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RESEARCH PAPER

FRACTIONAL PROBLEMS WITH CRITICAL NONLINEARITIES BY A SUBLINEAR PERTURBATION

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Abstract

In this paper, the existence of two nontrivial solutions for a fractional problem with critical exponent, depending on real parameters, is established. The variational approach is used based on a local minimum theorem due to G. Bonanno. In addition, a numerical estimate on the real parameters is provided, for which the two solutions are obtained.

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Key Words and Phrases: critical growth; fractional Laplacian problem; variational methods; local minimum

1. Introduction and main result

In this paper we consider the fractional problem

$$\begin{cases} (-\Delta)^s u = \lambda(|u|^{2_s^*-2}u + \mu|u|^{q-2}u) & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \qquad (P_{\lambda,\mu})$$

where Ω is a non-empty bounded open subset of the Euclidean space $(\mathbb{R}^N, |\cdot|)$, $N > 2s$, with Lipshcitz boundary $\partial\Omega$, $0 < s < 1$, $2_s^* = \frac{2N}{N-2s}$, $1 < q < 2$, λ, μ are positive parameters. Servadei and Valdinoci [21] study a fractional problems with a critical growth, which presents several difficulties. Indeed, the Palais-Smale condition, as well as the weak lower semi-continuity of the associated functional may fail because the Sobolev embedding is not

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compact. To be precise, consider the problem

$$\begin{cases} (-\Delta)^s u = |u|^{2_s^*-2}u + g(u), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{D}$$

where Ω is a non-empty bounded open subset of the Euclidean space $(\mathbb{R}^N, |\cdot|)$, $N \geq 2s$, with boundary of class C^1, and $g : \mathbb{R} \to \mathbb{R}$ is a non-zero continuous lower-order perturbation of $|u|^{2_s^*-2}u$. $0 \leq g(u) \leq \mu u^t$ for all $u \in \mathbb{R}$, for some $\mu > 0$ and $0 < t < 2_s^*$. A typical example for g is that $g(u) = \mu|u|^t$. Servadei and Valdinoci [21] study the problem (D) when $g \equiv 0$ following the well-known nonexistence result [10]. In particular, they established that, when $g(u) = \mu|u|^t$, problem (D) admits a solution for suitable values of μ, provided that $1 \leq t < 2_s^*$. Hence, a lower-order perturbation, which is linear or super-linear at zero, can reverse the situation highlighted by Pohozaev. For other result of this type of problem, we refer the reader to [7, 19, 15, 13, 14, 22] and references therein.

Subsequently, Barrios et al. [3] study a fractional equation with critical growth and a sub-linear perturbation following the idea of García-Azorero and Peral [11]. They proved that for the problem (D) with $0 < t < 1$, $g(u) = \mu u^t$ there is $\Lambda > 0$ such that for each $\mu \in]0, \Lambda[$ the problem has at least two weak solutions. Moreover, they also proved that if $\mu > \Lambda$, the previous problem admits no solution (see [3, Theorem 1.1]) and if $\mu = \Lambda$ the previous problem admits at least one solution.

In this paper, we investigate fractional problems with critical exponent. In this case, the Palais-Smale condition and the weak lower semi-continuity of the associated functional may fail and direct method theorems cannot be used to obtain nontrivial solutions. Our approach is due to Bonanno [4, 5] to ensures the existence of one positive solution. Then, as a consequence, the existence of two positive solutions are obtained. Firstly, we give the framework of the problem, and we establish Lemma 3.1 which is fundamental in the proof of Theorem 1.1.

The nonlocal operator $(-\Delta)^s$ is defined as follows:

$$(-\Delta)^s u(x) := C(N,s) \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy, \qquad x \in \mathbb{R}^N,$$

where $B_\varepsilon(x)$ is the ball centered at $x \in \mathbb{R}^N$ with radius ε and $C(N,s)$ is the following (positive) normalization constant:

$$C(N,s) := \left(\int_{\mathbb{R}^N} \frac{1 - \cos(\zeta_1)}{|\zeta|^{n+2s}} d\zeta \right),$$

with $\zeta = (\zeta_1, \zeta')$, $\zeta' \in \mathbb{R}^{N-1}$, see Section 2 of [17].

Denote by $H^s(\mathbb{R}^N)$ the usual fractional Sobolev space endowed with the so-called *Gagliardo norm*

$$\|g\|_{H^s(\mathbb{R}^N)} = \|g\|_{L^2(\mathbb{R}^N)} + \left(\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|g(x)-g(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\right)^{1/2}, \qquad (1.1)$$

while $X_0^s(\Omega)$ is the function space defined as

$$X_0^s(\Omega) = \left\{u \in H^s(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\right\}. \qquad (1.2)$$

We refer to [20] for a general definition of $X_0^s(\Omega)$ and its properties.

We can consider the following norm

$$\|v\| = \left(\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\right)^{1/2}.$$

We also recall that $(X_0^s(\Omega), \|\cdot\|)$ is a Hilbert space, with scalar product

$$(u,v) = \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u(x)-u(y))\,(v(x)-v(y))}{|x-y|^{n+2s}}\,dx\,dy\,. \qquad (1.3)$$

See for instance [20, Lemma 7].

Observe that by [9, Proposition 3.6] we have the following identity

$$\|u\| = \|(-\Delta)^{s/2}u\|_{L^2(\mathbb{R}^N)}. \qquad (1.4)$$

We say that $u \in X_0^s(\Omega)$ is a weak solution of $(P_{\lambda,\mu})$ if for every $\varphi \in X_0^s(\Omega)$, one has

$$\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}dxdy = \lambda\mu\int_\Omega u^{q-2}u\varphi\,dx + \lambda\int_\Omega u^{2_s^*-2}u\varphi\,dx.$$

Let

$$\Phi(u) = \frac{||u||^2}{2}, \qquad \Psi(u) = \int_\Omega \left(\frac{1}{2_s^*}|u(x)|^{2_s^*} + \mu\frac{1}{q}|u(x)|^q\right)dx \qquad (1.5)$$

for all $u \in X_0^s(\Omega)$. Recall that, by Sobolev embedding

$$||u||_{L^t(\Omega)} \le c_t||u||, \quad u \in X_0^s(\Omega), \quad t \in [1, 2_s^*].$$

The best Sobolev constant is

$$c_{2_s^*} = 2^{-2s}\pi^{-s}\frac{\Gamma((N-2s)/2)}{\Gamma((N+2s)/2)}\left(\frac{\Gamma(N)}{\Gamma(N/2)}\right)^{2s/N}, \qquad (1.6)$$

which obtained in [8]. Due to (1.6), as a simple consequence of Hölder's inequality, it follows that

$$c_t \le \operatorname{meas}(\Omega)^{\frac{2_s^*-t}{2_s^*t}}2^{-2s}\pi^{-s}\frac{\Gamma((N-2s)/2)}{\Gamma((N+2s)/2)}\left(\frac{\Gamma(N)}{\Gamma(N/2)}\right)^{2s/N}, \qquad (1.7)$$

where "meas(Ω)" denotes the Lebesgue measure of the set Ω and that the embedding $X_0^s(\Omega) \hookrightarrow L^t(\Omega)$ is not compact if $t = 2_s^*$.

Now, fix $r > 0$ and put

$$\lambda_r^* = \frac{r}{\frac{\mu}{q} c_q^q (2r)^{q/2} + \frac{(2r)^{2_s^*/2}}{2_s^*} c_{2_s^*}^{2_s^*}}, \quad \tilde{\lambda}_r = \frac{1}{c_{2_s^*}^{2_s^*}} \left(\frac{s}{2rN}\right)^{\frac{N-2s}{2s}},$$

$$\bar{\lambda}_r = \min\{\lambda_r^*, \tilde{\lambda}_r\},$$

where $c_{2_s^*}$, c_q are given by (1.6) and (1.7).

The main result of our paper is the following theorem.

THEOREM 1.1. *Fix $q \in]1, 2[$. Then, there exists $\mu^* > 0$, where*

$$\mu^* = \left(\frac{q}{c_q^q} \frac{1}{2^{\frac{q+2}{2}}}\right) \left(\min\left\{\left(\frac{2_s^*}{2^{\frac{2_s^*+2}{2}} c_{2_s^*}^{2_s^*}}\right)^{\frac{2}{2_s^*-2}} ; \frac{s}{3N} \left(\frac{1}{c_{2_s^*}^{2_s^*}}\right)^{\frac{N-2s}{2s}}\right\}\right)^{\frac{2-q}{2}}$$

and c_q, $c_{2_s^*}$ are given by (1.7) and (1.6), such that for each $\lambda \in]0, \overline{\lambda}_r[$ and $\mu \in]0, \mu^*[$ problem $(P_{\lambda,\mu})$ admits at least one positive weak solution. Let $\lambda = 1$ and u_μ be the positive weak solution. Then,

$$||u_\mu|| < \left(\frac{2_s^*}{c_{2_s^*}^{2_s^*}}\right)^{\frac{1}{2_s^*-2}},$$

and the mapping

$$\mu \mapsto \frac{1}{2}\|u_\mu\|^2 - \frac{1}{2_s^*}\int_\Omega |u_\mu|^{2_s^*} dx - \frac{\mu}{q}\int_\Omega |u_\mu|^q dx$$

is negative and strictly decreasing in $]0, \mu^*[$.

The proof of Theorem 1.1 was obtained by the variational method, that is, via a local minimum result Theorem 2.1. We also observe that, [3, Theorem 2.1] establishes, in particular, the existence of $\Lambda > 0$ such that problem $(P_{\lambda,\mu})$ admits a solution for each $\mu \in]0, \Lambda]$ and no solution for $\mu > \Lambda$. However, no estimate of Λ was pointed out in [3].

Finally, we obtain the following existence result of two solutions, where an estimate of parameters is also pointed out.

THEOREM 1.2. *Fix $q \in]1, 2[$. Then there exists $\mu^* > 0$, where*

$$\mu^* = \left(\frac{q}{c_q^q} \frac{1}{2^{\frac{q+2}{2}}}\right) \left(\min\left\{\left(\frac{2_s^*}{2^{\frac{2_s^*+2}{2}} c_{2_s^*}^{2_s^*}}\right)^{\frac{2}{2_s^*-2}} ; \frac{1}{3N} \left(\frac{1}{c_{2_s^*}^{2_s^*}}\right)^{\frac{N-2}{2}}\right\}\right)^{\frac{2-q}{2}}$$

and c_q, c_2 are given by (1.7) *and* (1.6), *such that for each $\mu \in]0, \mu^*[$ problem*

$$\begin{cases} (-\Delta)^s u = |u|^{2_s^*-2}u + \mu|u|^{q-2}u, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \qquad (P_\mu)$$

admits at least two positive solutions u_μ and w_μ such that $\|u_\mu\| < \left(\frac{2_s^*}{c_{2_s^*}^{2_s^*}}\right)^{\frac{1}{2_s^*-2}}$ *and $w_\mu > u_\mu$.*

We observe that the solution obtained in Theorem 1.1 is a local minimum for considered functional. So, to obtain the second solution is enough to apply the mountain pass theorem arguing as in part of the proof of [3, Theorem 1.1].

Example 1.1. Fix $N = 3$, $s = 1/2$, and let $\Omega = \{x \in \mathbb{R}^3 : |x| \leq 1\}$. Then, the problem

$$\begin{cases} (-\Delta)^{1/2} u = |u|^3 + \mu|u|^{1/2}, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

admits at least two solutions with $\mu \in]0, \frac{\pi^{19/6}}{6^{4/3}}[$. Actually, we have $2_s^* = 3$, $q = 1/2$, $c_{2_s^*} = \left(\frac{1}{2\pi^2}\right)^{1/3}$, $c_q = \left(\frac{1}{6\pi}\right)^{1/3}$, $\frac{s}{3N}\left(\frac{1}{c_{2_s^*}^{2_s^*}}\right)^{\frac{N-2s}{2s}} = \frac{2\pi^4}{9}$, $\left(\frac{2_s^*}{2^{\frac{2_s^*+2}{2}} c_{2_s^*}^{2_s^*}}\right)^{\frac{2}{2_s^*-2}} = \frac{9\pi^4}{8}$. So this result is obtained by Theorem 1.2.

2. Preliminaries

We present some definitions on differentiability of functionals and refer the reader to [4], Section 2. Let $(X, \cdot)$ be a real Banach space. We denote the dual space of X by X^*, while $\langle \cdot, \cdot \rangle$ stands for the duality pairing between X^* and X. A functional $I : X \to \mathbb{R}$ is called Gâteaux differentiable at $u \in X$ if there is $\varphi \in X^*$ (denoted by $I'(u)$) such that

$$\lim_{t \to 0^+} \frac{I(u+tv) - I(u)}{t} = \langle I'(u), v \rangle, \quad \forall v \in X.$$

It is called continuously Gâteaux differentiable if it is Gâteaux differentiable for any $u \in X$ and the functional $u \mapsto I(u)$ is a continuous map from X to its dual X^*.

Now, let Φ, $\Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals and put

$$I = \Phi - \Psi.$$

Fix r_1, $r_2 \in [-\infty, +\infty]$, with $r_1 < r_2$, and we say that the functional I verify the Palais-Smale condition cut off lower at r_1 and upper at r_2 (in short $(PS)_{[r_1]}^{[r_2]}$-condition) if any sequence $\{u_n\}$ such that

(1) $\{I(u_n)\}$ is bounded,
(2) $\lim_{n\to+\infty} ||I'(u_n)||_{X^*} = 0$,
(3) $r_1 < \Phi(u_n) < r_2 \ \forall n \in \mathbb{N}$,

has a convergent subsequence.

When we fix $r_2 = -\infty$, that is, $\Phi(u_n) < r_2 \ \forall n \in \mathbb{N}$, we denote this type of Palais-Smale condition with $(PS)^{[r_2]}$. When, in addition, $r_2 = +\infty$, it is the classical Palais-Smale condition.

Now, we recall the following local minimum theorem.

THEOREM 2.1 ([5], Theorem 3.3). *Let X be a real Banach space and let Φ, $\Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. Assume that there are $r \in \mathbb{R}$ and $\tilde{u} \in X$, with $0 < \Phi(\tilde{u}) < r$, such that*

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})} \tag{2.1}$$

and, for each $\lambda \in \left]\frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}\right[$, the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies $(PS)^{[r]}$-condition.

Then, for each $\lambda \in \left]\frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}\right[$, there is $u_\lambda \in \Phi^{-1}(]0, r[)$ (hence, $u_\lambda \neq 0$) such that $I_\lambda(u_\lambda) \leq I_\lambda(\tilde{u})$ for all $u \in \Phi^{-1}(]0, r[)$ and $I'(u_\lambda) = 0$.

The following computations are useful in proving some estimates on the norm of some truncated functions. Precisely, fix an element $x_0 \in \Omega$ (where $\Omega \subset \mathbb{R}^N$ is of class C^1, and choose $\tau > 0$ in such a way that

$$B(x_0, \tau) := \{x \in \mathbb{R}^N : |x - x_0| \leq \tau\} \subset \Omega, \tag{2.2}$$

Now let $\sigma \in (0, 1)$ and $t_0 \in \mathbb{R}$, and define $u_{t_0}^\sigma \in H^s(\mathbb{R}^N)$ as follows:

$$u_{t_0}^\sigma := \begin{cases} 0 & \text{if } x \in \mathbb{R}^N \setminus B(x_0, \tau) \\ \frac{t_0}{(1-\sigma)\tau}(\tau - |x - x_0|) & \text{if } x \in B(x_0, \tau) \setminus B(x_0, \sigma\tau) \\ t_0 & \text{if } x \in B(x_0, \sigma\tau), \end{cases} \tag{2.3}$$

where $B(x_0, r)$ denotes the N-dimensional ball with center $x_0 \in \Omega$ and radius $r > 0$. Set

$$\nu_0 := 1 + \frac{1}{\tilde{\lambda}_1}, \tag{2.4}$$

where

$$\tilde{\lambda}_1 := \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\|\nabla u\|^2_{L^2(\Omega)}}{\|u\|^2_{L^2(\Omega)}}.$$

The following result holds:

Proposition 2.1 ([16], Proposition 1.16). *Let σ, $s \in (0,1)$, $t_0 \in \mathbb{R}$, and τ be such that (2.2) is verified. Let $u^\sigma_{t_0}$ be the function given in (2.3), S_{N-2} be the Lebesgue measure of the unit sphere in $\mathbb{R}^{N-1}$, and $\Gamma(t) := \int_0^{+\infty} z^{t-1}e^{-z}dz$ $t > 0$, be the usual Gamma function. Then $u^\sigma_{t_0} \in H^s(\mathbb{R}^N)$, and one has*

$$\left(\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u^\sigma_{t_0}(x) - u^\sigma_{t_0}(y)|^2}{|x-y|^{N+2s}} dxdy\right)^{1/2} < \frac{|t_0|}{(1-\sigma)}\sqrt{\frac{\pi^{N/2}\tau^{N-2}(1-\sigma^N)}{\Gamma(1+\frac{N}{2})}\kappa_1\kappa_2},$$

where

$$\kappa_1 := \begin{cases} 2\nu_0 & \text{if } N = 1 \\ \left(\pi + \frac{4}{1+2s}\right)\nu_0 & \text{if } N = 2 \\ S_{n-2}\left(\frac{\pi}{2} + \frac{2}{1+2s}\right)\nu_0 & \text{if } N \geq 3 \end{cases} \quad \text{and } \kappa_2 := \frac{1}{2(1-s)} + \frac{2}{s}$$

with ν_0 given in (2.4).

3. Proof of the main results

Firstly, we establish the following result.

Lemma 3.1. *Let Φ and Ψ be the functional defined in (1.5) and fix $r > 0$. Then, for each $\lambda \in]0, \bar{\lambda}_r[$ the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies the $(PS)^{[r]}$-condition.*

P r o o f. Fix λ as in the conclusion and let $\{u_n\} \subseteq X_0^s(\Omega)$ be a sequence such that

(1) $\{I_\lambda(u_n)\}$ is bounded,
(2) $\lim_{n\to+\infty} ||I'_\lambda(u_n)||_{X_0^s(\Omega)^*} = 0$,
(3) $\Phi(u_n) < r$ $\forall n \in \mathbb{N}$.

In particular, from $\Phi(u_n) < r$ for all $n \in \mathbb{N}$ we obtain that $\{u_n\}$ is bounded in $X_0^s(\Omega)$. So, going to a subsequence if necessary. We can assume

$$u_n \rightharpoonup u_0 \text{ in } X_0^s(\Omega),$$
$$u_n \to u_0 \text{ in } L^t(\Omega),$$
$$u_n \to u_0 \text{ a.e. on } \Omega,$$

where $1 < t < 2_s^*$ and, taking (1) into account, $\lim_{n\to\infty} I_\lambda(u_n) = c$. Moreover, $\{u_n\}$ is bounded in $L^{2_s^*}(\Omega)$ by Sobolev embedding.

First step. We prove that u_0 is a weak solution of problem $(P_{\lambda,\mu})$.

Since $\{u_n\}$ is bounded in $L^{2_s^*}(\Omega)$, it follows that $\{u_n^{2_s^*-1}\}$ is bounded in $L^{\frac{2_s^*}{2_s^*-1}}(\Omega)$. Indeed, one has

$$\int_\Omega |u_n^{2_s^*-1}|^{\frac{2_s^*}{2_s^*-1}} dx \le \int_\Omega |u_n|^{2_s^*} dx.$$

Therefore, it follows that

$$u_n^{2_s^*-1} \rightharpoonup u_0^{2_s^*-1} \text{ in } L^{\frac{2_s^*}{2_s^*-1}}.$$

In fact, since $u_n \to u_0$ a.e. $x \in \Omega$, we obtain $u_n^{2_s^*-1} \to u_0^{2_s^*-1}$ a.e. $x \in \Omega$, and that, together with the boundedness of $\{u_n^{2_s^*-1}\}$ in $L^{\frac{2_s^*}{2_s^*-1}}$, ensures the weak convergence of $u_n^{2_s^*-1}$ to $u_0^{2_s^*-1}$ in $L^{\frac{2_s^*}{2_s^*-1}}$(see Willem [6, Remark (iii)]). Moreover, since $u_n \to u_0$ in $L^q(\Omega)$, define the composition operator $Au = u^{q-1}$ from L^q to $L^{q/(q-1)}$, one has that

$$u_n^{q-1} \to u_0^{q-1} \text{ in } L^{\frac{q}{q-1}}(\Omega).$$

So, in particular,

$$u_n^{q-1} \rightharpoonup u_0^{q-1} \text{ in } L^{\frac{q}{q-1}}(\Omega).$$

Due to what was seen before, that is,

$$u_n \rightharpoonup u_0 \text{ in } X_0^s(\Omega),$$
$$u_n^{2_s^*-1} \rightharpoonup u_0^{2_s^*-1} \text{ in } L^{\frac{2_s^*}{2_s^*-1}},$$
$$u_n^{q-1} \rightharpoonup u_0^{q-1} \text{ in } L^{\frac{q}{q-1}},$$

one has

$$\begin{aligned}
&\lim_{n\to\infty}\Bigg(\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u_n(x)-u_n(y))(v(x)-v(y))}{|x-y|^{n+2s}} dxdy \\
&\quad -\lambda\int_\Omega u_n(x)^{2_s^*-1}v(x)dx - \lambda\mu\int_\Omega u_n(x)^{q-1}v(x)dx\Bigg) \\
&= \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u_0(x)-u_0(y))(v(x)-v(y))}{|x-y|^{n+2s}} dxdy - \lambda\int_\Omega u_0(x)^{2_s^*-1}v(x)dx \\
&\quad - \lambda\mu\int_\Omega u_0(x)^{q-1}v(x)dx
\end{aligned}$$

for all $v \in X_0^s(\Omega)$. Therefore, owing to (2) we obtain that

$$0 = \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u_0(x) - u_0(y))(v(x) - v(y))}{|x - y|^{n+2s}} dxdy - \lambda \int_\Omega u_0^{2_s^*-1} v(x)dx$$
$$-\lambda\mu \int_\Omega u_0^{q-1} v(x)dx$$

for all $v \in X_0^s(\Omega)$, that is, u_0 is a weak solution of $(P_{\lambda,\mu})$.

Second step. We prove that

$$I_\lambda(u_0) > -r. \tag{3.1}$$

In fact, by Sobolev embeddings

$$\begin{aligned}\Psi(u) &= \int_\Omega \left(\frac{1}{2_s^*} |u(x)|^{2_s^*} + \mu \frac{1}{q} |u(x)|^q \right) dx \\ &= \frac{\mu}{q} \|u\|^q_{L^q(\Omega)} + \frac{1}{2_s^*} ||u||^{2_s^*}_{L^{2_s^*}(\Omega)} \\ &\leq \frac{\mu}{q} c_q^q ||u||^q + \frac{1}{2_s^*} c_{2_s^*}^{2_s^*} ||u||^{2_s^*}\end{aligned}$$

and

$$\Psi(u) \leq \frac{\mu}{q} c_q^q ||u||^q + \frac{1}{2_s^*} c_{2_s^*}^{2_s^*} ||u||^{2_s^*}, \quad \forall u \in X_0^s(\Omega).$$

Therefore, for all $u \in X_0^s(\Omega)$ such that $||u|| \leq \sqrt{2r}$ one has

$$\begin{aligned}I_\lambda(u) = \Phi(u) - \lambda\Psi(u) &\geq \frac{||u||^2}{2} - \lambda \left(\frac{\mu}{q} c_q^q ||u||^q + \frac{1}{2_s^*} c_{2_s^*}^{2_s^*} ||u||^{2_s^*} \right) \\ &\geq -\lambda \left(\frac{\mu}{q} c_q^q (2r)^{q/2} + \frac{1}{2_s^*} c_{2_s^*}^{2_s^*} (2r)^{2_s^*/2} \right) = -\lambda \frac{r}{\lambda_r^*} > -r.\end{aligned}$$

So, taking into account (3) and that Φ is sequentially weakly lower-semicontinuous, we have

$$||u_0|| \leq \liminf_{n\to\infty} ||u_n|| \leq \sqrt{2r}$$

and, hence,

$$I_\lambda(u_0) > -r.$$

Third step. Put $v_n = u_n - u_0$. We point that one has

$$c = \Phi(u_0) - \lambda\Psi(u_0) + \lim_{n\to\infty} \left(\frac{1}{2} ||v_n||^2 - \lambda \int_\Omega \frac{1}{2_s^*} |v_n|^{2_s^*} dx \right). \tag{3.2}$$

In fact, one has

$$||u_n||^2 = ||v_n + u_0||^2 = ||v_n||^2 + ||u_0||^2 + 2(v_n, u_0),$$

so, it follows that

$$||u_n||^2 = ||v_n||^2 + ||u_0||^2 + o(1).$$

Moreover, the Brezis-Lieb Lemma (see [6, Theorem 1]), leads to

$$\int_\Omega |u_n|^{2_s^*}dx = \int_\Omega |v_n|^{2_s^*}dx + \int_\Omega |u_0|^{2_s^*}dx + o(1).$$

Since $u_n \to u_0$ in $L^q(\Omega)$, one has

$$\int_\Omega |u_n|^q dx = \int_\Omega |u_0|^q dx + o(1).$$

Hence, by

$$c = \lim_{n\to\infty} (\Phi(u_n) - \lambda\Psi(u_n)),$$

one has

$$\begin{aligned} c &= \Phi(u_n) - \lambda\Psi(u_n) + o(1) \\ &= \frac{1}{2}||u_n||^2 - \lambda\frac{1}{2_s^*}\int_\Omega |u_n|^{2_s^*}dx - \lambda\mu\frac{1}{q}\int_\Omega |u_n|^q dx + o(1) \\ &= \frac{1}{2}||v_n||^2 + \frac{1}{2}||u_0||^2 - \lambda\frac{1}{2_s^*}\int_\Omega |v_n|^{2_s^*}dx - \lambda\frac{1}{2_s^*}\int_\Omega |u_0|^{2_s^*}dx \\ &\quad - \lambda\mu\frac{1}{q}\int_\Omega |u_0|^q dx + o(1) \\ &= \Phi(u_0) - \lambda\Psi(u_0) + \frac{1}{2}||v_n||^2 - \lambda\frac{1}{2_s^*}\int_\Omega |v_n|^{2_s^*}dx + o(1). \end{aligned}$$

Hence, (3.2) is proved.

Fourth step. We prove the following

$$\lim_{n\to\infty}\left(||v_n||^2 - \lambda\int_\Omega |v_n|^{2_s^*}dx\right) = 0. \tag{3.3}$$

From (2) we have $\lim_{n\to\infty}\langle I'(u_n), u_n\rangle = 0$. Then,

$$\|u_n\|^2 - \lambda\int_\Omega |u_n|^{2_s^*-1}u_n dx - \lambda\mu\int_\Omega |u_n|^{q-1}u_n dx = o(1).$$

Therefore, as seen in the proof of (3.2). Taking into account that

$$\int_\Omega |u_n|^{q-1}u_n dx = \int_\Omega |u_0|^{q-1}u_0 dx + o(1),$$

owing to the fact that $|u_n|^{q-1} \to |u_0|^{q-1}$ in $L^{\frac{q}{q-1}}(\Omega)$ (by the first step) and $u_n \to u_0$ in $L^q(\Omega)$, one has

$$||v_n||^2 + ||u_0||^2 - \lambda\int_\Omega |v_n|^{2_s^*}dx - \lambda\int_\Omega |u_0|^{2_s^*}dx - \lambda\mu\int_\Omega |u_0|^q dx = o(1),$$

that is,

$$||v_n||^2 - \lambda\int_\Omega |v_n|^{2_s^*}dx = -||u_0||^2 + \lambda\int_\Omega |u_0|^{2_s^*}dx + \lambda\mu\int_\Omega |u_0|^q dx + o(1).$$

Since u_0 is a weak solution of $(P_{\lambda,\mu})$, one has

$$||u_0||^2 - \lambda \int_\Omega |u_0|^{2_s^*} dx - \lambda\mu \int_\Omega |u_0|^q dx = 0.$$

Therefore,

$$||v_n||^2 - \lambda \int_\Omega |v_n|^{2_s^*} dx = o(1),$$

and (3.3) is proved.

Conclusion. Finally, we observe that $||v_n||^2$ is bounded in $\mathbb{R}$ since $\{u_n\}$ is bounded in $X_0^s(\Omega)$. Thus, there is a subsequence, called again $||v_n||^2$, which converges to $b \in \mathbb{R}$. Hence,

$$\lim_{n\to\infty} ||v_n||^2 = b.$$

If $b = 0$ we have proved the lemma. In fact, we have that $\lim_{n\to\infty} ||u_n - u_0|| = 0$, that is, u_n strongly converges to u_0 in $X_0^s(\Omega)$.

Assume that $b \neq 0$, arguing by contradiction. From (3.3) we obtain

$$\lim_{n\to\infty} \lambda \int_\Omega |v_n|^{2_s^*} dx = b.$$

Now, taking into account that

$$\int_\Omega |v_n|^{2_s^*} dx \leq c_{2_s^*}^{2_s^*} ||v_n||^{2_s^*},$$

and passing to the limit, one has $\frac{b}{\lambda} \leq c_{2_s^*}^{2_s^*} b^{2_s^*/2}$ and then, since $b \neq 0$, one has

$$b \geq \left(\frac{1}{\lambda}\right)^{\frac{N-2s}{2s}} \left(\frac{1}{c_{2_s^*}}\right)^{N/s}.$$

Now, taking (3.1) in to account, from (3.2) we have

$$c = \Phi(u_0) - \lambda\Psi(u_0) + \frac{1}{2}b - \frac{1}{2_s^*}b > -r + \left(\frac{1}{2} - \frac{1}{2_s^*}\right) b = -r + \frac{bs}{N},$$

that is,

$$c > -r + \frac{bs}{N}.$$

On the other hand, since $\frac{1}{2_s^*}|\xi|^{2_s^*} + \mu\frac{1}{q}|\xi|^q \geq 0$ for all $\xi \in \mathbb{R}$, one has

$$\Phi(u_n) - \lambda\Psi(u_n) < r$$

for all $n \in \mathbb{N}$. Hence, we have

$$c \leq r.$$

Thus,

$$-r + \frac{sb}{N} < c \leq r.$$

It follows that $\frac{bs}{N} < 2r$, that is,

$$b < \frac{2rN}{s}.$$

Therefore, one has

$$\left(\frac{1}{\lambda}\right)^{\frac{N-2s}{2s}}\left(\frac{1}{c_{2_s^*}}\right)^{N/s}\leq b<\frac{2rN}{s},$$

so, it follows that $\frac{1}{\lambda}<(\frac{2rN}{s}c_{2_s^*}^{N/s})^{\frac{2s}{N-2s}}$. Hence, one has

$$\lambda>\frac{1}{c_{2_s^*}^{2_s^*}}\left(\frac{s}{2rN}\right)^{\frac{N-2s}{2s}}=\tilde{\lambda}_r,$$

and this is a contradiction. □

Now, we give the proof of Theorem 1.1.

Proof of Theorem 1.1. Let

$$r=\min\left\{\left(\frac{2_s^*}{2^{\frac{2_s^*+2}{2}}c_{2_s^*}^{2_s^*}}\right)^{\frac{2}{2_s^*-2}};\frac{1}{3N}\left(\frac{1}{c_{2_s^*}^{2_s^*}}\right)^{\frac{N-2}{2}}\right\}$$

and

$$\mu^*=\left(\frac{q}{c_q^q}\frac{1}{2^{\frac{q+2}{2}}}\right)r^{\frac{2-q}{2}}.$$

Fix $0<\mu<\mu^*$, and one has $\bar{\lambda}_r>1$. Indeed,

$$\begin{aligned}\tilde{\lambda}_r&=\frac{1}{c_{2_s^*}^{2_s^*}}\left(\frac{s}{2rN}\right)^{\frac{N-2s}{2s}}\\&\geq\frac{1}{c_{2_s^*}^{2_s^*}[(2N)/s]^{\frac{2s}{N-2s}}\left[\frac{s}{3N}\left(\frac{1}{c_{2_s^*}^{2_s^*}}\right)^{\frac{N-2s}{2s}}\right]^{\frac{2s}{N-2s}}}=\left(\frac{3}{2}\right)^{\frac{2s}{N-2s}}>1\end{aligned}$$

and

$$\begin{aligned}\lambda_r^*&=\frac{1}{\frac{\mu}{q}c_q^q2^{q/2}r^{\frac{q-2}{2}}+\frac{2^{2_s^*/2}}{2_s^*}c_{2_s^*}^{2_s^*}r^{\frac{2_s^*-2}{2}}}\\&\geq\frac{1}{\frac{\mu}{q}c_q^q2^{q/2}r^{\frac{q-2}{2}}+\frac{2^{2_s^*/2}}{2_s^*}c_{2_s^*}^{2_s^*}\left[\left(\frac{2_s^*}{2^{\frac{2_s^*+2}{2}}c_{2_s^*}^{2_s^*}}\right)^{\frac{2}{2_s^*-2}}\right]^{\frac{2_s^*-2}{2}}}\\&>\frac{1}{\frac{\mu^*}{q}c_q^q2^{q/2}r^{\frac{q-2}{2}}+\frac{1}{2}}=1.\end{aligned}$$

From Lemma 3.1, the functional $I_\lambda=\Phi-\lambda\Psi$ satisfies the $(PS)^{[r]}$-condition for all $\lambda\in]0.\bar{\lambda}_r[$.

Now, fix $\lambda<\bar{\lambda}_r=\min\{\lambda_r^*,\tilde{\lambda}_r\}$. We claim that there is $v_0\in X_0^s(\Omega)$, with $0<\Phi(v_0)<r$, such that

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u)}{r} < \frac{\Psi(v_0)}{\Phi(v_0)}. \tag{3.4}$$

To this end, taking into account that $||u||_{L^t(\Omega)} \le c_t||u||$, $u \in X_0^s(\Omega)$, one has

$$\begin{aligned}\frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u)}{r} &\le \frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\left(\frac{\mu}{q}\|u\|^q_{L^q(\Omega)} + \frac{1}{2_s^*}\|u\|^{2_s^*}_{L^{2_s^*}(\Omega)}\right)}{r}\\ &\le \frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\left(\frac{\mu}{q}c_q^q\|u\|^q + \frac{1}{2_s^*}c_{2_s^*}^{2_s^*}\|u\|^{2_s^*}\right)}{r}\\ &\le \frac{\frac{\mu}{q}c_q^q(2r)^{q/2} + \frac{1}{2_s^*}c_{2_s^*}^{2_s^*}(2r)^{2_s^*/2}}{r} = \frac{1}{\lambda_r^*}.\end{aligned}$$

Hence, one has

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u)}{r} \le \frac{1}{\lambda_r^*} < \frac{1}{\lambda}.$$

Now, put a function v_δ^σ as in (2.3). By Proposition 1.1, one has

$$\Phi(v_\delta^\sigma) < \frac{1}{2}\frac{\delta^2}{(1-\sigma)^2}\frac{\pi^{N/2}\tau^{N-2}(1-\sigma^N)}{\Gamma(1+\frac{N}{2})}\kappa_1\kappa_2,$$

where Γ is the Gamma function. Moreover, one has

$$\begin{aligned}\Psi(v_\delta^\sigma) &= \int_\Omega\left(\frac{1}{2_s^*}|v_\delta^\sigma(x)|^{2_s^*} + \mu\frac{1}{q}|v_\delta^\sigma(x)|^q\right)dx\\ &\ge \int_{B(x_0,\sigma\tau)}\left(\frac{1}{2_s^*}|\delta|^{2_s^*} + \mu\frac{1}{q}|\delta|^q\right)dx\\ &\ge \left(\frac{1}{2_s^*}|\delta|^{2_s^*} + \mu\frac{1}{q}|\delta|^q\right)\frac{\pi^{N/2}}{\Gamma(1+N/2)}(\sigma\tau)^N\end{aligned}$$

and, hence

$$\frac{\Psi(v_\delta^\sigma)}{\Phi(v_\delta^\sigma)} \ge \frac{2(\sigma\tau)^2(1-\sigma)^2}{\delta^2\tau^{N-2}(1-\sigma^N)\kappa_1\kappa_2}\left(\frac{1}{2_s^*}|\delta|^{2_s^*} + \mu\frac{1}{q}|\delta|^q\right).$$

From

$$\lim_{t\to0^+}\frac{|t|^q}{t^2} = +\infty$$

it follows that

$$\limsup_{t\to0^+}\frac{\left(\frac{1}{2_s^*}|t|^{2_s^*} + \mu\frac{1}{q}|t|^q\right)}{t^2} = +\infty.$$

So, there is a $\bar\delta > 0$ such that

$$\frac{2(\sigma\tau)^2(1-\sigma)^2}{\tau^{N-2}(1-\sigma^N)\kappa_1\kappa_2}\frac{\left(\frac{1}{2_s^*}|\bar\delta|^{2_s^*} + \mu\frac{1}{q}|\bar\delta|^q\right)}{\bar\delta^2} > \frac{1}{\lambda}$$

and $\Phi(v^\sigma_{\bar\delta}) < r$. Therefore,

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u)}{r} < \frac{1}{\lambda} < \frac{2(\sigma\tau)^2(1-\sigma)^2}{\tau^{N-2}(1-\sigma^N)\kappa_1\kappa_2}\frac{\left(\frac{1}{2^*_s}|\bar\delta|^{2^*_s} + \mu\frac{1}{q}|\bar\delta|^q\right)}{\bar\delta^2} \leq \frac{\Psi(v^\sigma_{\bar\delta})}{\Phi(v^\sigma_{\bar\delta})}$$

with $0 < \Phi(v^\sigma_{\bar\delta}) < r$. Hence, our claim is proved.

Finally, from Theorem 2.1 then functional $\Phi - \lambda\Psi$ admits a critical point $u_{\lambda,\mu}$ such that $\frac{1}{2}||u_{\lambda,\mu}\|^2 > 0$, which is a positive weak solution for problem $(P_{\lambda,\mu})$. In particular, by choosing $\lambda = 1 < \bar\lambda_r$, a positive weak solution u_μ for problem (P_μ) is obtained. Moreover, one has

$$\frac{1}{2}\|u_\mu\|^2 < r,$$

from which

$$\frac{1}{2}\|u_\mu\|^2 < \left(\frac{2^*_s}{2^{\frac{2^*_s+2}{2}}c^{2^*_s}_{2^*_s}}\right)^{\frac{2}{2^*_s-2}},$$

that is,

$$\|u_\mu\| < \left(\frac{2^*_s}{c^{2^*_s}_{2^*_s}}\right)^{\frac{1}{2^*_s-2}}.$$

Now, since u_μ is a global minimum for I_1 in $\Phi^{-1}(]0,r[)$ again from Theorem 2.1, and $v^\sigma_{\bar\delta} \in \Phi^{-1}(]0,r[)$, one has

$$I_1(u_\mu) \leq I_1(v^\sigma_{\bar\delta}).$$

Taking into account that

$$\frac{\Psi(v^\sigma_{\bar\delta})}{\Phi(v^\sigma_{\bar\delta})} > \frac{1}{\lambda} = 1,$$

one has

$$I_1(u_\mu) \leq I_1(v^\sigma_{\bar\delta}) < 0.$$

Next, fix $0 < \mu_1 < \mu_2$. One has

$$\begin{aligned} I_1(u_{\mu_1}) &= \min_{u\in\Phi^{-1}(]0,r[)}\left(\frac{1}{2}\|u\|^2 - \frac{1}{2^*_s}\int_\Omega |u|^{2^*_s}dx - \mu_1\frac{1}{q}\int_\Omega |u|^q dx\right) \\ &> \min_{u\in\Phi^{-1}(]0,r[)}\left(\frac{1}{2}\|u\|^2 - \frac{1}{2^*_s}\int_\Omega |u|^{2^*_s}dx - \mu_2\frac{1}{q}\int_\Omega |u|^q dx\right) = I_1(u_{\mu_2}) \end{aligned}$$

and the conclusion is achieved. □

Now, we want to find a second positive solution of the problem. The proof of the theorem will be done in several steps.

Fix $\mu \in]0, \mu^*[$. From Theorem 1.1 there exists a positive weak solution u_μ of (P_μ) such that u_μ is a local minimum for the functional

$$I(u) = \Phi(u) - \Psi(u) = \frac{\|u\|^2}{2} - \int_\Omega F(u(x))dx,$$

where F is the primitive of $f(t) = t^{2_s^*-1} + \mu t^{q-1}$ if $t \geq 0$ and $f(t) = 0$ if $t < 0$. Now, consider the problem

$$\begin{cases} (-\Delta)^s v = (u_\mu + v)^{2_s^*-1} - u_\mu^{2_s^*-1} + \mu(u_\mu + v)^{q-1} - \mu u_\mu^{q-1}, & \text{in } \Omega, \\ v = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{3.5}$$

Clearly, if v_μ is a positive weak solution to (3.5), then $w_\mu = u_\mu + v_\mu$ is a weak solution of (P_μ) such that $w_\mu > u_\mu > 0$. Our aim is to prove that (3.5) admits at least one positive weak solution. Consider the functional J defined as

$$J(v) = \frac{\|v\|^2}{2} - \int_\Omega L(x, v(x))dx,$$

where

$$L(x, \xi) = \int_0^\xi l(x, t)dt$$

and

$$l(x,t) = (u_\mu(x) + t)^{2_s^*-1} - [u_\mu(x)]^{2_s^*-1} + \mu(u_\mu(x) + t)^{q-1} - \mu\,[u_\mu(x)]^{q-1}$$

if $t \geq 0$ and $l(x,t) = 0$ if $t < 0$. Clearly, non-zero critical points of J are positive weak solution of (3.5). It is easy to check that the functional $J(v)$ attains its absolute minimum in X at some point $v_0 \in X$.

Now, we observe that 0 is a local minimum of J. Indeed, since u_μ is a local minimum of I, one has

$$I(u_\mu + v) - I(u_\mu) \geq 0$$

for all $v \in X_0^s(\Omega)$ such that $\|v\| < \delta$ for some $\delta > 0$. So, taking into account that

$$J(v) = \frac{1}{2}\|v^-\|^2 + I(u_\mu + v^+) - I(u_\mu) \geq 0$$

for all $v \in X_0^s(\Omega)$, where v^+ and v^- denotes the positive part and negtive part of v, respectively. Now, one has $J(v) \geq 0$ for all $v \in X_0^s(\Omega)$ such that $\|v\| < \delta$, this result following the same as [2, Lemma 3.4].

Now, we will prove the functional J admits a positive critical point v_ν for which $w_\nu = u_\nu + v_\nu$ is the second weak solution of (P_μ), this strategy following from [3].

We have the following result from [3].

LEMMA 3.2 ([3], Lemma 2.10.). *If $u = 0$ is the only critical point of J in X, then J satisfies the $(PS)_{c_1}$ condition, provided $c_1 < c_0$, where c_0 is defined as*

$$c_0 = \frac{s}{N} c_{2_s^*}^{N/2s}.$$

Here $c_{2_s^*}$ denotes the Sobolev constant defined in (1.6).

The last, which we have to show is that there exists a Palais-Smale sequence below the critical level $\frac{s}{N} c_{2_s^*}^{N/2s}$. More precisely, we have the following lemma.

LEMMA 3.3. *If $v_0 = 0$ is the unique critical point of J, then there exists a Palais-Smale sequence such that*

$$\lim_{n\to\infty} J(v_n) = c < c_0 = \frac{s}{N} c_{2_s^*}^{N/2s}.$$

Proof. We assume for simplicity that $0 \in \Omega$. Consider the best constant of the Sobolev inclusion defined in (1.6). By [21] for $\Omega = \mathbb{R}^N$ the best constant is attained by the following function,

$$V_\epsilon(x) = K_1 \left(\frac{\epsilon}{\epsilon^2 + |x|^2} \right)^{\frac{N-2s}{2}}, \quad \epsilon > 0,$$

where $K_1 = 2^{\frac{N-2s}{2}} \frac{\Gamma\left(\frac{N+2s}{2}\right)}{\Gamma\left(\frac{N-2s}{2}\right)}$ and V_ϵ satisfies the problem $(-\Delta)^s u = u^{\frac{N+2s}{N-2s}}$ in $\mathbb{R}^N$ with $N > 2s$ and

$$c_{2_s^*}^{\frac{N}{2s}} = \int_{\mathbb{R}^N} \left| (-\Delta)^{s/2} V_1 \right|^2 dx = \int_{\mathbb{R}^N} |V_1|^{2_s^*} dx,$$

$$V_\epsilon(x) = \left(\frac{\epsilon}{\epsilon^2 + |x|^2} \right)^{\frac{N-2s}{2}}, \quad \epsilon > 0.$$

The idea is to perform a truncation with a cutoff smooth function $\rho(x) \geq 0$, such that, $\rho(x) = 1$ if $|x| < R$, $\rho(x) = 0$ if $|x| > 2R$; where we take $R > 0$ in such that $\{x : |x| \leq 2R\} \subset \Omega$. More precisely, define $v_\epsilon(x) = \rho(x) V_\epsilon(x)$. For ϵ small enough, the concentration of V_ϵ will give us that

$$\sup_{t\geq 0} J(tv_\epsilon) = c_\epsilon < \frac{s}{N} c_{2_s^*}^{N/2s}, \tag{3.6}$$

which is sufficient to have the result.

Now, we have the estimates,

$$\int_\Omega \left| (-\Delta)^{s/2} v_\epsilon \right|^2 dx = \int_{\mathbb{R}^N} \left| (-\Delta)^{s/2} V_1 \right|^2 dx + O\left(\epsilon^{N-2s}\right),$$

$$\int_\Omega |v_\epsilon|^{2_s^*} dx = \int_{\mathbb{R}^N} |V_1|^{2_s^*} dx + O\left(\epsilon^N\right)$$

and for some positive k,

$$\int_\Omega |v_\epsilon|^r\,dx = \begin{cases} k\epsilon^{\frac{(N-2s)r}{2}} + O\left(\epsilon^{\frac{(N-2s)r}{2}}\right) & \text{if} \quad r < \frac{N}{N-2s}, \\ k\epsilon^{N-\frac{(N-2s)r}{2}}|\log\epsilon| + O\left(\epsilon^{N-\frac{(N-2s)r}{2}}|\log\epsilon|\right) & \text{if } r = \frac{N}{N-2s}, \\ k\epsilon^{N-\frac{(N-2s)r}{2}} + O\left(\epsilon^{N-\frac{(N-2s)r}{2}}\right) & \text{if} \quad r > \frac{N}{N-2s}. \end{cases} \tag{3.7}$$

The key for the estimate (3.6) is

$$G_\lambda(x,s) \geq \frac{1}{2_s^*}s^{2_s^*} + u_\lambda(x)s^{2_s^*-1} + Cu_\lambda(x)^{2_s^*-l}s^l, \quad l \in \left(\frac{N}{N-2s}, \frac{N+2s}{N-2s}\right),$$

which is a consequence of the following inequality:

By J. García Azorero and A. I. Peral [12, Lemma A4(4)] if $r > 2$ then for given $l \in (1, r-1)$ there exists a constant $C > -\infty$ such that

$$\inf_{t>0}\left\{\frac{(1+t)^r - \left(1 + t^r + rt + rt^{r-1}\right)}{t^l}\right\} \geq C.$$

Now, from (3.7) we have

$$\begin{aligned} J(tv_\epsilon) \leq \frac{t^2}{2}\int_\Omega |(-\Delta)^{s/2}v_\epsilon|^2 dx - \frac{t^{2_s^*}}{2_s^*}\int_\Omega |v_\epsilon|^{2_s^*}dx - m_1 t^{2_s^*-1}\int_\Omega |v_\epsilon|^{2_s^*-1}dx \\ + |C|m_1^{2_s^*-l} - t^l\int_\Omega |v_\epsilon(x)|^l\,dx. \end{aligned}$$

Here we use that $0 < m_1 = \inf_{x\in B_{2R}} u_\lambda(x)$. Then

$$\begin{aligned} J(tv_\epsilon) \leq \frac{t^2}{2}\int_{\mathbb{R}^N}\left|(-\Delta)^{s/2}V_1\right|^2 dx - \frac{t^{2_s^*}}{2_s^*}\int_{\mathbb{R}^N}|V_1|^{2_s^*}dx - m_1 t^{2_s^*-1}k\epsilon^{\frac{N-2s}{2}} \\ + O\left(\epsilon^{\frac{N-2s}{2}}\right). \end{aligned}$$

Consider the function

$$\begin{aligned} h_\epsilon(t) = \frac{t^2}{2}\int_{\mathbb{R}^N}\left|(-\Delta)^{s/2}V_1\right|^2 dx - \frac{t^{2_s^*}}{2_s^*}\int_{\mathbb{R}^N}|V_1|^{2_s^*}dx - Ct^{2_s^*-1}\epsilon^{\frac{N-2s}{2}} \\ + O\left(\epsilon^{\frac{N-2s}{2}}\right). \end{aligned}$$

When $\varepsilon = 0$, h_0 attains its maximum in $[0,1)$ at t_0 and $h_0(t_0) = \frac{s}{N}c_{2_s^*}^{N/2s}$ by the relationship between V_1 and the best Sobolev constant $c_{2_s^*}$. It is clear that $h_\varepsilon(t) < h_0(t)$; hence we conclude that

$$\max_{t>0} h_\epsilon(t) < h_0(t_0) = \frac{s}{N}c_{2_s^*}^{N/2s}.$$

To finish the proof, we need to analyze the influence of the error term. If we denote by t_ε the point where h_ε attains its maximum, it is easily seen that $0 < t_\varepsilon < t_0$ and $t_\varepsilon \to t_0$ as $\varepsilon \to 0$. Therefore, we can write $t_\varepsilon = t_0 x_\varepsilon$, where $x_\varepsilon \to 1$ as $\varepsilon \to 0$. Taking into account that $h'_\varepsilon(t_\varepsilon) = 0$, we get

$$t_0 x_\epsilon \int_{\mathbb{R}^N} \left|(-\Delta)^{s/2} V_1\right|^2 dx - t_0^{2_s^*-1} x_\epsilon^{2_s^*-1} \int_{\mathbb{R}^N} |V_1|^{2_s^*} dx$$
$$= C\left(2_s^* - 1\right) t_0^{2_s^*-2} x_\epsilon^{2_s^*-2} \epsilon^{\frac{N-2s}{2}}.$$

Using the precise value of t_0, after some computations we arrive at

$$1 - x_\epsilon^{2_s^*-2} = A x_\epsilon^{2_s^*-3} \epsilon^{\frac{N-2s}{2s}},$$

where

$$A = C\left(2_s^* - 1\right) \frac{\left(\int_{\mathbb{R}^N} \left|(-\Delta)^{s/2} V_1\right|^2 dx\right)^{\frac{-1}{2_s^*-1}}}{\left(\int_{\mathbb{R}^N} |V_1|^{2_s^*} dx\right)^{1-\frac{1}{2_s^*-2}}}.$$

By Taylor's expansion:

$$\left(1 - x_\epsilon\right)\left(2_s^* - 2\right) x_\epsilon^{2_s^*-3} + o\left(1 - x_\epsilon\right) = A x_\epsilon^{2_s^*-3} \epsilon^{\frac{N-2s}{2}}.$$

Therefore, $1 - x_\epsilon = M \epsilon^{\frac{N-2s}{2}} + o\left(\epsilon^{\frac{N-2s}{2}}\right)$, for $M = \frac{A}{2_s^*-2}$.

Finally, this identity allows us to prove that

$$h_\epsilon\left(t_\epsilon\right) = \frac{s}{N} c_{2_s^*}^{N/2s} - C t_0^{2_s^*-1} \epsilon^{\frac{N-2s}{2}} + O\left(\epsilon^{\frac{N-2s}{2}}\right),$$

and the conclusion follows. □

End of proof of Theorem 1.2. Assume that v_0 is the unique critical point of J. Consider the function $w_\epsilon = r_\epsilon v_\epsilon$, with r_ϵ large enough, such that $J(w_\epsilon) < 0$ and the mini-max value

$$c_\epsilon = \inf_{\gamma \in \mathcal{P}} \max_{t \in [0,1]} J(\gamma(t)),$$

where

$$\mathcal{P} = \left\{\gamma : [0,1] \longrightarrow X : \text{ continuous}, \gamma(0) = 0, \gamma(1) = w_\epsilon\right\}.$$

Because $v_0 = 0$ is the local minimum, then $0 \le c_\epsilon < \frac{s}{N} c_{2_s^*}^{\frac{N}{2s}}$. If $c_\epsilon > 0$ the Mountain Pass Lemma by Ambrosetti and Rabinowitz, [1], gives us a second positive critical point, in contradiction with the hypothesis. In the case $c_\epsilon = 0$, we get the same contradiction by using a result by Pucci-Serrin, [18]. This contradiction finishes the proof. □

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