{-\frac{5}{2}})$. And,

$$_0D_x^{1+\alpha} f(x) = \frac{3}{2} \frac{x^{1-\alpha}}{\Gamma(2-\alpha)} + 2\Gamma\left(\frac{1}{2}\right)\left(\frac{1}{2} + \alpha\right) \frac{x^{-(\frac{3}{2}+\alpha)}}{\Gamma(\frac{1}{2} - \alpha)}.$$

Clearly, $_0D_x^{1+\alpha} f(x) > 0$ for $\alpha \leq \frac{1}{2}$. So, we finally conclude that f is classically convex in $(1, x)$ but α-fractionally convex in $(0, x)$ for $\alpha \in (0, \frac{1}{2}]$. We suggest the reader to look at the graph of $_0D_x^{1+\alpha} f(x)$ in Figure 3.5 for different values of $\alpha \in (0, 1)$ (also for $\alpha = 0, 1$, the classical case). One may also observe the behavior of α^{th}-order fractional derivative of f for monotonicity of f.

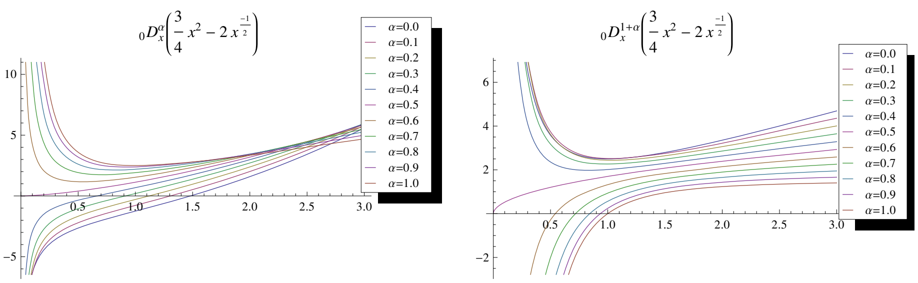

Fig. 3.5: Plot of α^{th},$(1+\alpha)^{\text{th}}$-order Riemann-Liouville derivatives of $f(x) = \frac{3}{4}x^2 - 2x^{-\frac{1}{2}}$, Example 3.5

EXAMPLE 3.6. Let $f(x) = -x$, classically convex for $x \in \mathbb{R}$. And,

$$_0D_x^{1+\alpha} f(x) = -\frac{x^{-\alpha}}{\Gamma(1-\alpha)}.$$

Clearly, $_0D_x^{1+\alpha} f(x) < 0$ for $\alpha \in (0, 1)$. Thus, we can conclude that f is not α-fractionally convex in the positive real axis. Figure 3.6 illustrates the non-negativity of $_0D_x^{1+\alpha} f(x)$ for all values of $\alpha \in (0, 1)$.

Next, we present the fractional-order analogue for the equivalent definitions of classical convex functions. Let us first give the Taylor-Riemann series for Riemann-Liouville fractional derivatives stated below.

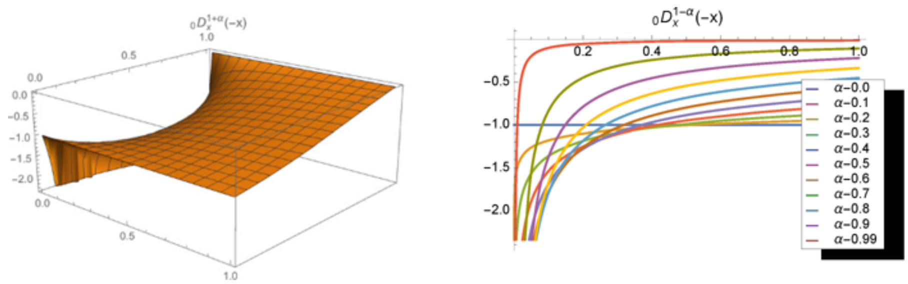

Fig. 3.6: Plot of $(1+\alpha)^{\text{th}}$-order Riemann-Liouville derivatives of $f(x) = -x$, Example 3.6

Theorem 3.10. (Taylor-Riemann series for fractional derivatives [18]) *Let $\alpha > 0$, $n \in \mathbb{Z}^+$ and $f(x) \in C^{[\alpha+n+1]}([a,b])$. Then,*

$$f(x) = \sum_{k=-n}^{n-1} \frac{{}_aD_x^{\alpha+k} f(x_0)}{\Gamma(\alpha+k+1)} (x-x_0)^{\alpha+k} + R_n(x), \ \forall a \le x_0 < x \le b, \quad (3.4)$$

where $R_n(x) = {}_aI_x^{\alpha+n}\, {}_aD_x^{\alpha+n} f(x)$ is the remainder.

Theorem 3.11. *Let $f \in C^2[a,b]$ and $\alpha \in (0,1)$. If f is α-fractionally convex, then*

$$f(x) \ge {}_aI_x^{1-\alpha} f(x_0) \frac{(x-x_0)^{-(1-\alpha)}}{\Gamma(\alpha)} + {}_aD_x^{\alpha} f(x_0) \frac{(x-x_0)^{\alpha}}{\Gamma(\alpha+1)}, \ \forall a \le x_0 < x \le b. \quad (3.5)$$

For $\alpha = 1$,

$$f(x) \ge f(x_0) + (x-x_0) f'(x_0), \ \forall\ a \le x_0 < x \le b,$$

that is, we regain the definition of classical convexity of f.

P r o o f. Let f be α-fractionally convex, $\alpha \in (0,1)$. By the Taylor-Riemann series for fractional derivatives (see Theorem 3.10), we have

$$f(x) = \sum_{k=-1}^{0} \frac{{}_aD_x^{\alpha+k} f(x_0)}{\Gamma(\alpha+k+1)} (x-x_0)^{\alpha+k} + R_1(x), \ \ \forall a \le x_0 < x \le b,$$

where $R_1(x) = {}_aI_x^{\alpha+1}\, {}_aD_x^{\alpha+1} f(x)$ is the remainder. Since f is α-fractionally convex, ${}_aD_x^{\alpha+1} f(x) \ge 0$ which implies $R_1(x) = {}_aI_x^{\alpha+1}\, {}_aD_x^{\alpha+1} f(x) \ge 0$. Thus, we arrive at

$$f(x) \geq {}_aI_x^{1-\alpha} f(x_0) \frac{(x-x_0)^{-(1-\alpha)}}{\Gamma(\alpha)} + {}_aD_x^{\alpha} f(x_0) \frac{(x-x_0)^{\alpha}}{\Gamma(\alpha+1)}, \ \forall a \leq x_0 < x \leq b.$$

□

THEOREM 3.12. *Let $f \in C^2[a,b]$ and $\alpha \in (0,1)$. If f is α-fractionally convex, for $\lambda \in [0,1]$,*

$${}_aI_x^{1-\alpha} f(\lambda x_1+(1-\lambda)x_2) \leq \lambda {}_aI_x^{1-\alpha} f(x_1)+(1-\lambda) {}_aI_x^{1-\alpha} f(x_2), \ \forall x_1, x_2 \in [a,x).$$

Clearly, for $\alpha = 1$, we arrive at the classical definition of convex functions.

P r o o f. Since f is α-fractionally convex, by using the condition stated in Theorem 3.11, we can write

$$\begin{aligned} f(x) &\geq {}_aI_x^{1-\alpha} f(x_0) \frac{(x-x_0)^{-(1-\alpha)}}{\Gamma(\alpha)} + {}_aD_x^{\alpha} f(x_0) \frac{(x-x_0)^{\alpha}}{\Gamma(\alpha+1)}, \\ \text{or,} \quad {}_aI_x^{1-\alpha} f(x) &\geq {}_aI_x^{1-\alpha} f(x_0) + {}_aD_x^{\alpha} f(x_0)\,(x-x_0). \end{aligned}$$

For $\lambda \in [0,1]$, let $x = x_1$, and $x_0 = \lambda x_1 + (1-\lambda)x_2$, we have

$${}_aI_x^{1-\alpha} f(x_1) \geq {}_aI_x^{1-\alpha} f(\lambda x_1+(1-\lambda)x_2)+(1-\lambda) {}_aD_x^{\alpha} f(\lambda x_1+(1-\lambda)x_2).(x_1-x_2). \tag{3.6}$$

Again, let $x = x_2$, and $x_0 = \lambda x_1 + (1-\lambda)x_2$

$${}_aI_x^{1-\alpha} f(x_2) \geq {}_aI_x^{1-\alpha} f(\lambda x_1+(1-\lambda)x_2)-\lambda {}_aD_x^{\alpha} f(\lambda x_1+(1-\lambda)x_2).(x_1-x_2). \tag{3.7}$$

Adding λ times of Eq. (3.6) to $(1-\lambda)$ times of Eq. (3.7), we obtain

$$\lambda\, {}_aI_x^{1-\alpha} f(x_1) + (1-\lambda)\, {}_aI_x^{1-\alpha} f(x_2) \geq {}_aI_x^{1-\alpha} f(\lambda x_1 + (1-\lambda)x_2),$$

which proves the result. □

Collectively, by virtue of Theorems 3.11-3.12, we can say that a function $f \in C^2[a,b]$ is said to be α-fractionally convex in $[a,x)$ if any one of the following conditions holds:

- ${}_aD_x^{1+\alpha} f(x) \geq 0, \ \alpha \in (0,1)$
- For all $a \leq x_0 < x \leq b$,

$$f(x) \geq {}_aI_x^{1-\alpha} f(x_0) \frac{(x-x_0)^{-(1-\alpha)}}{\Gamma(\alpha)} + {}_aD_x^{\alpha} f(x_0) \frac{(x-x_0)^{\alpha}}{\Gamma(\alpha+1)},$$

- For $x_1, x_2 \in [a,x)$, $\alpha \in (0,1)$, and $\lambda \in [0,1]$

$${}_aI_x^{1-\alpha} f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda {}_aI_x^{1-\alpha} f(x_1) + (1-\lambda) {}_aI_x^{1-\alpha} f(x_2),$$

Next, we characterize the condition for minimizing α-fractionally convex functions, which is very similar to a classical optimization problem.

THEOREM 3.13. *Let* $\min_{x\in\mathbb{R}} f(x)$ *be an unconstrained optimization problem, where* f *is* α*-fractionally convex for some* $\alpha \in (0,1)$. *Then, any point* $x^* \in [a,x)$ *satisfying* ${}_aD_x^\alpha f(x) = 0$, *is a unique minimum of* f.

P r o o f. Using Theorem 3.11 for α-fractionally convex functions,

$$f(x) \geq {}_aI_x^{1-\alpha} f(x_0)\frac{(x-x_0)^{-(1-\alpha)}}{\Gamma(\alpha)} + {}_aD_x^\alpha f(x_0)\frac{(x-x_0)^\alpha}{\Gamma(\alpha+1)},\ \forall a \leq x_0 < x \leq b.$$

In particular,

$$f(x) \geq {}_aI_x^{1-\alpha} f(x^*)\frac{(x-x^*)^{-(1-\alpha)}}{\Gamma(\alpha)} + {}_aD_x^\alpha f(x^*)\frac{(x-x^*)^\alpha}{\Gamma(\alpha+1)}.$$

Since ${}_aD_x^\alpha f(x^*) = 0$, we get

$$f(x) \geq {}_aI_x^{1-\alpha} f(x^*)\frac{(x-x^*)^{-(1-\alpha)}}{\Gamma(\alpha)}. \tag{3.8}$$

It suffices to show that $f(x^*) \leq f(x)$. Let $m(x) = f(x^*) - f(x) > 0$, then

$$\begin{aligned} {}_aI_x^{1-\alpha} m(x) &= {}_aI_x^{1-\alpha}[f(x^*) - f(x)] \geq 0, \\ \text{i.e.,}\quad {}_aI_x^{1-\alpha} f(x^*) &\geq {}_aI_x^{1-\alpha} f(x), \end{aligned}$$

which is a contradiction to Eq. (3.8). Note that ${}_aI_x^{1-\alpha} f(x^*) = {}_aI_x^{1-\alpha} f(x)$ if and only if $f(x^*) = f(x)$. Thus, $f(x^*) \leq f(x)$, and we conclude that x^* is a unique minimum of α-fractionally convex function f. □

4. Conclusions

We have investigated the monotonicity of a function in terms of Riemann-Liouville and Caputo fractional derivatives. The concept of α-fractionally convex functions has been introduced with relevant properties. Some of the equivalent results of the proposed α-fractionally convex functions are given so that the classical results are obtained for $\alpha = 1$. We characterize the condition for which a classically convex function becomes fractionally convex and vice versa. Few examples are provided to observe the cases when a function is (i) convex but not fractionally convex, (ii) fractionally convex but not convex, (iii) classically convex as well as fractionally convex. Also, we deduce the optimality condition for minimizing a fractionally convex function. A review of Karush-Kuhn-Tucker (KKT) optimality conditions, with α-fractionally convex functions, is in the future research of the author.

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