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RESEARCH PAPER

WELL-POSEDNESS OF TIME-FRACTIONAL ADVECTION-DIFFUSION-REACTION EQUATIONS

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Abstract

We establish the well-posedness of an initial-boundary value problem for a general class of linear time-fractional, advection-diffusion-reaction equations, allowing space- and time-dependent coefficients as well as initial data that may have low regularity. Our analysis relies on novel energy methods in combination with a fractional Gronwall inequality and properties of fractional integrals.

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1. Introduction

The main scope of this paper is to investigate the existence and uniqueness of the weak solution of a linear, time-fractional problem of the form

$$\partial_t u - \nabla \cdot \left(\kappa \nabla \partial_t^{1-\alpha} u - \vec{F} \partial_t^{1-\alpha} u - \vec{G} u\right) + a \partial_t^{1-\alpha} u + bu = g \tag{1.1}$$

for $x \in \Omega$ and $0 < t \leq T$. The parameter α in the fractional derivative lies in the range $0 < \alpha < 1$, and the spatial domain $\Omega \subseteq \mathbb{R}^d$ ($d \geq 1$) is bounded and Lipschitz. The transport coefficients $\vec{F}$ and $\vec{G}$, the reaction coefficients a and b, as well as the source term g, are assumed to be known functions of x and t, whereas the generalized diffusivity $\kappa = \kappa(x)$ may depend only on x but is permitted to be a real, symmetric positive-definite

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matrix. In (1.1), $-\nabla \cdot (\kappa \nabla \partial_t^{1-\alpha} u)$ is the non-local diffusion term, whereas $\nabla \cdot \big(\vec{F} \partial_t^{1-\alpha} u + \vec{G} u\big)$ is the non-local/local advection term, and $a \partial_t^{1-\alpha} u + bu$ is the non-local/local reaction term.

We impose homogeneous Dirichlet boundary conditions,

$$u(x,t) = 0 \quad \text{for } x \in \partial\Omega \text{ and } 0 \le t \le T, \tag{1.2}$$

and the initial condition

$$u(x,0) = u_0(x) \quad \text{for } x \in \Omega. \tag{1.3}$$

The Riemann–Liouville fractional *derivative* [33] of order $1-\alpha$ is defined via the fractional *integral* of order α: with $\omega_\alpha(t) = t^{\alpha-1}/\Gamma(\alpha)$ we have

$$\partial_t^{1-\alpha} v(x,t) = \partial_t \mathcal{I}^\alpha v(x,t) \quad \text{where} \quad \mathcal{I}^\alpha v(x,t) = \int_0^t \omega_\alpha(t-s) v(x,s)\, ds.$$

We denote by $W_p^k(\Omega)$ the usual Sobolev space of functions whose partial derivatives of order k or less belong to $L_p(\Omega)$. The following regularity assumptions on the coefficients will be used:

$$\begin{gathered} \kappa \in L_\infty(\Omega)^{d\times d}, \qquad \vec{F}, \vec{G} \in C^2\big([0,T]; W^1_\infty(\Omega)^d\big), \\ a, b \in C^1\big([0,T]; L_\infty(\Omega)\big). \end{gathered} \tag{1.4}$$

In addition, to ensure that the spatial operator $v \mapsto -\nabla \cdot (\kappa \nabla v)$ is uniformly elliptic on Ω, we assume that the minimal eigenvalue of $\kappa(x)$ is bounded away from zero, uniformly for $x \in \Omega$.

Based on physical models of various subdiffusive transport processes, different classes of time-fractional PDEs arise as special cases of (1.1), including

- fractional Fokker–Planck equations [4, 10, 16, 30], when $\vec{G} = \mathbf{0}$, $a = b = 0$ and $g = 0$;
- fractional reaction-diffusion equations [11, 12], when $\vec{F} = \vec{G} = \mathbf{0}$;
- fractional cable equations [19], when $\vec{F} = \vec{G} = \mathbf{0}$;
- fractional advection-dispersion (or fractional convection-diffusion) equations [25], when $\vec{F} = \vec{F}(x)$, $\vec{G} = \mathbf{0}$ and $a = b = 0$.

Consider the simplest non-trivial case, when κ is the identity matrix with $\vec{F} = \vec{G} = \mathbf{0}$, $a = b = 0$ and $g = 0$, so that (1.1) reduces to the fractional subdiffusion equation: $\partial_t u - \nabla^2 \partial_t^{1-\alpha} u = 0$. Let β denote a Dirichlet eigenfunction of the Laplacian on Ω, with corresponding eigenvalue $\lambda > 0$, that is, $-\nabla^2 \beta = \lambda \beta$ in Ω with $\beta|_{\partial\Omega} = 0$. For the special choice of initial data $u_0 = \beta(x)$, the solution of the initial-boundary value problem (1.1)–(1.3) has the separable form $u(x,t) = E_\alpha(-\lambda t^\alpha)\beta(x)$, where $E_\alpha(z) = \sum_{n=0}^\infty z^n/\Gamma(1+n\alpha)$ is the Mittag–Leffler function [33]. Notice that $\partial_t^m u = O(t^{\alpha-m})$ as $t \to 0$. Moreover, we can extend the classical method of separation of variables for the heat equation to construct a series solution for arbitrary initial data $u_0 \in L_2(\Omega)$, and the regularity properties of the solution u follow from this representation [28].

Such an explicit construction is no longer possible for the solution of the general equation (1.1). Instead, we proceed by formally integrating (1.1) in time, multiplying both sides by a test function v, and applying the first Green identity over Ω to arrive at the weak formulation

$$\begin{aligned}\langle u(t), v\rangle &+ \int_0^t \langle \kappa\nabla\partial_s^{1-\alpha}u(s) - \vec{F}(s)\partial_s^{1-\alpha}u(s) - \vec{G}(s)u(s), \nabla v\rangle\, ds \\ &+ \int_0^t \langle a(s)\partial_s^{1-\alpha}u(s) + b(s)u(s), v\rangle\, ds = \langle u_0, v\rangle + \int_0^t \langle g(s), v\rangle\, ds \qquad (1.5)\end{aligned}$$

for all $v \in H_0^1(\Omega)$, where we have suppressed the dependence of the functions on x, and where $\langle \cdot,\cdot\rangle$ denotes the inner product in $L_2(\Omega)$ or $L_2(\Omega)^d$.

Numerical methods for particular cases of (1.1) were extensively studied over the last two decades, see for example [1, 18, 23, 36, 38] for finite differences, [14, 20, 31] for continuous and discontinuous finite elements, and also see [8, 13] for more references. However, due to various types of mathematical difficulties, proof of the well-posedness of the continuous problem is almost missing despite its importance, apart from the case [28] when $\vec{F} = \vec{G} = \mathbf{0}$ and $a = b = 0$. In this paper, we address these fundamental questions. A related paper [21] treats the fractional Fokker–Planck equation (that is, the case $\vec{G} = \mathbf{0}$ and $a = b = 0$) via a different, and somewhat simpler, chain of estimates that, for instance, does not use the quadratic operator $\mathcal{Q}_1^{\mu}$ defined below in Section **2**.

If the coefficients $\vec{F}$ and a are independent of t, and if $\vec{G} = \mathbf{0}$ and $b = 0$, then by applying the fractional integration operator $\mathcal{I}^{1-\alpha}$ to both sides of (1.1) we obtain

$$^{\mathrm{C}}\partial_t^{\alpha} u - \nabla\cdot(\kappa\nabla u - \vec{F}u) + au = g, \qquad (1.6)$$

where $^{\mathrm{C}}\partial_t^{\alpha} u = \mathcal{I}^{1-\alpha}\partial_t u$ denotes the Caputo fractional derivative [33] and where $g = \mathcal{I}^{1-\alpha} g$. Existence and uniqueness results for (1.6) were studied by several authors, including Zacher [39], Alikhanov [2], Sakamoto and Yamamoto [34] and Kubica and Yamamoto [17]. Further, the reader can refer to [15, 22, 27, 35]. Some of these papers include results for time-dependent coefficients, but in that case (1.6) is no longer equivalent to (1.1).

To recast the weak formulation (1.5) as a Volterra integral equation, we introduce two bounded linear operators, firstly $K_1(t) : H_0^1(\Omega) \to H^{-1}(\Omega)$ defined by

$$\langle K_1(t)v, w\rangle = \langle \kappa\nabla v, \nabla w\rangle - \langle \vec{F}(t)v, \nabla w\rangle + \langle a(t)v, w\rangle \quad \text{for } v,\, w \in H_0^1(\Omega),$$

and secondly $K_2(t) : L_2(\Omega) \to H^{-1}(\Omega)$ by

$$\langle K_2(t)v, w\rangle = \langle b(t)v, w\rangle - \langle \vec{G}(t)v, \nabla w\rangle \quad \text{for } v \in L_2(\Omega) \text{ and } w \in H_0^1(\Omega).$$

The variational problem (1.5), subject to the initial condition (1.3), can then be written more succinctly as

$$u(t) + \int_0^t \bigl(K_1(s)\partial_s^{1-\alpha}u(s) + K_2(s)u(s)\bigr)\, ds = f(t) \equiv u_0 + \int_0^t g(s)\, ds. \quad (1.7)$$

Assuming u is sufficiently regular that $(\mathcal{I}^\alpha u)(0) = 0$, and using a dash to indicate a derivative in time, integration by parts leads to

$$\begin{aligned}\int_0^t K_1(s)\partial_s^{1-\alpha}u(s)\, ds &= K_1(t)\mathcal{I}^\alpha u(t) - \int_0^t K_1'(s)\mathcal{I}^\alpha u(s)\, ds\\ &= \int_0^t \Bigl(\omega_\alpha(t-s)K_1(t) - \int_s^t \omega_\alpha(z-s)K_1'(z)\, dz\Bigr) u(s)\, ds,\end{aligned}$$

with $K_1'(t) : H_0^1(\Omega) \to H^{-1}(\Omega)$ given by

$$\langle K_1'(t)v, w\rangle = -\langle \vec{F}'(t)v, \nabla w\rangle + \langle a'(t)v, w\rangle.$$

Thus, u satisfies

$$u(t) + \int_0^t K(t,s)u(s)\, ds = f(t) \quad \text{for } 0 \le t \le T, \quad (1.8)$$

where $K(t,s) : H_0^1(\Omega) \to H^{-1}(\Omega)$ is the weakly-singular, operator-valued kernel

$$K(t,s) = \omega_\alpha(t-s)K_1(t) + K_2(s) - \int_s^t \omega_\alpha(z-s)K_1'(z)\, dz. \quad (1.9)$$

Following some technical preliminaries in Section **2**, we apply the Galerkin method in Section **3** to project the problem (1.8) to a finite dimensional subspace $X \subseteq H_0^1(\Omega)$, thereby obtaining an approximate solution $u_X : [0,T] \to X$. Using delicate energy arguments and a fractional Gronwall inequality, we prove *a priori* estimates for u_X that are uniform with respect to the dimension of X, allowing us in Section **4** (Theorems **4.1** and **4.2**) to establish the existence and uniqueness of a weak solution u to the original problem (1.1)–(1.3), provided (1.4) holds.

The regularity of the weak solution u will be studied in a companion paper [29].

2. Preliminaries and notations

Our subsequent analysis makes frequent use of two quadratic operators defined, for $\mu \ge 0$ and $0 \le t \le T$, by

$$\mathcal{Q}_1^\mu(\phi, t) = \int_0^t \langle \phi, \mathcal{I}^\mu \phi\rangle\, ds \quad \text{and} \quad \mathcal{Q}_2^\mu(\phi, t) = \int_0^t \|\mathcal{I}^\mu \phi\|^2\, ds.$$

These operators coincide when $\mu = 0$ because $\mathcal{I}^0 \quad = \quad$, and so we write $\mathcal{Q}^0 = \mathcal{Q}_1^0 = \mathcal{Q}_2^0$. If we put $\ (t) = 0$ for $t > T$, then the Laplace transform $\ (z) = \int_0^T e^{-zt} \ (t)\, dt$ is an entire function and $\mathcal{I}^\mu \ (z) = z^{-\mu} \ (z)$, so it follows by the Plancherel Theorem that

$$\mathcal{Q}_1^\mu(\ , T) = \frac{\cos(\ \mu/2)}{\ } \int_0 \ y^{-\mu} \ \ (iy) \ ^2 \, dy \geq 0, \tag{2.1}$$

assuming that is real-valued; see also [32, Theorem 2]. Note that because $\ _\mu \in L_1(0, T)$, the fractional integral defines a bounded linear operator

$$\mathcal{I}^\mu : L_p\big((0,T); L_2(\Omega)\big) \qquad L_p\big((0,T); L_2(\Omega)\big) \quad \text{for } 1 \leq p \leq \quad . \tag{2.2}$$

Also, $\mathcal{I}^{\mu+\beta} = \mathcal{I}^\mu \mathcal{I}^\beta$ because $\ _\mu \quad _\beta = \ _{\mu+\beta}$ for $\mu > 0$ and $\ > 0$; here, denotes the Laplace convolution.

The next four lemmas establish key inequalities satisfied by $\mathcal{Q}_1^\mu$ and $\mathcal{Q}_2^\mu$.

LEMMA **2.1**. *If* $0 < \alpha < 1$ *and* $\ > 0$, *then*

$$\int_0^t \ , \mathcal{I}^\alpha \quad ds \leq \frac{\mathcal{Q}_1^\alpha(\ , t)}{4 \ (1-\alpha)^2} + \ \mathcal{Q}_1^\alpha(\ , t), \tag{2.3}$$

$$\mathcal{Q}_2^\alpha(\ , t) \leq \frac{2t^\alpha}{1-\alpha} \mathcal{Q}_1^\alpha(\ , t), \tag{2.4}$$

$$\mathcal{Q}_1^\alpha(\ , t) \leq 2t^\alpha \mathcal{Q}^0(\ , t), \tag{2.5}$$

$$\int_0^t \ , \mathcal{I}^\alpha \quad ds \leq \frac{t^\alpha \mathcal{Q}^0(\ , t)}{2 \ (1-\alpha)^2} + \ \mathcal{Q}_1^\alpha(\ , t). \tag{2.6}$$

P r o o f. The first three inequalities are proved by Le, McLean and Mustapha [20, Lemma 3.2]. The fourth inequality follows from (2.3) and (2.5). □

For the next result, note that if $\ \in W_1^1\big((0,T); X\big)$ for a normed space X, then $\ : [0, T] \quad X$ is absolutely continuous and

$$(\partial_t \mathcal{I}^\alpha \quad - \mathcal{I}^\alpha \partial_t \ )(t) = \ (0) \ _\alpha(t) \quad \text{for } 0 < t \leq T. \tag{2.7}$$

LEMMA **2.2**. *If* $0 < \alpha \leq 1$, *then for* $\ \in L_2\big((0,t), L_2(\Omega)\big)$,

$$\mathcal{Q}_2^\alpha(\ , t) \leq 2 \int_0^t \ _\alpha(t-s) \mathcal{Q}_1^\alpha(\ , s)\, ds.$$

P r o o f. Assume first that $\ \in W_1^1\big((0,T), L_2(\Omega)\big)$ and let $\ = \mathcal{I}^\alpha \ $. Since $\ (0) = 0$, the Caputo fractional derivative of is

$${}^{\mathrm{C}}\partial_t^\alpha \quad = \mathcal{I}^{1-\alpha}(\quad) = (\mathcal{I}^{1-\alpha} \quad) - \ (0) \ _{1-\alpha} = (\mathcal{I}^1 \quad) = \quad .$$

Recalling an identity of Alikhanov [3, Corollary1],

$$2\quad (t), {}^{\mathrm{C}}\partial_t^\alpha \quad (t) \quad = {}^{\mathrm{C}}\partial_t^\alpha\big(\qquad {}^2\big)(t)$$
$$+ \frac{\alpha}{2\quad(1-\alpha)} \int_0^t \frac{1}{(t-s)^{1-\alpha}} \int_0^s \frac{\quad(q)\,dq}{(t-q)^\alpha} \quad {}^2 \, ds,$$

we see that

$$2 \quad , \mathcal{I}^\alpha \quad = 2 \; {}^{\mathrm{C}}\partial_t^\alpha \quad , \quad \geq {}^{\mathrm{C}}\partial_t^\alpha\big(\qquad {}^2\big) = \mathcal{I}^{1-\alpha}\big(\; \mathcal{I}^\alpha \quad {}^2\big)\,, \tag{2.8}$$

and thus

$$\mathcal{I}^1\big(\; \mathcal{I}^\alpha \quad {}^2\big) = \mathcal{I}^2\big(\; \mathcal{I}^\alpha \quad {}^2\big) = \mathcal{I}^{1+\alpha}\mathcal{I}^{1-\alpha}\big(\; \mathcal{I}^\alpha \quad {}^2\big)$$
$$\leq 2\mathcal{I}^{1+\alpha}\big(\quad , \mathcal{I}^\alpha \quad \big) = 2\mathcal{I}^\alpha\mathcal{I}^1\big(\quad , \mathcal{I}^\alpha \quad \big),$$

which is equivalent to the desired inequality.

Now let $\in L_2\big((0,T), L_2(\Omega)\big)$, and choose $_n \in W_1^1\big((0,T), L_2(\Omega)\big)$ such that $\int_0^T \quad _n(t) - \quad (t) \quad {}^2\,dt \qquad 0$ as $n \qquad$. Using (2.2) with $\mu = \alpha$ and $p = 2$, it follows that $\mathcal{Q}_1^\alpha(\;_n, t) \qquad \mathcal{Q}_1^\alpha(\;, t)$ and $\mathcal{Q}_2^\alpha(\;_n, t) \qquad \mathcal{Q}_2^\alpha(\;, t)$, uniformly for $t \in [0,T]$, which implies the result in the general case. □

The next lemma will eventually enable us to establish pointwise (in time) estimates for $u(t)$.

LEMMA **2.3**. *Let $0 \leq \mu < \alpha \leq 1$. If the function $\; : [0,T] \qquad L_2(\Omega)$ is continuous with $\;(0) = 0$, and if its restriction to $(0,T]$ is differentiable with $\quad (t) \quad \leq Ct^{-\mu}$ for $0 < t \leq T$, then $\quad (t) \quad {}^2 \leq 2 \quad _{2-\alpha}(t)\, \mathcal{Q}_1^\alpha(\;, t)$.*

P r o o f. For $\alpha = 1$, equality holds:

$$2 \quad _1(t)\mathcal{Q}_1^1(\quad, t) = 2 \int_0^t \quad , \quad ds = \quad (t) \quad {}^2.$$

For $0 < \alpha < 1$, put $\quad (t) = \mathcal{I}^\alpha$ and note that $\quad (t) \quad \leq Ct^{\alpha-\mu}$. By following similar arguments, one can show that (2.8) holds with in place of , that is $2 \quad , \mathcal{I}^\alpha \quad = \mathcal{I}^{1-\alpha}\big(\; \mathcal{I}^\alpha \quad {}^2\big)$, for almost all $t > 0$. Now, applying the operator $\mathcal{I}^1$ to both sides, and using $\mathcal{I}^\alpha \quad (0) = \quad (0) = 0$, we observe that

$$\mathcal{I}^{1-\alpha}\big(\; \mathcal{I}^\alpha \qquad {}^2\big)(t) = 2\mathcal{Q}_1^\alpha(\quad, t) \quad \text{for } t > 0. \tag{2.9}$$

Since $\quad = \mathcal{I}^1 \quad = \mathcal{I}^{1-\alpha} \quad ,$

$$
\begin{aligned}
(t) \;^{2} &\leq \;\; {}_{0}^{t} \;\; {}_{1-\alpha}(t-s) \quad (s) \;\; ds \;\; ^{2} \\
&\leq \;\; {}_{0}^{t} \;\; {}_{1-\alpha}(t-s)\,ds \;\; {}_{0}^{t} \;\; {}_{1-\alpha}(t-s) \quad (s) \;^{2}\,ds \\
&= \;\; {}_{2-\alpha}(t)\,\mathcal{I}^{1-\alpha} \quad \mathcal{I}^{\alpha} \quad ^{2} \; (t),
\end{aligned}
$$

and hence the desired result follows immediately after using (2.9). □

Lemma **2.4**. *If* $0 \leq \mu \leq \quad \leq 1$, *then* $\mathcal{Q}_2^{\beta}(\;\, , t) \leq 2t^{2(\beta-\mu)}\mathcal{Q}_2^{\mu}(\;\, , t)$.

P r o o f. See Le, McLean and Mustapha [20, Lemma 3.1]. □

We will make essential use of the following fractional Gronwall inequality.

Lemma **2.5**. *Let* $\quad > 0$ *and* $T > 0$. *Assume that* a *and* b *are non-negative, non-decreasing functions on the interval* $[0, T]$. *If* $\mathsf{q} : [0, T] \quad \mathbb{R}$ *is an integrable function satisfying*

$$
0 \leq \mathsf{q}(t) \leq \mathsf{a}(t) + \mathsf{b}(t) \;\; {}_{0}^{t} \quad (t-s)\mathsf{q}(s)\,ds \quad \text{for } 0 \leq t \leq T,
$$

then

$$
\mathsf{q}(t) \leq \mathsf{a}(t) E \;\; \big(\mathsf{b}(t) t \;\; \big) \quad \text{for } 0 \leq t \leq T.
$$

P r o o f. See Dixon and McKee [9, Theorem 3.1]. □

Let $\mathcal{M}$ denote the operator of pointwise multiplication by t, that is, $(\mathcal{M} \quad)(t) = t \quad (t)$, and note the commutator property

$$
\mathcal{M}\mathcal{I}^{\mu} - \mathcal{I}^{\mu}\mathcal{M} = \mu\mathcal{I}^{\mu+1}, \tag{2.10}
$$

for any real $\mu \geq 0$. We will need the following estimates involving the linear operator B^{μ} defined (for suitable and) by

$$
(B^{\mu} \quad)(t) = \quad (t)\,\mathcal{I}^{\mu} \quad (t) - \;\; {}_{0}^{t} \quad (s)\,\mathcal{I}^{\mu} \quad (s)\,ds. \tag{2.11}
$$

Lemma **2.6**. *If* $\quad \in W^1 \big((0, T); L \quad (\Omega)^d\big)$ *and* $\quad \in W_1^1\big((0, T); L_2(\Omega)\big)$, *then there is a constant* C *(depending only on* $\;\;,$ μ *and* T*) such that for* $0 \leq t \leq T$,

$$\mathcal{Q}^0(B^\mu \ \ ,t) \le C\mathcal{Q}_2^\mu(\ \ ,t), \tag{2.12}$$

$$\mathcal{Q}^0(\mathcal{M}B^\mu \ \ ,t) + \mathcal{Q}^0(\mathcal{I}^1B^\mu \ \ ,t) \le Ct^2\mathcal{Q}_2^\mu(\ \ ,t), \tag{2.13}$$

$$\mathcal{Q}^0\big((\mathcal{M}B^\mu \ \ ) \ ,t\big) \le C\mathcal{Q}_2^\mu\big((\mathcal{M} \ \ ) \ ,t\big) + C\mathcal{Q}_2^\mu(\mathcal{M} \ \ ,t) + C\mathcal{Q}_2^\mu(\ \ ,t). \tag{2.14}$$

P r o o f. The assumption on implies that

$$(B^\mu \ \ )(t) \ ^2 \le C \ (\mathcal{I}^\mu \ \ )(t) \ ^2 + C \ \ _0^t \ (\mathcal{I}^\mu \ \ )(s) \ ^2\,ds,$$

and (2.12) follows after integrating in time. By the Cauchy–Schwarz inequality,

$$(\mathcal{M}B^\mu \ \ )(t) \ ^2 + \ (\mathcal{I}^1B^\mu \ \ )(t) \ ^2 \le t^2 \ (B^\mu \ \ )(t) \ ^2 + t \ \ _0^t \ (B^\mu \ \ )(s) \ ^2\,ds,$$

and (2.13) follows after integrating in time. The third identity in (2.10) implies that

$$\mathcal{M}B^\mu \ \ = \ \big(\mathcal{I}^\mu\mathcal{M} \ \ + \mu\mathcal{I}^{\mu+1} \ \ \big) - \mathcal{M}\mathcal{I}^1(\ \ \mathcal{I}^\mu \ \ )$$

and therefore, differentiating with respect to t,

$$(\mathcal{M}B^\mu \ \ ) \ = \ \big(\mathcal{I}^\mu\mathcal{M} \ \ +\mu\mathcal{I}^{\mu+1} \ \ \big) + \ \big((\mathcal{I}^\mu\mathcal{M} \ \ ) \ +\mu\mathcal{I}^\mu \ \ \big) - (\mathcal{I}^1+\mathcal{M})(\ \ \mathcal{I}^\mu \ \ ).$$

Thus, noting that $(\mathcal{I}^\mu\mathcal{M} \ \ ) \ = \mathcal{I}^\mu(\mathcal{M} \ \ )$ by (2.7), with

$$\mathcal{I}^{\mu+1} \ \ (t) \ ^2 = \ \mathcal{I}^1(\mathcal{I}^\mu \ \ )(t) \ ^2 \le t\mathcal{Q}_2^\mu(\ \ ,t)$$

and $\mathcal{I}^1(\ \ \mathcal{I}^\mu \ \ )(t) \ ^2 \le Ct\mathcal{Q}_2^\mu(\ \ ,t)$, we have

$$\begin{aligned}(\mathcal{M}B^\mu \ \ ) \ (t) \ ^2 &\le C \ \mathcal{I}^\mu(\mathcal{M} \ \ )(t) \ ^2 + C \ \mathcal{I}^\mu(\mathcal{M} \ \ ) \ (t) \ ^2 \\ &\quad + C \ (\mathcal{I}^\mu \ \ )(t) \ ^2 + Ct\mathcal{Q}_2^\mu(\ \ ,t),\end{aligned}$$

so (2.14) follows after integrating in time. □

3. The projected equation

Suppose that X is a finite-dimensional subspace of $H_0^1(\Omega)$, equipped with the induced norm: $v \ _X = \ v \ _{H_0^1(\ )}$. We define a bounded linear operator $K_X(t,s) : X \quad X$ in terms of $K(t,s)$ in (1.9) by

$$K_X(t,s)v, w \ = \ K(t,s)v, w \quad \text{for } v,\ w \in X \text{ and } 0 \le s \le t \le T,$$

and let $f_X(t)$ denote the L_2-projection onto X of $f(t)$ from (1.7), that is,

$$f_X(t), w \ = \ f(t), w \quad \text{for } w \in X \text{ and } 0 \le t \le T.$$

In this way, we arrive at a finite dimensional reduction of the Volterra equation (1.8),

$$u_X(t) + \int_0^t K_X(t,s)u_X(s)\,ds = f_X(t) \quad \text{for } 0 \le t \le T. \tag{3.1}$$

In the next theorem, we outline a self-contained proof of existence and uniqueness under relaxed assumptions on the coefficients in the fractional PDE (1.1). Similar results for scalar-valued kernels are shown by Linz [24, §3.4], Becker [5], and Brunner [6].

Henceforth, C will denote a generic constant that may depend on the coefficients in (1.1), the spatial domain Ω, the time interval $[0,T]$, the fractional exponent α, the parameter , and the integer m in (1.4). However, any dependence on the subspace X is indicated explicitly by writing C_X. We let $Y = C([0,T];X)$ with the norm $\|v\|_Y = \max_{0 \le t \le T} \|v(t)\|_X$.

Theorem **3.1**. *Assume that the coefficients in* (1.1) *satisfy*

$$\kappa \in L_\infty(\Omega)^{d\times d}, \quad \vec{F} \in W^1_\infty\big((0,T);L_\infty(\Omega)^d\big), \quad \vec{G} \in L_\infty\big((0,T);L_\infty(\Omega)^d\big),$$
$$a \in W^1_\infty\big((0,T);L_\infty(\Omega)\big), \quad b \in L_\infty\big((0,T);L_\infty(\Omega)\big).$$

Assume, in addition, that the source term $g : (0,T] \to L_2(\Omega)$ is a measurable function satisfying

$$\|g(t)\| \le Mt^{\ -1} \quad \text{for } 0 < t \le T, \tag{3.2}$$

where M and   are positive constants, and that the initial data $u_0 \in L_2(\Omega)$. Then, the weakly-singular Volterra integral equation (3.1) *has a unique solution $u_X \in Y$, and moreover $\|u_X\|_Y \le C_X \|f_X\|_Y \le C_X(\|u_0\| + M)$.*

P r o o f. Our assumptions on u_0 and g ensure that $f_X \in Y$. The kernel (1.9) has the form

$$K(t,s) = \omega_\alpha(t-s)G(t,s) + H(t,s),$$

where

$$G(t,s) = K_1(t) - \Gamma(\alpha)(t-s)\int_0^1 \omega_\alpha(y)K_1\big(s+(t-s)y\big)\,dy$$

and $H(t,s) = K_2(s)$ for $0 \le s \le t \le T$. Our assumptions on the coefficients of the fractional PDE (1.1) ensure that G and H are continuous mappings from the closed triangle $\Delta = \{\,(t,s) : 0 \le s \le t \le T\,\}$ into the space of bounded linear operators $H_0^1(\Omega) \to H^{-1}(\Omega)$. Likewise,

$$K_X(t,s) = \omega_\alpha(t-s)G_X(t,s) + H_X(t,s),$$

where $G_X(t,s) : X \to X$ and $H_X(t,s) : X \to X$ are defined by

$$\langle G_X(t,s)v, w\rangle = \langle G(t,s)v, w\rangle \quad \text{and} \quad \langle H_X(t,s)v, w\rangle = \langle H(t,s)v, w\rangle$$

for $(t,s) \in \Delta$ and v, $w \in X$. Since X is finite dimensional, G_X and H_X are continuous functions from Δ into the space of bounded linear operators $X \to X$. Hence, there is a positive constant C_X such that

$$\|K_X(t,s)v\|_X \le C_X\, \omega_\alpha(t-s)\, \|v\|_X \quad \text{for } (t,s) \in \Delta \text{ and } v \in X,$$

so we can define the Volterra operator $\mathcal{K}_X : Y \to Y$ by

$$\mathcal{K}_X v(t) = \int_0^t K_X(t,s)v(s)\,ds \quad \text{for } 0 \le t \le T \text{ and } v \in Y.$$

We see that $\|\mathcal{K}_X v\|_Y \le C_X\, \omega_{1+\alpha}(T)\, \|v\|_Y$. In fact, using the semigroup property,

$$\int_0^t \omega_\alpha(t-s)\, \omega_\mu(s)\,ds = \omega_{\alpha+\mu}(t),$$

we obtain the following estimate for the operator norm of the nth power of $\mathcal{K}_X$,

$$\|\mathcal{K}_X^n\|_{Y \to Y} \le C_X^n \max_{0 \le t \le T} \int_0^t \omega_{n\alpha}(t-s)\,ds = C_X^n\, \omega_{1+n\alpha}(T) \quad \text{for } n \ge 1.$$

It follows that the sum $\mathcal{R}_X = \sum_{n=1}^\infty (-1)^{n+1}\mathcal{K}_X^n$ defines a bounded linear operator with

$$\|\mathcal{R}_X\|_{Y \to Y} \le \sum_{n=1}^\infty \omega_{1+n\alpha}(T)\, C_X^n = E_\alpha(C_X T^\alpha) - 1.$$

The existence and uniqueness of $u_X \in Y$ is seen by noting

$$u_X + \mathcal{K}_X u_X = f_X \quad \text{if and only if} \quad u_X = f_X - \mathcal{R}_X f_X,$$

from which we also deduce the *a priori* estimate claimed in the theorem. □

For a scalar, weakly-singular, second-kind Volterra equation, it is known that if f_X admits an expansion in powers of t and t^α, then so does the solution u_X; see Lubich [26, Corollary 3], and also Brunner, Pedas and Vainikko [7, Theorem 2.1] (with $\beta = 1-\alpha$). To outline a proof that a similar result holds for *systems* of Volterra equations, let $C_\alpha^m = C_\alpha^m\big([0,T];X\big)$ denote the space of continuous functions $v : [0,T] \to X$ that are C^m on the half-open interval $(0,T]$ and for which the seminorm

$$|v|_{j,\alpha} = \sup_{0<t \le T} t^{j-\alpha} \|v^{(j)}(t)\|_X \quad \text{is finite for } 1 \le j \le m.$$

We make C_α^m into a Banach space by defining the obvious norm:

$$\|v\|_{m,\alpha} = \|v\|_Y + \sum_{j=1}^m |v|_{j,\alpha}.$$

THEOREM **3.2**. *Let $m \geq 1$, and strengthen the assumptions* (1.4) *by requiring*

$$\vec{F}, \vec{G} \in C^{m+1}\big([0,T]; W^1\ (\Omega)^d\big) \quad \text{and} \quad a, b \in C^m\big([0,T]; L\ \ (\Omega)\big).$$

If $u_0 \in L_2(\Omega)$ *and* $g : (0,T] \quad X$ *is* C^m *with* $g^{(i-1)}(t) \ \leq Mt^{\alpha-i}$ *for* $1 \leq i \leq m$, *then* $u_X \in C^m_\alpha$ *and* $u_X \ {}_{m,\alpha} \leq C_X \ f_X \ {}_{m,\alpha} \leq C_X(\ u_0 \ + M)$.

P r o o f. Our assumptions on u_0 and g imply that $f_X \in C^m_\alpha$. Using the substitution $z = s + (t-s)y$ in (1.9), we find that if $j + k \leq m$ and $0 \leq s < t \leq T$, then

$$\partial_t^k(\partial_t + \partial_s)^j K(t,s)v \ {}_{H^{-1}(\ )} \leq C_X(t-s)^{\alpha-1-k} \ v \ {}_{H^1_0(\ )} \quad \text{for } v \in H^1_0(\Omega),$$

and, since X is finite dimensional,

$$\partial_t^k(\partial_t + \partial_s)^j K_X(t,s)v \ {}_X \leq C_X(t-s)^{\alpha-1-k} \ v \ {}_X \quad \text{for } v \in X.$$

Hence, the Volterra operator $\mathcal{K}_X : C^m_\alpha \quad C^m_\alpha$ is compact [37, Theorem 6.1]. Theorem **3.1** implies that the homogeneous equation, $u_X + \mathcal{K}_X u_X = 0$, has only the trivial solution $u_X = 0$, and therefore the inhomogeneous equation $u_X + \mathcal{K}_X u_X = f_X$ is well-posed not only in Y but also in C^m_α. □

Our goal in the remainder of this section is to obtain bounds for $\ u_X(t)$ and $\ \nabla u_X(t)$ with constants that are independent of X. Our proof relies on a sequence of technical lemmas. To simplify our estimates, we rescale the time variable, if necessary, so that the minimal eigenvalue of κ is bounded below by unity:

$$\min\big(\kappa(x)\big) \geq 1 \quad \text{for } x \in \Omega. \tag{3.3}$$

In this way, $\ \kappa\nabla v, \nabla v \ \geq \ \nabla v \ ^2$ for $v \in H^1_0(\Omega)$, and we see from (2.1) that for (real-valued) $\ \in C\big([0,T]; H^1_0(\Omega)\big)$,

$$\begin{aligned} {}_0^t \ \kappa\mathcal{I}^\mu\nabla \ , \nabla \ \ ds &= \frac{\cos(\ \mu/2)}{} \ {}_0 \ y^{-\mu} \ \kappa\nabla \ (iy), \overline{\nabla \ (iy)} \ dy \\ &\geq \frac{\cos(\ \mu/2)}{} \ {}_0 \ y^{-\mu} \ \nabla \ (iy) \ ^2 \, dy, \end{aligned}$$

so

$$ {}_0^t \ \kappa\mathcal{I}^\mu\nabla \ , \nabla \ \ ds \geq {}_0^t \ \mathcal{I}^\mu\nabla \ , \nabla \ \ ds = \mathcal{Q}_1^\mu(\nabla \ , t). \tag{3.4}$$

Since (1.7) is equivalent to (1.8), if $v \in X$ then

$$\int_0^t K_X(t,s)u_X(s)\,ds, v$$
$$= \int_0^t K_1(s)\partial_s^{1-\alpha}u_X, v \;\; ds + \int_0^t K_2(s)u_X(s), v \;\; ds$$
$$= \kappa(\mathcal{I}^\alpha \nabla u_X)(t), \nabla v \; - \; (B_1 u_X)(t), \nabla v \; + \; (B_2 u_X)(t), v \;\; ,$$

where

$$\vec{B}_1 \;\; (t) = \int_0^t \vec{F}(s)\partial_s^{1-\alpha} \;\; (s) + \vec{G}(s) \;\; (s) \;\; ds,$$
$$B_2 \;\; (t) = \int_0^t a(s)\partial_s^{1-\alpha} \;\; (s) + b(s) \;\; (s) \;\; ds. \tag{3.5}$$

Assuming $\in C^1_\alpha([0,T];X)$, we may integrate by parts and use the notation (2.11) to write

$$\vec{B}_1 = B^\alpha_F + B^1_G \quad \text{and} \quad B_2 = B^\alpha_a + B^1_b. \tag{3.6}$$

Thus, the solution of (3.1) satisfies

$$u_X(t), v \; + \; \kappa\nabla\mathcal{I}^\alpha u_X(t), \nabla v \; - \; (\vec{B}_1 u_X)(t), \nabla v \; + \; (B_2 u_X)(t), v$$
$$= \; f_X(t), v \quad \text{for } v \in X, \tag{3.7}$$

which yields the following estimates (with C independent of X).

LEMMA **3.1**. *For $0 \le t \le T$, the solution u_X of the Volterra equation* (3.1) *satisfies the* a priori *estimates*

$$\mathcal{Q}^\alpha_1(u_X,t) + \mathcal{Q}^\alpha_2(\nabla u_X,t) \le Ct^\alpha \mathcal{Q}^0(f_X,t)$$

and

$$\mathcal{Q}^0(u_X,t) + \mathcal{Q}^\alpha_1(\nabla u_X,t) \le C\mathcal{Q}^0(f_X,t).$$

P r o o f. From (3.7),

$$u_X(t), v \; + \; \kappa\nabla\mathcal{I}^\alpha u_X(t), \nabla v \;\le \tfrac{1}{2} \;\; \nabla v \;\; ^2 + \tfrac{1}{2} \;\; \vec{B}_1 u_X(t) \;\; ^2 + \tfrac{1}{2} \;\; B_2 u_X(t) \;\; ^2$$
$$+ \tfrac{1}{2} \;\; v \;\; ^2 + \;\; f_X(t), v \;\; .$$

Choosing $v = \mathcal{I}^\alpha u_X(t)$ we have $\kappa\nabla\mathcal{I}^\alpha u_X(t), \nabla v \; = \; \kappa\nabla v, \nabla v \; \ge \; \nabla v \;\; ^2$ because of (3.3). Thus, after canceling the term $\frac{1}{2} \;\; \nabla v \;\; ^2$ and integrating in time, we see that

$$\mathcal{Q}^\alpha_1(u_X,t) + \tfrac{1}{2}\mathcal{Q}^\alpha_2(\nabla u_X,t) \le \tfrac{1}{2}\mathcal{Q}^0(\vec{B}_1 u_X,t) + \tfrac{1}{2}\mathcal{Q}^0(B_2 u_X,t) + \tfrac{1}{2}\mathcal{Q}^\alpha_2(u_X,t)$$
$$+ \int_0^t \;\; f_X(s), \mathcal{I}^\alpha u_X(s) \;\; ds. \tag{3.8}$$

Using the representation (3.6) and the achieved estimate (2.12),

$$\begin{aligned}\mathcal{Q}^0(\vec{B}_1 u_X, t) &\le 2\mathcal{Q}^0(B_F^\alpha u_X, t) + 2\mathcal{Q}^0(B_G^1 u_X, t)\\ &\le C\mathcal{Q}_2^\alpha(u_X, t) + C\mathcal{Q}_2^1(u_X, t) \le C\mathcal{Q}_2^\alpha(u_X, t),\end{aligned}$$

where, in the final step, we used Lemma **2.4**. In the same way,

$$\mathcal{Q}^0(B_2 u_X, t) \le C\mathcal{Q}_2^\alpha(u_X, t).$$

Using (2.6) with $= f_X$, $= u_X$ and $= 1/2$, we deduce that

$$\mathcal{Q}_1^\alpha(u_X, t) + \tfrac{1}{2}\mathcal{Q}_2^\alpha(\nabla u_X, t) \le C\mathcal{Q}_2^\alpha(u_X, t) + Ct^\alpha\mathcal{Q}^0(f_X, t) + \tfrac{1}{2}\mathcal{Q}_1^\alpha(u_X, t).$$

Hence, applying Lemma **2.2** with $= u_X$, we can show that the function $\mathsf{q}(t) = \mathcal{Q}_1^\alpha(u_X, t) + \mathcal{Q}_2^\alpha(\nabla u_X, t)$ satisfies

$$\mathsf{q}(t) \le Ct^\alpha\mathcal{Q}^0(f_X, t) + C\int_0^t {}_\alpha(t-s)\mathcal{Q}_1^\alpha(u_X, s)\,ds.$$

Since $\mathcal{Q}_1^\alpha(u_X, s) \le \mathsf{q}(s)$, Lemma **2.5** implies the first estimate.

To show the second estimate, use $-\langle(\vec{B}_1 u_X)(t), \nabla v\rangle = \langle\nabla\cdot\vec{B}_1 u_X(t), v\rangle$ in (3.7) to obtain

$$\begin{aligned}\langle u_X(t), v\rangle + \langle\kappa\nabla\mathcal{I}^\alpha u_X(t), \nabla v\rangle \le \tfrac{1}{2}\|v\|^2 &+ \tfrac{3}{2}\|\nabla\cdot(\vec{B}_1 u_X)(t)\|^2\\ &+ \tfrac{3}{2}\|(B_2 u_X)(t)\|^2 + \tfrac{3}{2}\|f_X(t)\|^2.\end{aligned}$$

Choosing $v = u_X(t)$, integrating in time, and using (3.4), we have

$$\tfrac{1}{2}\mathcal{Q}^0(u_X, t) + \mathcal{Q}_1^\alpha(\nabla u_X, t) \le C\mathcal{Q}^0(\nabla\cdot\vec{B}_1 u_X, t) + C\mathcal{Q}^0(B_2 u_X, t) + C\mathcal{Q}^0(f_X, t).$$

Since

$$\begin{aligned}\nabla\cdot(B_F^\alpha u_X)(t) &= \big(\nabla\cdot\vec{F}(t)\big)\mathcal{I}^\alpha u_X(t) + \vec{F}(t)\cdot\mathcal{I}^\alpha\nabla u_X(t)\\ &\quad - \int_0^t \Big(\big(\nabla\cdot\vec{F}\,'(s)\big)\mathcal{I}^\alpha u_X(s) + \vec{F}\,'(s)\cdot\mathcal{I}^\alpha\nabla u_X(s)\Big)\,ds \qquad (3.9)\end{aligned}$$

it follows that

$$\begin{aligned}\|\nabla\cdot(B_F^\alpha u_X)(t)\|^2 &\le C\|\mathcal{I}^\alpha u_X(t)\|^2 + C\|\mathcal{I}^\alpha\nabla u_X(t)\|^2\\ &\quad + C\int_0^t \Big(\|\mathcal{I}^\alpha u_X(s)\|^2 + \|\mathcal{I}^\alpha\nabla u_X(s)\|^2\Big)\,ds,\end{aligned}$$

implying that $\mathcal{Q}^0(\nabla\cdot B_F^\alpha u_X, t) \le C\mathcal{Q}_2^\alpha(u_X, t) + C\mathcal{Q}_2^\alpha(\nabla u_X, t)$. In the same way, $\mathcal{Q}^0(\nabla\cdot B_G^1 u_X, t) \le C\mathcal{Q}_2^1(u_X, t) + C\mathcal{Q}_2^1(\nabla u_X, t)$ and therefore, by Lemma **2.4**,

$$\mathcal{Q}^0(\nabla\cdot\vec{B}_1 u_X, t) \le C\mathcal{Q}_2^\alpha(u_X, t) + C\mathcal{Q}_2^\alpha(\nabla u_X, t).$$

Recall $\mathcal{Q}^0(B_2u_X,t) \le C\mathcal{Q}_2^\alpha(u_X,t)$ and let $\mathsf{q}(t) = \mathcal{Q}^0(u_X,t) + \mathcal{Q}_1^\alpha(\nabla u_X,t)$. It follows using Lemma **2.2** and (2.5) that

$$\begin{aligned}\mathsf{q}(t) &\le C\mathcal{Q}_2^\alpha(u_X,t) + C\mathcal{Q}_2^\alpha(\nabla u_X,t) + C\mathcal{Q}^0(f_X,t)\\ &\le C\mathcal{Q}^0(f_X,t) + C\int_0^t \omega_\alpha(t-s)\,\big(\mathcal{Q}_1^\alpha(u_X,s) + \mathcal{Q}_1^\alpha(\nabla u_X,s)\big)\,ds\\ &\le C\mathcal{Q}^0(f_X,t) + Ct^\alpha\int_0^t \omega_\alpha(t-s)\mathsf{q}(s)\,ds.\end{aligned}$$

We may now apply Lemma **2.5** to complete the proof. □

The function $\mathcal{M}u_X(t) = tu_X(t)$ satisfies a similar estimate to the first one in Lemma **3.1**, but with an additional factor t^2 on the right-hand side.

LEMMA **3.2**. *The solution u_X of* (3.1) *satisfies*

$$\mathcal{Q}_1^\alpha(\mathcal{M}u_X,t) + \mathcal{Q}_2^\alpha(\mathcal{M}\nabla u_X,t) \le Ct^{2+\alpha}\mathcal{Q}^0(f_X,t) \quad \textit{for } 0 \le t \le T.$$

P r o o f. Multiplying both sides of (3.7) by t, and applying the third identity in (2.10), we find that (since κ is independent of t)

$$\begin{aligned}\langle \mathcal{M}u_X', v\rangle &+ \langle \kappa(\mathcal{I}^\alpha\mathcal{M} + \alpha\mathcal{I}^{\alpha+1})\nabla u_X, \nabla v\rangle\\ &= \langle \mathcal{M}\vec{B}_1u_X, \nabla v\rangle + \langle \mathcal{M}(f_X - B_2u_X), v\rangle, \quad (3.10)\end{aligned}$$

whereas integrating (3.7) in time gives

$$\langle \kappa\mathcal{I}^{\alpha+1}\nabla u_X, \nabla v\rangle = \langle \mathcal{I}^1\vec{B}_1u_X, \nabla v\rangle + \langle \mathcal{I}^1(f_X - u_X - B_2u_X), v\rangle,$$

so, after eliminating $\langle \kappa\mathcal{I}^{\alpha+1}\nabla u_X, \nabla v\rangle$,

$$\begin{aligned}\langle \mathcal{M}u_X', v\rangle + \langle \kappa\mathcal{I}^\alpha\mathcal{M}\nabla u_X, \nabla v\rangle &= \langle (\mathcal{M} - \alpha\mathcal{I}^1)\vec{B}_1u_X, \nabla v\rangle\\ &\quad + \langle (\mathcal{M} - \alpha\mathcal{I}^1)(f_X - B_2u_X) + \alpha\mathcal{I}^1u_X, v\rangle\\ \le \tfrac12\|\nabla v\|^2 + \tfrac12\|\vec{B}_3u_X\|^2 &+ \tfrac12\|B_4u_X\|^2 + \tfrac12\|v\|^2 + \langle (\mathcal{M}-\alpha\mathcal{I}^1)f_X + \alpha\mathcal{I}^1u_X, v\rangle,\end{aligned}$$

where $\vec{B}_3 = (\mathcal{M} - \alpha\mathcal{I}^1)\vec{B}_1$ and $B_4 = (\mathcal{M} - \alpha\mathcal{I}^1)B_2$. By choosing $v = \mathcal{I}^\alpha\mathcal{M}u_X$, we have $\langle \kappa\mathcal{I}^\alpha\mathcal{M}\nabla u_X, \nabla v\rangle = \langle \kappa\nabla v, \nabla v\rangle \ge \|\nabla v\|^2$ so, after canceling the term $\frac12\|\nabla v\|^2$ and integrating in time,

$$\begin{aligned}&\mathcal{Q}_1^\alpha(\mathcal{M}u_X,t) + \tfrac12\mathcal{Q}_2^\alpha(\mathcal{M}\nabla u_X,t)\\ &\quad\le \tfrac12\mathcal{Q}^0(B_3u_X,t) + \tfrac12\mathcal{Q}^0(B_4u_X,t) + \tfrac12\mathcal{Q}_2^\alpha(\mathcal{M}u_X,t)\\ &\qquad + \int_0^t \langle (\mathcal{M} - \alpha\mathcal{I}^1)f_X, \mathcal{I}^\alpha\mathcal{M}u_X\rangle\,ds + \alpha\int_0^t \langle \mathcal{I}^1u_X, \mathcal{I}^\alpha\mathcal{M}u_X\rangle\,ds.\end{aligned}$$

Using (2.6), we find that

$$\int_0^t (\mathcal{M}-\alpha\mathcal{I}^1)f_X, \mathcal{I}^\alpha\mathcal{M}u_X \; ds \le Ct^\alpha\mathcal{Q}^0\big((\mathcal{M}-\alpha\mathcal{I}^1)f_X,t\big)+\tfrac14\mathcal{Q}_1^\alpha(\mathcal{M}u_X,t)$$

and

$$\int_0^t \mathcal{I}^1u_X, \mathcal{I}^\alpha\mathcal{M}u_X \; ds \le Ct^\alpha\mathcal{Q}^0(\mathcal{I}^1u_X,t)+\tfrac14\mathcal{Q}_1^\alpha(\mathcal{M}u_X,t),$$

so

$$\mathcal{Q}_1^\alpha(\mathcal{M}u_X,t)+\mathcal{Q}_2^\alpha(\mathcal{M}\nabla u_X,t) \le \mathcal{Q}^0(B_3u_X,t)+\mathcal{Q}^0(B_4u_X,t) + 2\mathcal{Q}_2^\alpha(\mathcal{M}u_X,t)+Ct^\alpha\mathcal{Q}^0\big((\mathcal{M}-\alpha\mathcal{I}^1)f_X,t\big)+Ct^\alpha\mathcal{Q}^0(\mathcal{I}^1u_X,t).$$

Since

$$\vec{B}_3 = (\mathcal{M}-\alpha\mathcal{I}^1)B_{\vec{F}}^\alpha + (\mathcal{M}-\alpha\mathcal{I}^1)B_{\vec{G}}^1$$

and

$$B_4 = (\mathcal{M}-\alpha\mathcal{I}^1)B_a^\alpha + (\mathcal{M}-\alpha\mathcal{I}^1)B_b^1,$$

the estimate (2.13) gives

$$\mathcal{Q}^0(\vec{B}_3u_X,t)+\mathcal{Q}^0(B_4u_X,t) \le Ct^2\mathcal{Q}_2^\alpha(u_X,t)+Ct^2\mathcal{Q}_2^1(u_X,t) \le Ct^2\mathcal{Q}_2^\alpha(u_X,t),$$

where, in the last step, we used Lemma **2.4** with $\mu=\alpha$ and $\ = 1$. We easily verify that

$$\mathcal{Q}^0\big((\mathcal{M}-\alpha\mathcal{I}^1)f_X,t\big) \le Ct^2\mathcal{Q}^0(f_X,t),$$

and by Lemma **2.4** with $\mu=0$ and $\ =1$,

$$\mathcal{Q}^0(\mathcal{I}^1u_X,t) = \mathcal{Q}_2^1(u_X,t) \le t^2\mathcal{Q}^0(u_X,t).$$

Thus, the function $\mathsf{q}(t)=\mathcal{Q}_1^\alpha(\mathcal{M}u_X,t)+\mathcal{Q}_2^\alpha(\mathcal{M}\nabla u_X,t)$ satisfies

$$\mathsf{q}(t) \le Ct^2\mathcal{Q}_2^\alpha(u_X,t)+2\mathcal{Q}_2^\alpha(\mathcal{M}u_X,t)+Ct^{2+\alpha}\mathcal{Q}^0(f_X,t)+Ct^{2+\alpha}\mathcal{Q}^0(u_X,t).$$

By (2.4) and Lemma **3.1**,

$$t^2\mathcal{Q}_2^\alpha(u_X,t)+t^{2+\alpha}\mathcal{Q}^0(u_X,t) \le Ct^{2+\alpha}\mathcal{Q}^0(u_X,t) \le Ct^{2+\alpha}\mathcal{Q}(f_X,t),$$

and therefore, using Lemma **2.2** with $\ =\mathcal{M}u_X$,

$$\mathsf{q}(t) \le Ct^{2+\alpha}\mathcal{Q}^0(f_X,t)+C\int_0^t {}_\alpha(t-s)\mathsf{q}(s)\,ds,$$

The result now follows by applying Lemma **2.5**. □

Lemma **3.3**. *The solution u_X of* (3.1) *satisfies, for $0\le t\le T$,*

$$\mathcal{Q}_1^\alpha\big((\mathcal{M}u_X)\,,t\big)+\mathcal{Q}_2^\alpha\big((\mathcal{M}\nabla u_X)\,,t\big) \le Ct^\alpha\mathcal{Q}^0(f_X,t)+Ct^\alpha\mathcal{Q}^0\big((\mathcal{M}f_X)\,,t\big).$$

P r o o f. By differentiating (3.10) with respect to t, we have

$$(\mathcal{M}u_X)\,,v \;+\; \kappa\nabla(\mathcal{I}^\alpha\mathcal{M}u_X)\,,\nabla v \;=\; \vec{B}_5 u_X - \alpha\kappa\mathcal{I}^\alpha\nabla u_X, \nabla v$$
$$+\;\; (\mathcal{M}f_X) \;- B_6 u_X, v \;, \quad (3.11)$$

where $\vec{B}_5 \;\; = (\mathcal{M}\vec{B}_1 \;\;)$ and $B_6 \;\; = (\mathcal{M}B_2 \;\;)$. Hence,

$$(\mathcal{M}u_X)\,,v \;+\; \kappa\nabla(\mathcal{I}^\alpha\mathcal{M}u_X)\,,\nabla v \;\le \tfrac{1}{2}\;\nabla v\;^2 + \;\vec{B}_5 u_X\;^2 + \tfrac{1}{2}\;B_6 u_X\;^2$$
$$+\tfrac{1}{2}\;v\;^2 + C\;\mathcal{I}^\alpha\nabla u_X\;^2 + \;(\mathcal{M}f_X)\,,v\;.$$

Putting $v = \mathcal{I}^\alpha(\mathcal{M}u_X)$, we can cancel $\frac{1}{2}\;\nabla v\;^2$ because $v = (\mathcal{I}^\alpha\mathcal{M}u_X)$ by (2.7). Thus, by integrating in time and using (2.6) to show

$$\int_0^t (\mathcal{M}f_X)\,,\mathcal{I}^\alpha(\mathcal{M}u_X)\;\;ds \le Ct^\alpha\mathcal{Q}^0\big((\mathcal{M}f_X)\,,t\big) + \tfrac{1}{2}\mathcal{Q}_1^\alpha\big((\mathcal{M}u_X)\,,t\big),$$

and using (3.4), we arrive at the estimate

$$\mathcal{Q}_1^\alpha\big((\mathcal{M}u_X)\,,t\big) + \mathcal{Q}_2^\alpha\big((\mathcal{M}\nabla u_X)\,,t\big) \le 2\mathcal{Q}^0(\vec{B}_5 u_X, t) + \mathcal{Q}^0(B_6 u_X, t)$$
$$+\,\mathcal{Q}_2^\alpha\big((\mathcal{M}u_X)\,,t\big) + C\mathcal{Q}_2^\alpha(\nabla u_X, t) + Ct^\alpha\mathcal{Q}^0\big((\mathcal{M}f_X)\,,t\big).$$

Since

$$\vec{B}_5 u_X = (\mathcal{M}B_F^\alpha u_X)\; + (\mathcal{M}B_G^1 u_X)$$

and

$$B_6 u_X = (\mathcal{M}B_a^\alpha u_X)\; + (\mathcal{M}B_b^1 u_X)\,,$$

it follows from (2.14) that

$$\mathcal{Q}^0(\vec{B}_5 u_X, t) + \mathcal{Q}^0(B_6 u_X, t) \le C\mathcal{Q}_2^\alpha\big((\mathcal{M}u_X)\,,t\big) + C\mathcal{Q}_2^\alpha(\mathcal{M}u_X, t)$$
$$+\,C\mathcal{Q}_2^\alpha(u_X, t).$$

By Lemmas **2.4**, **3.1** and **3.2**,

$$\mathcal{Q}_2^\alpha(\mathcal{M}u_X, t) + \mathcal{Q}_2^\alpha(u_X, t) \le Ct^\alpha\mathcal{Q}_1^\alpha(\mathcal{M}u_X, t) + Ct^\alpha\mathcal{Q}_1^\alpha(u_X, t)$$
$$\le C(t^{2+2\alpha} + t^{2\alpha})\mathcal{Q}^0(f_X, t)$$

and $\mathcal{Q}_2^\alpha(\nabla u_X, t) \le Ct^\alpha\mathcal{Q}^0(f_X, t)$. Hence, the function

$$\mathsf{q}(t) = \mathcal{Q}_1^\alpha\big((\mathcal{M}u_X)\,,t\big) + \mathcal{Q}_2^\alpha\big((\mathcal{M}\nabla u_X)\,,t\big)$$

satisfies

$$\mathsf{q}(t) \le Ct^\alpha\mathcal{Q}^0(f_X, t) + Ct^\alpha\mathcal{Q}^0\big((\mathcal{M}f_X)\,,t\big) + C\mathcal{Q}_2^\alpha\big((\mathcal{M}u_X)\,,t\big).$$

Finally, by Lemma **2.2**,

$$\mathcal{Q}_2^\alpha\big((\mathcal{M}u_X)\,,t\big) \le C\int_0^t \;_\alpha(t-s)\mathcal{Q}_1^\alpha\big((\mathcal{M}u_X)\,,s\big)\,ds \le C\int_0^t \;_\alpha(t-s)\mathsf{q}(s)\,ds,$$

and the desired estimate follows by Lemma **2.5**. □

LEMMA **3.4**. *The solution u_X of* (3.1) *satisfies, for $0 \le t \le T$,*

$$\mathcal{Q}^0\big((\mathcal{M}u_X)\,,t\big) + \mathcal{Q}_1^\alpha\big((\mathcal{M}\nabla u_X)\,,t\big) \le C\mathcal{Q}^0(f_X,t) + C\mathcal{Q}^0\big((\mathcal{M}f_X)\,,t\big)$$

P r o o f. Using $-\ \vec{B}_5 u_X, \nabla v\ =\ \nabla\cdot\vec{B}_5 u_X(t), v$ in (3.11), we obtain

$$(\mathcal{M}u_X)\,, v\ +\ \kappa\mathcal{I}^\alpha(\mathcal{M}\nabla u_X)\,, \nabla v\ \le \tfrac{1}{2}\ v\ ^2 + 2\ \nabla\cdot\vec{B}_5 u_X\ ^2 + 2\ B_6 u_X\ ^2$$
$$+\ \ (\mathcal{M}f_X)\ ^2 - \alpha\ \kappa\mathcal{I}^\alpha\nabla u_X, \nabla v\ .$$

Choosing $v = (\mathcal{M}u_X)$, integrating in time, and using (3.4) yields

$$\tfrac{1}{2}\mathcal{Q}^0\big((\mathcal{M}u_X)\,,t\big) + \mathcal{Q}_1^\alpha\big((\mathcal{M}\nabla u_X)\,,t\big) \le 2\mathcal{Q}^0\big(\nabla\cdot\vec{B}_5 u_X, t\big) + 2\mathcal{Q}^0(B_6 u_X, t)$$
$$+\ \mathcal{Q}^0\big((\mathcal{M}f_X)\,,t\big) - \alpha\int_0^t\ (\mathcal{M}\nabla u_X)\,(s), \kappa\mathcal{I}^\alpha\nabla u_X(s)\ \ ds.$$

Recall from (3.9) that $\nabla\cdot B_F^\alpha\ = B^\alpha{}_{\cdot F} + B_{F\cdot}^\alpha\nabla$, where we have used the notation

$$B_{F\cdot}^\alpha\nabla\ \ = \vec{F}(t)\cdot\mathcal{I}^\alpha\nabla\ \ -\int_0^t \vec{F}\ (s)\cdot\mathcal{I}^\alpha\nabla\ \ (s)\,ds.$$

Thus,

$$\begin{aligned}\nabla\cdot\vec{B}_5 u_X = \nabla\cdot\big(\mathcal{M}\vec{B}_1 u_X\big)\ &= \big(\mathcal{M}\nabla\cdot\vec{B}_1 u_X\big)\\ &= \big(\mathcal{M}\nabla\cdot B_F^\alpha u_X\big)\ + \big(\mathcal{M}\nabla\cdot B_G^1 u_X\big)\\ &= \big(\mathcal{M}B^\alpha{}_{\cdot F} u_X\big)\ + \big(\mathcal{M}B_{F\cdot}^\alpha\nabla u_X\big)\\ &\quad + \big(\mathcal{M}B^1{}_{\cdot G} u_X\big)\ + \big(\mathcal{M}B_{G\cdot}^1\nabla u_X\big)\,,\end{aligned}$$

and so, by (2.14),

$$\mathcal{Q}^0\big(\nabla\cdot\vec{B}_5 u_X, t\big) + \mathcal{Q}^0(B_6 u_X, t) \le C\mathcal{Q}_2^\alpha\big((\mathcal{M}u_X)\,,t\big) + C\mathcal{Q}_2^\alpha(\mathcal{M}u_X, t)$$
$$+\ C\mathcal{Q}_2^\alpha(u_X, t) + C\mathcal{Q}_2^\alpha\big((\mathcal{M}\nabla u_X)\,,t\big) + C\mathcal{Q}_2^\alpha(\mathcal{M}\nabla u_X, t) + C\mathcal{Q}_2^\alpha(\nabla u_X, t).$$

By (2.3),

$$\int_0^t\ (\mathcal{M}\nabla u_X)\,(s), \kappa\mathcal{I}^\alpha\nabla u_X(s)\ \ ds \le \tfrac{1}{2}\mathcal{Q}_1^\alpha\big((\mathcal{M}\nabla u_X)\,,t\big) + C\mathcal{Q}_1^\alpha(\nabla u_X, t),$$

and thus the function $\mathsf{q}(t) = \mathcal{Q}^0\big((\mathcal{M}u_X)\,,t\big) + \mathcal{Q}_1^\alpha\big((\mathcal{M}\nabla u_X)\,,t\big)$ satisfies

$$\begin{aligned}\mathsf{q}(t) &\le C\mathcal{Q}_2^\alpha\big((\mathcal{M}u_X)\,,t\big) + C\mathcal{Q}_2^\alpha(\mathcal{M}u_X, t) + C\mathcal{Q}_2^\alpha(u_X, t)\\ &\quad + C\mathcal{Q}_2^\alpha\big((\mathcal{M}\nabla u_X)\,,t\big) + C\mathcal{Q}_2^\alpha(\mathcal{M}\nabla u_X, t) + C\mathcal{Q}_2^\alpha(\nabla u_X, t)\\ &\quad + C\mathcal{Q}^0\big((\mathcal{M}f_X)\,,t\big) + C\mathcal{Q}_1^\alpha(\nabla u_X, t)\\ &\le C\mathcal{Q}_2^\alpha\big((\mathcal{M}u_X)\,,t\big) + Ct^\alpha\mathcal{Q}_1^\alpha(\mathcal{M}u_X, t) + Ct^\alpha\mathcal{Q}_1^\alpha(u_X, t)\\ &\quad + C\mathcal{Q}_2^\alpha\big((\mathcal{M}\nabla u_X)\,,t\big) + Ct^{2+\alpha}\mathcal{Q}^0(f_X, t) + Ct^\alpha\mathcal{Q}^0(f_X, t)\\ &\quad + C\mathcal{Q}^0\big((\mathcal{M}f_X)\,,t\big) + C\mathcal{Q}^0(f_X, t),\end{aligned}$$

where, in the second step, we used Lemmas **2.2**, **3.1** and **3.2**. A further application of Lemmas **3.1** and **3.2** yields

$$\mathsf{q}(t) \le C\mathcal{Q}^0\big((\mathcal{M}f_X)\ ,t\big) + C\mathcal{Q}^0(f_X,t) + C\mathcal{Q}_2^\alpha\big((\mathcal{M}u_X)\ ,t\big) + C\mathcal{Q}_2^\alpha\big((\mathcal{M}\nabla u_X)\ ,t\big).$$

Lemma **2.2** implies that $\mathcal{Q}_2^\alpha\big((\mathcal{M}u_X)\ ,t\big) + \mathcal{Q}_2^\alpha\big((\mathcal{M}\nabla u_X)\ ,t\big)$ is bounded by

$$C \quad {}_0^t \quad {}_\alpha(t-s) \quad \mathcal{Q}_1^\alpha\big((\mathcal{M}u_X)\ ,s\big) + \mathcal{Q}_1^\alpha\big((\mathcal{M}\nabla u_X)\ ,s\big) \quad ds \le C \quad {}_0^t \quad {}_\alpha(t-s)\mathsf{q}(s)\,ds,$$

where we used $\mathcal{Q}_1^\alpha\big((\mathcal{M}u_X)\ ,s\big) \le Ct^\alpha\mathcal{Q}^0\big((\mathcal{M}u_X)\ ,s\big)$, which follows by Lemma **2.4**. Finally, Lemma **2.5** implies the desired estimate. □

The preceding lemmas yield the main result for this section.

THEOREM **3.3**. *Assume that the coefficients satisfy* (1.4), *that the initial data* $u_0 \in L_2(\Omega)$ *and that the source term satisfies* (3.2). *Then, the solution* u_X *of the projected Volterra equation* (3.1) *satisfies (with* C *independent of* X*)*

$$u_X(t) \quad ^2 + t^\alpha \quad \nabla u_X(t) \quad ^2 \le C\big(\quad u_0 \quad ^2 + M^2t^2 \quad \big) \quad \textit{for } 0 \le t \le T.$$

P r o o f. The function $\quad = \mathcal{M}u_X$ satisfies $\quad (t) \quad \le Ct^\alpha$ by Theorem **3.2** so, applying Lemma **2.3** with $\mu = 0$, we see that Lemma **3.3** gives

$$t^2 \quad u_X(t) \quad ^2 = \quad \mathcal{M}u_X(t) \quad ^2 \le Ct^{1-\alpha}\mathcal{Q}_1^\alpha\big((\mathcal{M}u_X)\ ,t\big) \le Ct\mathcal{Q}^0(f_X,t) + Ct\mathcal{Q}^0\big((\mathcal{M}f_X)\ ,t\big).$$

Define $g_X : [0,T] \quad X$ by $\quad g_X(t), v \quad = \quad g(t), v \quad$ for $v \in X$, and observe that $f_X = u_0 + \mathcal{I}^1 g_X$ and $(\mathcal{M}f_X) \quad = f_X + \mathcal{M}f_X = f_X + \mathcal{M}g_X$. We find using (3.2) that

$$\mathcal{Q}^0(f_X,t) + \mathcal{Q}^0\big((\mathcal{M}f_X)\ ,t\big) \le C \quad {}_0^t \quad u_0 \quad ^2 + \quad \mathcal{I}^1 g \quad ^2 + \quad \mathcal{M}g \quad ^2 \quad ds \le Ct\big(\quad u_0 \quad ^2 + M^2t^2 \quad \big), \tag{3.12}$$

so the estimate for the first term $\quad u_X(t) \quad ^2$ follows at once. Similarly, applying Lemma **2.3** with $\quad = \mathcal{M}\nabla u_X$ followed by Lemma **3.4**, we have

$$t^{2+\alpha} \quad \nabla u_X(t) \quad = t^\alpha \quad \mathcal{M}\nabla u_X(t) \quad ^2 \le Ct\mathcal{Q}_1^\alpha\big((\mathcal{M}\nabla u_X)\ ,t\big) \le Ct\mathcal{Q}^0(f_X,t) + Ct\mathcal{Q}^0\big((\mathcal{M}f_X)\ ,t\big),$$

implying the estimate for the second term $t^\alpha \quad \nabla u_X(t) \quad ^2$. □

4. The weak solution

We will now establish that the weak formulation (1.5) of the initial-boundary value problem (1.1)–(1.3) is well-posed. The proof relies on our estimates from Section **3** and also the following local Hölder continuity properties of u_X.

Lemma **4.1**. *If* $0 < \epsilon \le t_1 < t_2 \le T$, *then*

$$\|u_X(t_2) - u_X(t_1)\|^2 \le C\epsilon^{-2} t_2\big(\|u_0\|^2 + M^2 t_2^2\big)(t_2 - t_1)$$

and

$$\|\mathcal{I}^\alpha \nabla u_X(t_2) - \mathcal{I}^\alpha \nabla u_X(t_1)\| \le C\big(\|u_0\| + Mt_2\big)\big[\epsilon^{\alpha-2}(t_2 - t_1) + \epsilon^{-\alpha/2}(t_2 - t_1)^\alpha\big].$$

P r o o f. The Cauchy–Schwarz inequality implies that

$$\|u_X(t_2) - u_X(t_1)\|^2 = \Big\|\int_{t_1}^{t_2} u_X(s)\,ds\Big\|^2 \le (t_2 - t_1)\int_{t_1}^{t_2} \|u_X(s)\|^2\,ds,$$

and by the second inequality of Lemma **3.1**, together with Lemma **3.4**,

$$\begin{aligned}
\int_{t_1}^{t_2} \|u_X(s)\|^2\,ds &= \int_{t_1}^{t_2} s^{-2}\|(\mathcal{M}u_X)(s) - u_X(s)\|^2\,ds \\
&\le 2\epsilon^{-2}\int_0^{t_2}\big(\|(\mathcal{M}u_X)\|^2 + \|u_X\|^2\big)\,ds \\
&= 2\epsilon^{-2}\big[\mathcal{Q}^0\big((\mathcal{M}u_X), t_2\big) + \mathcal{Q}^0(u_X, t_2)\big] \\
&\le C\epsilon^{-2}\big[\mathcal{Q}^0\big(\mathcal{M}f_X), t_2\big) + \mathcal{Q}^0(f_X, t_2)\big].
\end{aligned}$$

The first result now follows from (3.12). To prove the second, we write

$$\mathcal{I}^\alpha \nabla u_X(t_2) - \mathcal{I}^\alpha \nabla u_X(t_1) = \int_0^{t_1-\epsilon/2} \big[\omega_\alpha(t_2 - s) - \omega_\alpha(t_1 - s)\big]\nabla u_X(s)\,ds$$

$$+ \int_{t_1-\epsilon/2}^{t_1} \big[\omega_\alpha(t_2 - s) - \omega_\alpha(t_1 - s)\big]\nabla u_X(s)\,ds + \int_{t_1}^{t_2} \omega_\alpha(t_2 - s)\nabla u_X(s)\,ds,$$

and deduce from Theorem **3.3** that

$$\|\mathcal{I}^\alpha \nabla u_X(t_2) - \mathcal{I}^\alpha \nabla u_X(t_1)\| \le C\big(\|u_0\| + Mt_2\big)\big(I_1 + I_2 + I_3\big),$$

where

$$I_1 = {}_{0}^{t_1- \ /2} \quad {}_{\alpha}(t_1 - s) - \ {}_{\alpha}(t_2 - s) \ s^{-\alpha/2}\, ds,$$

$$I_2 = {}_{t_1- \ /2}^{t_1} \quad {}_{\alpha}(t_1 - s) - \ {}_{\alpha}(t_2 - s) \ s^{-\alpha/2}\, ds,$$

$$I_3 = {}_{t_1}^{t_2} \quad {}_{\alpha}(t_2 - s)s^{-\alpha/2}\, ds.$$

By the mean value theorem,

$${}_{\alpha}(t_1 - s) - \ {}_{\alpha}(t_2 - s) = (t_2 - t_1)| \ {}_{\alpha-1}(\ )| \quad \text{with} \quad t_1 - s < \quad < t_2 - s,$$

and if $0 < s < t_1 - \ /2$ then $t_1 - s > \ /2$ so

$$\begin{aligned} I_1 &\le (t_2 - t_1)| \ {}_{\alpha-1}(\ /2)| \ {}_{0}^{t_1- \ /2} \frac{ds}{s^{\alpha/2}} \\ &\le \ \frac{2}{\ }^{2-\alpha} \ \frac{1-\alpha}{1-\alpha/2}\, \frac{(t_1 - \ /2)^{1-\alpha/2}}{(\alpha)}\, (t_2 - t_1). \end{aligned}$$

Moreover,

$$\begin{aligned} I_2 &\le (\ /2)^{-\alpha/2} \ {}_{t_1- \ /2}^{t_1} \quad {}_{\alpha}(t_1 - s) - \ {}_{\alpha}(t_2 - s) \ ds \\ &= (2/ \ )^{\alpha/2} \quad {}_{\alpha+1}(t_2 - t_1) + \ {}_{\alpha+1}(\ /2) - \ {}_{\alpha+1}(t_2 - t_1 + \ /2) \\ &\le (2/ \ )^{\alpha/2} \ {}_{\alpha+1}(t_2 - t_1) \end{aligned}$$

and $I_3 \le \ ^{-\alpha/2} \ {}_{t_1}^{t_2} \ {}_{\alpha}(t_2 - s)\, ds = \ ^{-\alpha/2} \ {}_{\alpha+1}(t_2 - t_1)$. □

Our existence theorem is stated as follows. Note the weak continuity at $t = 0$ asserted in part 5; we show in the companion paper [29] that the solution u is continuous on the closed interval $[0, T]$ provided $u_0 \in H^{\mu}(\Omega)$ for some $\mu > 0$.

THEOREM **4.1**. *Assume that the coefficients satisfy* (1.4), *that the source term satisfies* (3.2), *and that the initial data* $u_0 \in L_2(\Omega)$. *Then, the initial-boundary value problem* (1.1)–(1.3) *has a weak solution* u. *More precisely, there exists a function* $u : [0, T] \quad L_2(\Omega)$ *with the following properties*:

(1) *The restriction* $u : (0, T] \quad L_2(\Omega)$ *is continuous.*

(2) *If* $0 < t \le T$, *then* $u(t) \in H_0^1(\Omega)$ *with*

$$u(t) \ + t^{\alpha/2} \ \nabla u(t) \ \le C\big(\ u_0 \ + Mt \ \big).$$

(3) *The functions* $\mathcal{I}^\alpha u$ *and* $B_2 u$ *are continuous from the closed interval* $[0,T]$ *to* $L_2(\Omega)$*. Likewise,* $\mathcal{I}^\alpha \nabla u$ *and* $\vec{B}_1 u$ *are continuous from* $[0,T]$ *to* $L_2(\Omega)^d$*.*

(4) *At* $t=0$ *we have* $\mathcal{I}^\alpha u = B_2 u = 0$*,* $\mathcal{I}^\alpha \nabla u = \vec{B}_1 u = 0$ *and* $u(0)=u_0$*.*

(5) *If* $t \quad 0$*, then* $u(t), v \qquad u(0), v$ *for each* $v \in L_2(\Omega)$*.*

() *For* $0 \le t \le T$ *and* $v \in H_0^1(\Omega)$*,*

$$u(t), v \;+\; \kappa(\mathcal{I}^\alpha \nabla u)(t), \nabla v \;-\; (\vec{B}_1 u)(t), \nabla v \;+\; (B_2 u)(t), v \;=\; f(t), v \,. \tag{4.1}$$

P r o o f. Let $\;_1, \;_2, \;_3, \ldots$ be a sequence of functions spanning a dense subspace of $H_0^1(\Omega)$. For each integer $n \ge 1$, let $X_n = \operatorname{span}\{\;_1, \;_2, \ldots, \;_n\}$ and for brevity denote the solution of (3.7) with $X = X_n$ by $u_n = u_X$, and likewise write $f_n = f_X$, so that

$$u_n(t), v \,+\, \kappa(\mathcal{I}^\alpha \nabla u_n)(t), \nabla v \,-\, (B_1 u_n)(t), \nabla v \,+\, (B_2 u_n)(t), v \;=\; f_n(t), v \tag{4.2}$$

for $v \in X_n$ and $0 < t \le T$. We see from Theorem **3.3** and Lemma **4.1** that, whenever $0 < \quad < T$, the sequence of functions u_n is bounded and equicontinuous in $C\big([\;,T]; L_2(\Omega)\big)$. By choosing a sequence of values of tending to zero we can select a subsequence, again denoted by u_n, such that $u_n(t)$ converges in $L_2(\Omega)$ for $0 < t \le T$. We may therefore define

$$u(t) = \lim_{n} u_n(t) \quad \text{for } 0 < t \le T,$$

and this function satisfies Property 1 because, given any fixed $\;\in (0,T)$, the limit is uniform for $t \in [\;,T]$. Similarly, the functions $\mathcal{I}^\alpha \nabla u_n$ are bounded and equicontinuous in $C\big([\;,T]; L_2(\Omega)^d\big)$ so $\mathcal{I}^\alpha \nabla u : (0,T] \qquad L_2(\Omega)^d$ is continuous. In fact, it will follow from (4.4) below that $\mathcal{I}^\alpha \nabla u(t) \qquad 0$ as $t \quad 0$, so $\mathcal{I}^\alpha \nabla u : [0,T] \qquad L_2(\Omega)^d$ is continuous.

By Theorem **3.3**,

$$u_n(t) \;\le C\big(\; u_0 \;+ Mt\;\big) \quad \text{for } 0 < t \le T,$$

so by sending $n \qquad$ we conclude that $u(t) \;\le C\big(\; u_0 \;+ Mt\;\big)$. Also, for $0 < t \le T$,

$$|\; u_n(t), v\;| \le C \; u_n(t) \;_{H_0^1(\;)} \; v \;_{H^{-1}(\;)} \le Ct^{-\alpha/2}\big(\; u_0 \;+ Mt\;\big) \; v \;_{H^{-1}(\;)}$$

and sending $n \qquad$ it follows that

$$|\; u(t), v\;| \le Ct^{-\alpha/2}\big(\; u_0 \;+ Mt\;\big) \; v \;_{H^{-1}(\;)} \quad \text{for all } v \in L_2(\Omega),$$

so $u(t) \in H_0^1(\Omega)$ with $\; u(t) \;_{H_0^1(\;)} \le Ct^{-\alpha/2}\big(\; u_0 \;+ Mt\;\big)$, establishing Property 2.

Since $\; u(t)\;$ is bounded, $\mathcal{I}^\alpha u$ is continuous on $[0,T]$ with

$$\begin{aligned}\|\mathcal{I}^\alpha u(t)\| &\le \int_0^t \omega_\alpha(t-s)\,\|u(s)\|\,ds\\ &\le C\int_0^t (t-s)^{\alpha-1}\big(\|u_0\|+Ms\big)\,ds \le C\big(\|u_0\|+Mt\big)t^\alpha,\end{aligned}\tag{4.3}$$

and similarly

$$\|\mathcal{I}^\alpha \nabla u(t)\| \le C\int_0^t (t-s)^{\alpha-1}s^{-\alpha/2}\big(\|u_0\|+Ms\big)\,ds \le C\big(\|u_0\|+Mt\big)t^{\alpha/2}.\tag{4.4}$$

Likewise, for $n \ge 1$,

$$\|\mathcal{I}^\alpha u_n(t)\| \le C\big(\|u_0\|+Mt\big)t^\alpha \quad \text{and} \quad \|\mathcal{I}^\alpha \nabla u_n(t)\| \le C\big(\|u_0\|+Mt\big)t^{\alpha/2}.\tag{4.5}$$

Continuity of $\vec{B}_1 u$ and $B_2 u$ follow from (2.11) and (3.6), completing the proof of property 3, with

$$\begin{aligned}\|(\vec{B}_1 u)(t)\| + \|(B_2 u)(t)\| &\le C\|(\mathcal{I}^\alpha u)(t)\| + C\int_0^t \big(\|(\mathcal{I}^\alpha u)(s)\| + \|u(s)\|\big)\,ds\\ &\le C\big(\|u_0\|+M\big)t^\alpha.\end{aligned}\tag{4.6}$$

Property 4 follows from the estimates (4.3), (4.4) and (4.6).

If $0 \le \epsilon < t \le T$, then

$$\begin{aligned}\|(\mathcal{I}^\alpha u_n)(t) - (\mathcal{I}^\alpha u)(t)\| &\le \int_0^t \omega_\alpha(t-s)\,\|u_n(s)-u(s)\|\,ds\\ &\le C\Big(\int_0^\epsilon (t-s)^{\alpha-1}\big(\|u_0\|+Ms\big)\,ds + \int_\epsilon^t (t-s)^{\alpha-1}\|u_n(s)-u(s)\|\,ds\Big)\\ &\le C\Big(\epsilon^\alpha\big(\|u_0\|+M\epsilon\big) + \alpha^{-1}(t-\epsilon)^\alpha \max_{\epsilon\le s\le t}\|u_n(s)-u(s)\|\Big),\end{aligned}$$

showing that $\mathcal{I}^\alpha u_n(t) \to \mathcal{I}^\alpha u(t)$ in $L_2(\Omega)$, uniformly for $t \in [\epsilon, T]$. In fact, the convergence is uniform for $t \in [0,T]$, owing to the estimates (4.3) and (4.5). Therefore, we see using (2.11) and (3.6) that, for $v \in H_0^1(\Omega)$,

$$\langle(\vec{B}_1 u_n)(t), \nabla v\rangle \to \langle(\vec{B}_1 u)(t), \nabla v\rangle \quad \text{and} \quad \langle(B_2 u_n)(t), v\rangle \to \langle(B_2 u)(t), v\rangle.$$

Since $\langle f_n, \phi_j\rangle = \langle f, \phi_j\rangle$ for $j \le n$, we have

$$\lim_{n\to\infty}\langle f_n(t), \phi_j\rangle = \langle f(t), \phi_j\rangle \quad \text{for all } j \ge 1 \text{ and } 0 \le t \le T,$$

and therefore $\langle f_n(t), v\rangle \to \langle f(t), v\rangle$ for all $v \in L_2(\Omega)$. Thus, by sending $n \to \infty$ in (4.2), it follows that (4.1) holds for $v \in H_0^1(\Omega)$ and $0 < t \le T$. In light of (4.6) and (4.4), the variational equation (4.1) is satisfied when $t = 0$ if and only if $\langle u(0), v\rangle = \langle u_0, v\rangle$ for all $v \in H_0^1(\Omega)$, which is the case if and only if we define $u(0) = u_0$. Moreover, if $t \to 0$ then $\langle u(t), v\rangle \to \langle f(0), v\rangle =$

$\langle u_0, v\rangle$, for each $v \in H_0^1(\Omega)$, and hence by density for each $v \in L_2(\Omega)$, establishing Properties 5 and 6. □

REMARK **4.1**. Since our estimates rely on Lemma **2.1**, the constant C in part 2 of Theorem **4.1** becomes unbounded as $\alpha \to 1$. However, this behavior appears to be an artifact of our method of proof. In the limiting case when $\alpha = 1$ and (1.1) reduces to a parabolic PDE, a simple energy argument combined with the classical Gronwall inequality yields the *a priori* estimate

$$\|u(t)\| \le C\left(\|u_0\| + \int_0^t \|g(s)\|\,ds\right) \quad \text{for } 0 \le t \le T;$$

see also the alternative analysis [21] of the fractional Fokker–Planck equation.

THEOREM **4.2**. *The weak solution of the initial-boundary value problem* (1.1)–(1.3) *is unique. More precisely, under the same assumptions as Theorem* **4.1**, *there is at most one function u that satisfies* (4.1) *and is such that u and $\mathcal{I}^\alpha u$ belong to $L_2\big((0,T); L_2(\Omega)\big)$, and $\mathcal{I}^\alpha \nabla u$ belongs to $L_2\big((0,T); L_2(\Omega)^d\big)$.*

P r o o f. The problem is linear, so it suffices to show that if $u_0 = 0$ and $g(t) \equiv 0$ then $u(t) \equiv 0$. Thus, suppose that

$$\langle u'(t), v\rangle + \langle \kappa(\mathcal{I}^\alpha \nabla u)(t), \nabla v\rangle - \langle (\vec{B}_1 u)(t), \nabla v\rangle + \langle (B_2 u)(t), v\rangle = 0$$

for $0 < t \le T$ and $v \in H_0^1(\Omega)$. Proceeding as in the proof of (3.8), we have

$$\mathcal{Q}_1^\alpha(u,t) + \tfrac{1}{2}\mathcal{Q}_2^\alpha(\nabla u,t) \le \tfrac{1}{2}\mathcal{Q}^0(\vec{B}_1 u,t) + \tfrac{1}{2}\mathcal{Q}^0(B_2 u,t) + \tfrac{1}{2}\mathcal{Q}_2^\alpha(u,t)$$
$$\le C\mathcal{Q}_2^\alpha(u,t),$$

where the final step used (2.11), (2.12) and Lemma **2.4**. Thus, applying Lemma **2.2**, the function $\mathsf{q}(t) = \mathcal{Q}_1^\alpha(u,t) + \mathcal{Q}_2^\alpha(\nabla u,t)$ satisfies

$$\mathsf{q}(t) \le C\mathcal{Q}_2^\alpha(\nabla u,t) \le C\int_0^t \omega_\alpha(t-s)\mathsf{q}(s)\,ds,$$

and therefore $\mathsf{q}(t) = 0$ for $0 \le t \le T$ by Lemma **2.5**. In particular, $\mathcal{Q}_1^\alpha(u,T) = 0$, so if we put $u(t) = 0$ for $t > T$ then the Laplace transform of u satisfies $\hat{u}(iy) = 0$ for $-\infty < y < \infty$ by (2.1), implying that $u(t) = 0$ for $0 \le t \le T$. □

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References

[1] E. A. Abdel-Rehim, Implicit difference scheme of the space-time fractional advection diffusion equation. *Fract. Calc. Appl. Anal.* **18**, No 6 (2015), 1452–1469; DOI: 10.1515/fca-2015-0084; https://www.degruyter.com/view/j/fca.2015.18.issue-6/issue-files/fca.2015.18.issue-6.xml.

[2] A. A. Alikhanov, A priori estimates for solutions of boundary value problems for fractional-order equations. *Di er. Equ.* **46** (2010), 660–666; DOI: 10.1134/S0012266110050058.

[3] A. A. Alikhanov, Boundary value problems for the diffusion equation of the variable order in differential and difference settings. *Appl. Math. Comput.* **219** (2012), 3938–3946; DOI: 10.1016/j.amc.2012.10.029.

[4] C. N. Angstmann, B. I. Henry, B. A. Jacobs, and A. V. McGann, A time-fractional generalised advection equation from a stochastic process. *Chaos, Solitons and Fractals* **102** (2017), 175–183; DOI: 10.1016/j.chaos.2017.04.040.

[5] L. C. Becker, Resolvents and solutions of weakly singular linear Volterra integral equations. *Nonlinear Anal.* **74** (2011), 1892–1912; DOI: 10.1016/j.na.2010.10.060.

[6] H. Brunner, *Volterra Integral Equations: an Introduction to Theory and Applications*. Cambridge University Press (2017); DOI: 10.1017/9781316162491.

[7] H. Brunner, A. Pedas, and G. Vainikko, The piecewise polynomial collocation method for nonlinear weakly singular Volterra equations. *Math. Comp.* **68** (1999), 1079–1095; DOI: 10.1090/S0025-5718-99-01073-X.

[8] J. Cao, Ch. Li, and Y.-Q. Chen, High-order approximation to Caputo derivatives and Caputo-type advection-diffusion equations (ii). *Fract. Calc. Appl. Anal.* **18**, No 3 (2015), 735–761; DOI: 10.1515/fca-2015-0045; https://www.degruyter.com/view/j/fca.2015.18.issue-3/issue-files/fca.2015.18.issue-3.xml.

[9] J. Dixon and S. McKee, Weakly singular Gronwall inequalities. *ZAMM Z. Angew. Math. Mech.* **66** (1986), 535–544; DOI: 10.1002/zamm.19860661107.

[10] B. I. Henry, T. A. M. Langlands, and P. Straka, Fractional Fokker–Planck equations for subdiffusion with space- and time-dependent forces. *Phys. Rev. Lett.* **105** (2010), 170602; DOI: 10.1103/PhysRevLett.105.170602.

[11] B. I. Henry, T. A. M. Langlands, and S. L. Wearne, Anomalous diffusion with linear reaction dynamics: From continuous time random walks to fractional reaction-diffusion equations. *Phys. Rev. E* **74** (2006), 031116; DOI: 10.1103/PhysRevE.74.031116.

[12] B. I. Henry and S. L. Wearne, Fractional reaction-diffusion. *Phys. A* **276** (2000), 448–455; DOI: 10.1016/S0378-4371(99)00469-0.

[13] B. Jin, B. Li, and Z. Zhou, Discrete maximal regularity of time-stepping schemes for fractional evolution equations. *Numer. Math.* **138** (2018), 101–131; DOI: 10.1007/s00211-017-0904-8.

[14] S. Karaa and A. K. Pani, Error analysis of a FVEM for fractional order evolution equations with nonsmooth initial data. *ESAIM: M2AN* **52**, No 2 (2018), 773–801; DOI: 10.1051/m2an/2018029.

[15] J. Kemppainen, Existence and uniqueness of the solution for a time-fractional diffusion equation. *Fract. Calc. Appl. Anal.* **14**, No 3 (2011), 411–417; DOI: 10.2478/s13540-011-0025-5; https://www.degruyter.com/view/j/fca.2011.14.issue-3/issue-files/fca.2011.14.issue-3.xml.

[16] J. Klafter and I. M. Sokolov, *First Steps in Random Walks.* Oxford University Press (2011); DOI: 10.1093/acprof:oso/9780199234868.001.0001.

[17] A. Kubica and M. Yamamoto, Initial-boundary value problems for fractional diffusion equations with time-dependent coefficients. *Fract. Calc. Appl. Anal.* **21**, No 2 (2018), 276–311; DOI: 10.1515/fca-2018-0018; https://www.degruyter.com/view/j/fca.2018.21.issue-2/issue-files/fca.2018.21.issue-2.xml.

[18] T. A. M. Langlands and B. I. Henry, The accuracy and stability of an implicit solution method for the fractional diffusion equation. *J. Comput. Phys.* **205**, No 2 (2005), 719–736; DOI: 10.1016/j.jcp.2004.11.025.

[19] T. A. M. Langlands, B. I. Henry, and S. L. Wearne, Fractional cable equation models for anomalous electrodiffusion in nerve cells. *SIAM J. Appl. Math.* **71**, No 4 (2011), 1168–1203; DOI: 10.1137/090775920.

[20] Kim Ngan Le, W. McLean, and K. Mustapha, A semidiscrete finite element approximation of a time-fractional Fokker–Planck equation with non-smooth initial data. *SIAM J. Sci. Comput.* **40**, No 6, (2018), A3831–3852; DOI: 10.1137/17M1125261.

[21] Kim Ngan Le, W. McLean, and M. Stynes, Existence, uniqueness and regularity of the solution of the time-fractional Fokker–Planck equation

with general forcing. *Commun. Pure Appl. Anal.* (to appear); DOI: 10.13140/RG.2.2.30053.24801.

[22] Y. Li and Q. Zhang, Blow-up and global existence of solutions for a time fractional diffusion equation. *Fract. Calc. Appl. Anal.* **21**, No 6 (2018), 1619–1640; DOI: 10.1515/fca-2018-0085; https://www.degruyter.com/view/j/fca.2018.21.issue-6/issue-files/fca.2018.21.issue-6.xml.

[23] H.-L. Liao, D. Li, and J. Zhang, Sharp error estimate of the nonuniform l1 formula for linear reaction-subdiffusion equations. *SIAM J. Numer. Anal.* **66**, No 2 (2018), 1112–1133; DOI: 10.1137/17M1131829.

[24] P. Linz, *Analytical and Numerical Methods for Volterra Equations.* Ser. Studies in Applied and Numerical Mathematics, SIAM, Philadelphia (1985); DOI: 10.1137/1.9781611970852.

[25] F. Liu, V. V. Anh, I. Turner, and P. Zhuang, Time fractional advection-dispersion equation. *J. Appl. Math. Computing* **13** (2003), 233–245; DOI: 10.1007/BF02936089.

[26] Ch. Lubich, Runge–Kutta theory for Volterra and Abel integral equations of the second kind. *Math. Comp.* **41** (1983), 87–102; DOI: 10.1090/S0025-5718-1983-0701626-6.

[27] Y. Luchko and M. Yamamoto, General time-fractional diffusion equation: some uniqueness and existence results for the initial-boundary-value problems. *Fract. Calc. Appl. Anal.* **19**, No 3 (2016), 676–695; DOI: 10.1515/fca-2016-0036; https://www.degruyter.com/view/j/fca.2016.19.issue-3/issue-files/fca.2016.19.issue-3.xml.

[28] W. McLean, Regularity of solutions to a time-fractional diffusion equation. *ANZIAM J.* **52** (2010), 123–138; DOI: 10.1017/S1446181111000617.

[29] W. McLean, K. Mustapha, R. Ali, and O. M. Knio, Regularity theory for time–fractional advection–diffusion–reaction equations. *Comput. Math. Appl.* (2019), Available online Aug. 2019 (In press); DOI: 10.1016/j.camwa.2019.08.008.

[30] R. Metzler, E. Barkai, and J. Klafter, Deriving fractional Fokker–Planck equations from a generalised master equation. *Europhys. Lett.* **46** (1999), 431–436; DOI: 10.1209/epl/i1999-00279-7.

[31] K. Mustapha, Time-stepping discontinuous Galerkin methods for fractional diffusion problems. *Numer. Math.* **130**, No 3 (2015), 497–516; DOI: 10.1007/s00211-014-0669-2.

[32] J. A. Nohel and D. F. Shea, Frequency domain methods for Volterra equations. *Adv. Math.* **22** (1976), 278–304; DOI: 10.1016/0001-8708(76)90096-7.

[33] I. Podlubny, *Fractional Di erential Equations.* Academic Press, San Diego (1999).

[34] K. Sakamoto and M. Yamamoto, Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems. *J. Math. Anal. Appl.* **382**, No 1 (2011), 426–447; DOI: 10.1016/j.jmaa.2011.04.058.

[35] C.-S. Sin and L. Zheng, Existence and uniqueness of global solutions of Caputo-type fractional differential equations. *Fract. Calc. Appl. Anal.* **19**, No 3 (2016), 765–774. DOI: 10.1515/fca-2016-0040; https://www.degruyter.com/view/j/fca.2016.19.issue-3/issue-files/fca.2016.19.issue-3.xml.

[36] M. Stynes, E. O Riordan, and J. Luis Gracia, Error analysis of a finite difference method on graded meshes for a time-fractional diffusion equation. *SIAM J. Numer. Anal.* **55**, No 2 (2017), 1057–1079; DOI: 10.1137/16M1082329.

[37] G. Vainikko, *Weakly Singular Integral Equations.* Lecture Notes, University of Tartu, Helsinki University of Technology (2006–2007).

[38] S. B. Yuste and L. Acedo, An explicit finite difference method and a new von Neumann stability analysis for fractional diffusion equations. *SIAM. J. Numer. Anal.* **42**, No 5 (2005), 1862–1874; DOI: 10.1137/030602666.

[39] R. Zacher, Weak solutions of abstract evolutionary integro-differential equations in Hilbert spaces. *Funkcial. Ekvac.* **52** (2009), 1–18; DOI: 10.1619/fesi.52.1.

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