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**RESEARCH PAPER**

# ON SOLUTIONS OF LINEAR FRACTIONAL DIFFERENTIAL EQUATIONS AND SYSTEMS THEREOF

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## Abstract

We derive exact solutions to classes of linear fractional differential equations and systems thereof expressed in terms of generalized Wright functions and Fox $H$-functions. These solutions are invariant solutions of diffusion-wave equations obtained through certain transformations, which are briefly discussed. We show that the solutions given in this work contain previously known results as particular cases.

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*Key Words and Phrases*: fractional differential equations, exact solutions, generalized Wright function, Fox $H$-function

## 1. Introduction

For the last several decades, the application of fractional differentiation to the mathematical modeling of physical problems has become increasingly common [6, 14, 15]. In particular, anomalous diffusion processes in complex systems, from charge transport in amorphous semiconductors to bacterial motion, have been successfully modeled with fractional diffusion-wave equations [12].

In this work, we derive exact solutions expressed in terms of well-known special functions for fractional ordinary differential equations (FODEs) of


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the form

$$\frac{d^\alpha}{dz^\alpha}\ (z) = \frac{a_m}{\ \ m} z^m \frac{d^m}{dz^m}\ (z) + \frac{a_{m-1}}{\ \ m-1} z^{m-1} \frac{d^{m-1}}{dz^{m-1}}\ (z) + \cdots + a_0\ (z), \quad (1.1)$$

where $\ \ \mathbb{R}_+$, $a_i \ \ \mathbb{R}$ $(i = 0, \ldots, m)$ and $a_m = 0$, and for systems of the form

$$\begin{cases} \frac{d^\alpha}{dz^\alpha}\ (z) = \frac{a_{m_1}}{\alpha^{m_1}} z^{m_1} \frac{d^{m_1}}{dz^{m_1}}\ (z) + \frac{a_{m_1-1}}{\alpha^{m_1-1}} z^{m_1-1} \frac{d^{m_1-1}}{dz^{m_1-1}}\ (z) + \cdots + a_0\ (z), \\ \frac{d^\alpha}{dz^\alpha}\ (z) = \frac{b_{m_2}}{\alpha^{m_2}} z^{m_2} \frac{d^{m_2}}{dz^{m_2}}\ (z) + \frac{b_{m_2-1}}{\alpha^{m_2-1}} z^{m_2-1} \frac{d^{m_2-1}}{dz^{m_2-1}}\ (z) + \cdots + b_0\ (z), \end{cases} \quad (1.2)$$

where $\ \ \mathbb{R}_+$, $a_i, b_j \ \ \mathbb{R}$ $(i = 0, \ldots, m_1; j = 0, \ldots, m_2)$ and $a_{m_1} b_{m_2} = 0$. Here, for $\ \ \mathbb{R}_+$, fractional differentiation is defined in the Riemann-Liouville manner:

$$\frac{d^\alpha}{dz^\alpha}\ (z) := \begin{cases} \frac{d^n}{dz^n}\ (z), & \ = n \ \ \mathbb{N}, \\ \frac{1}{\ (n-\alpha)} \frac{d^n}{dz^n} \int_0^z (z \ \ s)^{n-\alpha-1}\ (s) ds, & \ \ (n \ \ 1, n) \text{ with } n \ \ \mathbb{N}. \end{cases}$$

Throughout this work, we consider $n \ \ \mathbb{N}$ satisfying $0 \ \ n \ \ 1 < \ < n$.

It is interesting to consider the forms taken by (1.1) and (1.2) in the particular cases that $m = 2$ and $m_1 = m_2 = 1$, because these are the cases most commonly considered in scientific and engineering fields. In these cases, we obtain the FODE

$$\frac{d^\alpha}{dz^\alpha}\ (z) = a\ (z) + \frac{b}{\ } z \frac{d}{dz}\ (z) + \frac{c}{\ \ 2} z^2 \frac{d^2}{dz^2}\ (z), \text{ where } a, b, c \ \ \mathbb{R} \quad (1.3)$$

and the system of FODEs

$$\begin{cases} \frac{d^\alpha}{dz^\alpha}\ (z) = a_1\ (z) + \frac{b_1}{\alpha} z \frac{d}{dz}\ (z), \\ \frac{d^\alpha}{dz^\alpha}\ (z) = a_2\ (z) + \frac{b_2}{\alpha} z \frac{d}{dz}\ (z), \end{cases} \quad \text{where } a_1, a_2, b_1, b_2 \ \ \mathbb{R}. \quad (1.4)$$

We can derive (1.3) and (1.4) from the fractional diffusion-wave equation

$$\frac{\ ^\alpha u}{\ t^\alpha} = c(x)^2 u_{xx} \quad (1.5)$$

and the system

$$\begin{cases} \frac{\partial^\alpha u}{\partial t^\alpha} = c(x)^2 v_x, \\ \frac{\partial^\alpha v}{\partial t^\alpha} = u_x \end{cases} \quad (1.6)$$

with variable diffusion coe cient $c(x) = A(x + B)^k$ or with $c(x) = Ae^{kx}$, where $A$, $B$ and $k$ are real constants, through scaling transformations with similarity variable $z = (x + B)^{\frac{s}{\alpha}} t$ (where $s$ is suitably chosen real number) or with $z = e^{\frac{k}{\alpha} x} t$. Thus, we can obtain exact invariant solutions to (1.5) and (1.6) by obtaining exact solutions to (1.3) and (1.4), respectively.

In the following Section 2, we briefly introduce the special functions that are used to express the solutions of (1.1) and (1.2). In Section 3, we present the solutions of (1.3) and (1.4) in two propositions. The results obtained there hint at a general pattern for the solutions to (1.1) and (1.2). In Section 4, we explicitly identify this pattern and derive exact solutions to (1.1) and (1.2) using the roots of the characteristic polynomials of the right-hand sides of (1.1) and (1.2) in analogy to the well-known method of solving Cauchy-Euler differential equations. In Section 5, we derive exact solutions of diffusion-wave equations and systems with variable coe cients using the results obtained in Section 3 and show that our solutions correspond to known solutions in some cases.

## 2. **Preliminaries**

We express the solutions of (1.3) and (1.4) in terms of generalized Wright functions and Fox $H$-functions, which are defined as follows:

Definition 2.1. 1) The Wright function is an entire function, defined by

$$(z;\ ,\ ) = \sum_{i=0}^{\infty} \frac{z^i}{i!\ (\ i+\ )},$$

for $z$ $\mathbb{C}$ and for real satisfying $>$ 1 and $\mathbb{C}$ [4].

2) The generalized Wright function

$$ {}_p\ {}_q\left[z\left|\begin{matrix}(A_i,\ {}_i)_{1,p}\\(B_j,\ {}_j)_{1,q}\end{matrix}\right.\right] = \sum_{k=0}^{\infty}\frac{\prod\limits_{i=1}^{p}\ (A_i+\ {}_ik)}{\prod\limits_{j=1}^{q}\ (B_j+\ {}_jk)}\frac{z^k}{k!},$$

is defined for $z$ $\mathbb{C}, p, q$ $\mathbb{N}_0 = \{0, 1, 2, \ldots\}, A_i, B_j$ $\mathbb{C}$ and ${}_i,\ {}_j$ $\mathbb{R}\backslash\{0\}$ $(i = 1, \ldots, p; j = 1, \ldots, q)$, is absolutely convergent, and thus it is an entire function for $= \sum\limits_{j=1}^{q}\ {}_j\ \sum\limits_{i=1}^{p}\ {}_i >$ 1, [5].

3) The Fox $H$-function

$$H^{m,l}_{p,q}\left[z\left|\begin{matrix}(A_i,\ {}_i)_{1,p}\\(B_j,\ {}_j)_{1,q}\end{matrix}\right.\right] = \frac{1}{2\ i}\int_L \mathcal{H}^{m,l}_{p,q}(s)z^s ds,$$

$$\text{with}\quad \mathcal{H}^{m,l}_{p,q}(s) = \frac{\prod\limits_{j=1}^{m}\ (B_j\quad {}_js)\prod\limits_{i=1}^{l}\ (1\quad A_i+\ {}_is)}{\prod\limits_{i=l+1}^{p}\ (A_i\quad {}_is)\prod\limits_{j=m+1}^{q}\ (1\quad B_j+\ {}_js)},$$

is defined for $z$ $\mathbb{C}\setminus\{0\}$, $m, l, p, q$ $\mathbb{N}_0$ with $(m, l) = (0, 0)$, ${}_i,\ {}_j$ $\mathbb{R}_+$ and $A_i, B_j$ $\mathbb{R}$ $(i = 1, \ldots, p; j = 1, \ldots, q)$. The contour $L$ separates the poles of the gamma functions $(B_j\quad {}_js)$ $(j = 1, \ldots, m)$ from the poles of the gamma functions $(1\quad A_i+\ {}_is)$ $(i = 1, \ldots, l)$, [9].

In this work, we take $L$ as $L_{\gamma+i\infty}$, a contour that extends from the point $i$ to the point $+i$ , where is chosen such that $L$ separates the poles as stated above. The above integral converges under the conditions ([10])

$$\mu = \sum_{i=1}^{l} {}_i \quad \sum_{i=l+1}^{p} {}_i + \sum_{j=1}^{m} {}_j \quad \sum_{j=m+1}^{q} {}_j > 0 \text{ and } |\arg z| < \frac{\mu}{2}.$$

With regard to expressions for solutions to (1.1) and (1.2), we are particularly interested in the case $l = 0$ of the $H$-function. In this case, the $H$-function vanishes exponentially for large $z$, [9].

Let us formulate the following known results as lemmas for the generalized Wright functions [5, 7] and Fox $H$-functions [10, 3] for the convenience to present our results.

LEMMA 2.1. *Let* $= \sum_{j=1}^{q} B_j \quad \sum_{i=1}^{p} A_i > \quad 1$. *Then the following equalities hold for* $\mathbb{R}_+$ *and* $a \quad \mathbb{R}$.

*(1) If* ${}_1 > 0$, $B_1 > 0$ *and* $A_1 = \ {}_1 = 1$, *then we have*

$$\frac{d^\alpha}{dz^\alpha}\left(z^{B_1-1} {}_p \quad {}_q \left[az^{\beta_1}\middle| \begin{array}{c} (1,1),(A_i, \ {}_i)_{2,p} \\ (B_j, \ {}_j)_{1,q} \end{array}\right]\right)$$
$$= \quad a^m z^{B_1+m\beta_1-1-\alpha} {}_p \quad {}_q \left[az^{\beta_1}\middle| \begin{array}{c} (1,1),(A_i+m \ {}_i, \ {}_i)_{2,p} \\ (B_1+m \ {}_1 \qquad , \ {}_1),(B_j+m \ {}_j, \ {}_j)_{2,q} \end{array}\right],$$

*where $m$ is the smallest non-negative integer such that $B_1 + m \ {}_1 \qquad 1$ is not a negative integer.*

*(2) For* $\mathbb{R}\setminus\{0\}$ *and* $R \quad \mathbb{R}$, *the following equality holds*

$$\left(\frac{1}{\ }z\frac{d}{dz}+R\right)\left(z^{\frac{A_1\sigma}{\alpha_1}-\alpha R} {}_p \quad {}_q \left[az^\sigma\middle| \begin{array}{c} (A_i, \ {}_i)_{1,p} \\ (B_j, \ {}_j)_{1,q} \end{array}\right]\right)$$
$$= \frac{\ }{{}_1} z^{\frac{A_1\sigma}{\alpha_1}-\alpha R} {}_p \quad {}_q \left[az^\sigma\middle| \begin{array}{c} (A_1+1, \ {}_1),(A_i, \ {}_i)_{2,p} \\ (B_j, \ {}_j)_{1,q} \end{array}\right].$$

LEMMA 2.2. *Let*

$$= \sum_{j=1}^{q} {}_j \quad \sum_{i=1}^{p} {}_i > 0, \ \mu = \sum_{j=1}^{m} {}_j \quad \sum_{j=m+1}^{q} {}_j \quad \sum_{i=1}^{p} {}_i > 0,$$

$\mathbb{R}_+$ *and* $a \quad \mathbb{R}\setminus\{0\}$. *Then the following equalities hold*:

*(1) For* $a > 0$,

$$\frac{d^\alpha}{dz^\alpha} H^{m,0}_{p,q}\left[ az^{-\alpha_p} \middle| \begin{matrix} (A_i, \quad {}_i)_{1,p-1}, (1, \quad {}_p) \\ (B_j, \quad {}_j)_{1,q} \end{matrix} \right]$$
$$= z^{-\alpha} H^{m,0}_{p,q}\left[ az^{-\alpha_p} \middle| \begin{matrix} (A_i, \quad {}_i)_{1,p-1}, (1 \qquad , \quad {}_p) \\ (B_j, \quad {}_j)_{1,q} \end{matrix} \right], \quad z > 0.$$

*(2) If* $m \quad 1$, *then*

$$\left( \frac{1}{\ \ _p} z \frac{d}{dz} + B_1 \right) H^{m,0}_{p,q}\left[ az^{-\alpha_p} \middle| \begin{matrix} (A_i, \quad {}_i)_{1,p-1}, (1, \quad {}_p) \\ (B_j, \quad {}_j)_{1,q} \end{matrix} \right]$$
$$= H^{m,0}_{p,q}\left[ az^{-\alpha_p} \middle| \begin{matrix} (A_i, \quad {}_i)_{1,p-1}, (1, \quad {}_p) \\ (B_1 + 1, \quad {}_1), (B_j, \quad {}_j)_{2,q} \end{matrix} \right].$$

In the next section, we present exact solutions of (1.3) and (1.4) in propositions using the aforementioned special functions.

## 3. **Construction of exact solutions**

We express solutions to (1.3) and (1.4) using two kinds of special functions. Which of these functions we use in any given case depends on the right-hand side and order of the fractional derivative of (1.3) or (1.4). Before moving on to the formulation of exact solutions of (1.3) and (1.4), we introduce the following notation.

Let us consider the case that $c = 0$ and $b = 0$ in (1.3). Then writing $\frac{a}{b}$ as $\bar{s}$, we can rewrite the right-hand side of (1.3) as

$$a \quad + \frac{b}{\ } z \frac{d}{dz} = b \left( \frac{1}{\ } z \frac{d}{dz} \quad \bar{s} \right) \quad .$$

Then, in the case of (1.4), assuming $b_1 b_2 = 0$ and introducing the quantities

$$s_1 = \quad \frac{a_1}{b_1}, \quad s_2 = \quad \frac{a_2}{b_2},$$

we rewrite the right-hand side of (1.4) as follows:

$$a_1 \quad + \frac{b_1}{\ } z \frac{d}{dz} = b_1 \left( \frac{1}{\ } z \frac{d}{dz} \quad s_1 \right) \quad ,$$

$$a_2 \quad + \frac{b_2}{\ } z \frac{d}{dz} = b_2 \left( \frac{1}{\ } z \frac{d}{dz} \quad s_2 \right) \quad .$$

Now, let us assume $c = 0$. Then the characteristic equation of the right-hand side of (1.3) is

$$s^2 + \left( \frac{b}{c} \quad \frac{1}{\ } \right) s + \frac{a}{c} = 0. \tag{3.1}$$

We write the determinant and roots of (3.1) as $D = \frac{1}{\alpha^2} \quad \frac{2b}{\alpha c} + \frac{b^2}{c^2} \quad \frac{4a}{c}$ and $s_{1,2} = \frac{1}{2}\left(\frac{1}{\alpha} \quad \frac{b}{c} \pm \quad \overline{D}\right)$, respectively. Then we can rewrite the right-hand side of (1.3) in the factorized differential form

$$a \quad + \frac{b}{\phantom{\alpha}} z \frac{d}{dz} + \frac{c}{\phantom{\alpha}^2} z^2 \frac{d^2}{dz^2} = c\left(\frac{1}{\phantom{\alpha}} z \frac{d}{dz} \quad s_1\right)\left(\frac{1}{\phantom{\alpha}} z \frac{d}{dz} \quad s_2\right) \quad . \tag{3.2}$$

This notation is useful for at least two reasons. First, it reveals the uniformity in given solutions of (1.3) and (1.4) with different orders of fractional derivatives. In particular, in the case $c = 0$, we can avoid a tedious computation by simply rewriting the right-hand side of (1.3) in factorized operator form. Second, using this notation, we can easily generalize (1.3) and (1.4) into (1.1) and (1.2). We will discuss this generalization in the next section.

3.1. **Solutions expressed in terms of generalized Wright functions.** We now formulate the solutions of (1.3) and (1.4) as follows.

Proposition 3.1. *We have the following solutions expressed in terms of the generalized Wright function.*

(1) *For $\quad > 1$ and $a, b \quad \mathbb{R}$ with $b = 0$, the equation*

$$\frac{d^\alpha}{dz^\alpha} = a \quad + \frac{b}{\phantom{\alpha}} z \frac{d}{dz}, \qquad z \quad \mathbb{R} \tag{3.3}$$

*has as a solution*

$$(z) = \sum_{k=1}^{n} c_k z^{\alpha-k} {}_2 \quad {}_1\left[ b z^\alpha \,\middle|\, \begin{matrix} \left(1 \quad \frac{k}{\alpha} \quad \bar{s}, 1\right), (1,1) \\ (1 + \quad k, \quad ) \end{matrix} \right],$$

*where $\bar{s} = \quad \frac{a}{b}$, and $c_k$ $(k = 1, \ldots, n)$ are constants.*

(2) *For $\quad > 2$ and $a, b, c \quad \mathbb{R}$ with $c = 0$, the equation*

$$\frac{d^\alpha}{dz^\alpha} = a \quad + \frac{b}{\phantom{\alpha}} z \frac{d}{dz} + \frac{c}{\phantom{\alpha}^2} z^2 \frac{d^2}{dz^2}, \qquad z \quad \mathbb{R} \tag{3.4}$$

*has as a solution*

$$(z) = \sum_{k=1}^{n} c_k z^{\alpha-k} {}_3 \quad {}_1\left[ c z^\alpha \,\middle|\, \begin{matrix} \left(1 \quad \frac{k}{\alpha} \quad s_1, 1\right), \left(1 \quad \frac{k}{\alpha} \quad s_2, 1\right), (1,1) \\ (1 + \quad k, \quad ) \end{matrix} \right],$$

*where $s_{1,2} = \frac{1}{2}\left(\frac{1}{\alpha} \quad \frac{b}{c} \pm \quad \overline{D}\right)$, $D = \frac{1}{\alpha^2} \quad \frac{2b}{\alpha c} + \frac{b^2}{c^2} \quad \frac{4a}{c}$, and $c_k$ $(k = 1, \ldots, n)$ are constants.*

(3) *For $\quad > 1$ and $a_1, a_2, b_1, b_2 \quad \mathbb{R}$ with $b_1 b_2 = 0$, the system*

$$\begin{cases} \frac{d^\alpha \varphi}{dz^\alpha} = a_1 \quad + \frac{b_1}{\alpha} z \frac{d\psi}{dz}, \\ \frac{d^\alpha \psi}{dz^\alpha} = a_2 \quad + \frac{b_2}{\alpha} z \frac{d\varphi}{dz}, \end{cases} \qquad z \quad \mathbb{R} \tag{3.5}$$

*has as a solution*

$$
\begin{aligned}
(z) = &\sum_{k=1}^{n} c_{k,1} z^{\alpha-k} {}_3 \quad {}_1 \left[ Az^{2\alpha} \middle| \begin{matrix} \left(1 \quad \frac{k}{2\alpha} \quad \frac{s_1}{2},1\right), \left(\frac{1}{2} \quad \frac{k}{2\alpha} \quad \frac{s_2}{2},1\right), (1,1) \\ (1+ \quad k, 2 \quad ) \end{matrix} \right] \\
&+ 2b_1 \sum_{k=1}^{n} c_{k,2} z^{2\alpha-k} {}_3 \quad {}_1 \left[ Az^{2\alpha} \middle| \begin{matrix} \left(\frac{3}{2} \quad \frac{k}{2\alpha} \quad \frac{s_1}{2},1\right), \left(1 \quad \frac{k}{2\alpha} \quad \frac{s_2}{2},1\right), (1,1) \\ (1+2 \quad k, 2 \quad ) \end{matrix} \right],
\end{aligned}
$$

$$
\begin{aligned}
&(z) = \\
&2b_2 \sum_{k=1}^{n} c_{k,1} z^{2\alpha-k} {}_3 \quad {}_1 \left[ Az^{2\alpha} \middle| \begin{matrix} \left(1 \quad \frac{k}{2\alpha} \quad \frac{s_1}{2},1\right), \left(\frac{3}{2} \quad \frac{k}{2\alpha} \quad \frac{s_2}{2},1\right), (1,1) \\ (1+2 \quad k, 2 \quad ) \end{matrix} \right] \\
&+ \sum_{k=1}^{n} c_{k,2} z^{\alpha-k} {}_3 \quad {}_1 \left[ Az^{2\alpha} \middle| \begin{matrix} \left(\frac{1}{2} \quad \frac{k}{2\alpha} \quad \frac{s_1}{2},1\right), \left(1 \quad \frac{k}{2\alpha} \quad \frac{s_2}{2},1\right), (1,1) \\ (1+ \quad k, 2 \quad ) \end{matrix} \right],
\end{aligned}
$$

*where* $A = 4b_1b_2$, $s_1 = \quad \frac{a_1}{b_1}$, $s_2 = \quad \frac{a_2}{b_2}$, *and* $c_{k,1}$, $c_{k,2}$ $(k = 1, \dots, n)$ *are constants.*

P r o o f. The proof can be carried out similarly in all three cases using Lemma 2.1. For this reason, we present proof only for the second case. From the linearity of (3.4), it is su  cient to show that a single summand,

$$
{}_k(z) = z^{\alpha-k} {}_3 \quad {}_1 \left[ cz^{\alpha} \middle| \begin{matrix} \left(1 \quad \frac{k}{\alpha} \quad s_1,1\right), \left(1 \quad \frac{k}{\alpha} \quad s_2,1\right), (1,1) \\ (1+ \quad k, \quad ) \end{matrix} \right],
$$

of the solution $\quad (z)$ satisfies (3.4). Because $1 + \quad k > 0$ for any $k$, by Lemma 2.1, we have the following identity for the left-hand side of (3.4):

$$
\frac{d^{\alpha} \quad {}_k}{dz^{\alpha}} = cz^{\alpha-k} {}_3 \quad {}_1 \left[ cz^{\alpha} \middle| \begin{matrix} (2 \quad \frac{k}{\alpha} \quad s_1,1), (2 \quad \frac{k}{\alpha} \quad s_2,1), (1,1) \\ (1+ \quad k, \quad ) \end{matrix} \right]. \tag{3.6}
$$

Then, by virtue of (3.2) and the second assertion of Lemma 2.1, the right-hand side of (3.4) becomes

$$
\begin{aligned}
c\left(\frac{1}{\,}z\frac{d}{dz} \quad s_1\right)\left(\frac{1}{\,}z\frac{d}{dz} \quad s_2\right) \quad {}_k = c\left(\frac{1}{\,}z\frac{d}{dz} \quad s_1\right) \\
\times \left( z^{\alpha-k} {}_3 \quad {}_1 \left[ cz^{\alpha} \middle| \begin{matrix} \left(1 \quad \frac{k}{\alpha} \quad s_1,1\right), \left(2 \quad \frac{k}{\alpha} \quad s_2,1\right), (1,1) \\ (1+ \quad k, \quad ) \end{matrix} \right] \right),
\end{aligned}
$$

which is equal to the left-hand side of (3.6). □

3.2. **Solutions expressed in terms of Fox $H$-functions.** Unlike the previously presented solutions expressed in terms of generalized Wright functions, we present the solutions expressed in terms of Fox $H$-functions for $z > 0$.

Proposition 3.2. *For the following cases, we have solutions expressed in terms of Fox $H$-functions.*

(1) *For* $0 < \quad < 1$ *and* $a, b \quad \mathbb{R}$ *with* $b > 0$, *the equation*

$$\frac{d^\alpha}{dz^\alpha} = a \quad + \frac{b}{\ } z \frac{d}{dz}, \qquad z > 0 \tag{3.7}$$

*has as a solution*

$$(z) = c_1 H^{1,0}_{1,1}\left[\frac{z^{-\alpha}}{b}\,\middle|\, \begin{matrix} (1, \quad ) \\ ( \quad \bar{s}, 1) \end{matrix}\right],$$

*where* $\bar{s} = \quad \frac{a}{b}$, *and* $c_1$ *is a constant.*

(2) *For* $0 < \quad < 2$ *and* $a, b, c \quad \mathbb{R}$ *with* $c > 0$, *the equation*

$$\frac{d^\alpha}{dz^\alpha} = a \quad + \frac{b}{\ } z \frac{d}{dz} + \frac{c}{\ ^2} z^2 \frac{d^2}{dz^2}, \qquad z > 0 \tag{3.8}$$

*has as a solution*

$$(z) = c_1 H^{2,0}_{1,2}\left[\frac{z^{-\alpha}}{c}\,\middle|\, \begin{matrix} (1, \quad ) \\ ( \quad s_1, 1), ( \quad s_2, 1) \end{matrix}\right],$$

*where* $s_{1,2} = \frac{1}{2}\left(\frac{1}{\alpha} \quad \frac{b}{c} \pm \quad \overline{D}\right)$, $D = \frac{1}{\alpha^2} \quad \frac{2b}{\alpha c} + \frac{b^2}{c^2} \quad \frac{4a}{c}$, *and* $c_1$ *is a constant.*

(3) *For* $0 < \quad < 1$ *and* $a_1, a_2, b_1, b_2 \quad \mathbb{R}$ *with* $b_1 b_2 > 0$, *the system*

$$\begin{cases} \frac{d^\alpha \varphi}{dz^\alpha} = a_1 \quad + \frac{b_1}{\alpha} z \frac{d\psi}{dz}, \\ \frac{d^\alpha \psi}{dz^\alpha} = a_2 \quad + \frac{b_2}{\alpha} z \frac{d\varphi}{dz}, \end{cases} \qquad z > 0 \tag{3.9}$$

*has as a solution*

$$\begin{aligned} (z) &= c_1 \operatorname{sgn}(b_1) H^{2,0}_{1,2}\left[\frac{z^{-2\alpha}}{4b_1 b_2}\,\middle|\, \begin{matrix} (1, 2 \quad ) \\ (\frac{1}{2} \quad \frac{s_1}{2}, 1), ( \quad \frac{s_2}{2}, 1) \end{matrix}\right], \\ (z) &= c_1 \sqrt{\frac{b_2}{b_1}} H^{2,0}_{1,2}\left[\frac{z^{-2\alpha}}{4b_1 b_2}\,\middle|\, \begin{matrix} (1, 2 \quad ) \\ ( \quad \frac{s_1}{2}, 1), (\frac{1}{2} \quad \frac{s_2}{2}, 1) \end{matrix}\right], \end{aligned}$$

*where* $s_1 = \quad \frac{a_1}{b_1}$, $s_2 = \quad \frac{a_2}{b_2}$, *and* $c_1$ *is a constant.*

P r o o f. Analogously to the proof of Proposition 3.1, the three assertions of this proposition can be proved in a similar manner by using Lemma 2.2. For this reason, we consider only the second assertion with $c_1 = 1$, without loss of generality. Because the convergence condition of $H$-functions holds (i.e. $\mu = 2 \qquad > 0$), we can apply the first assertion of Lemma 2.2. We thereby obtain

$$\frac{d^\alpha}{dz^\alpha} H^{2,0}_{1,2}\left[\frac{z^{-\alpha}}{c}\,\middle|\, \begin{matrix} (1, \quad ) \\ ( \quad s_1, 1), ( \quad s_2, 1) \end{matrix}\right] = z^{-\alpha} H^{2,0}_{1,2}\left[\frac{z^{-\alpha}}{c}\,\middle|\, \begin{matrix} (1 \qquad , \quad ) \\ ( \quad s_1, 1), ( \quad s_2, 1) \end{matrix}\right]$$

for the left-hand side of (3.8), which is further simplified into the form

$$\frac{d^\alpha}{dz^\alpha} H^{2,0}_{1,2}\left[\frac{z^{-\alpha}}{c}\,\middle|\, \begin{matrix} (1, \quad ) \\ ( \quad s_1, 1), ( \quad s_2, 1) \end{matrix}\right] = c H^{2,0}_{1,2}\left[\frac{z^{-\alpha}}{c}\,\middle|\, \begin{matrix} (1, \quad ) \\ (1 \quad s_1, 1), (1 \quad s_2, 1) \end{matrix}\right].$$

The right-hand side of (3.8) is obtained in analogy to the second assertion of Proposition 3.1, by using (3.2) and interchanging the parameters $(\quad s_1, 1)$ and $(\quad s_2, 1)$ in the solution $\ (z)$ in accordance with the second assertion of Lemma 2.2. □

We now show that for a special case of (3.8), we have solutions expressed in terms of Wright functions.

Corollary 3.1. *Let the determinant of* (3.8) *be* $D = \frac{1}{\alpha^2} \quad \frac{2b}{\alpha c} + \frac{b^2}{c^2} \quad \frac{4a}{c} = \frac{1}{4}$, *and suppose* $c = 0$.

(1) *For* $0 < \quad < 2$ *and* $c > 0$, (3.8) *has a solution of the following form:*

$$(z) = c_1 z^{\frac{1}{2}\left(\frac{1}{\alpha} - \frac{b}{c} + \frac{1}{2}\right)\alpha} \quad \left( \frac{2z^{-\frac{\alpha}{2}}}{\overline{c}}; \quad \frac{}{2}, \frac{1}{2}\left(\frac{3}{\ } \quad \frac{b}{c} + \frac{1}{2}\right) \right).$$

(2) *For* $\quad > 2$, (3.8) *has a solution of the following form:*

$$(z) = \sum_{k=1}^{n} c_k z^{\alpha - k}{}_2 \quad {}_1\left( \frac{cz^{\alpha}}{4} \,\middle|\, \begin{matrix} \left(\frac{3}{2} \quad \frac{2k+1}{\alpha} + \frac{b}{c}, 2\right), (1,1) \\ (1 + \quad k, \ ) \end{matrix} \right).$$

P r o o f. To prove the first assertion, we need to show that $\ (z)$ corresponds to the solution given in the second assertion of Proposition 3.2. When $D = \frac{1}{4}$ and $0 < \quad < 2$, the roots of the characteristic equation (3.1) become $s_1 = \frac{1}{2}\left(\frac{1}{\alpha} \quad \frac{b}{c} + \frac{1}{2}\right)$ and $s_2 = s_1 \quad \frac{1}{2}$. In this case, the solution given in the second assertion of Proposition 3.2 is

$$(z) = c_1 H^{2,0}_{1,2}\left[ \frac{z^{-\alpha}}{c} \,\middle|\, \begin{matrix} (1, \ ) \\ (\quad s_1, 1), (\quad s_1 + \frac{1}{2}, 1) \end{matrix} \right].$$

Then, applying the duplication formula for the Gamma function

$$(\quad s_1 \quad s) \quad \left( \quad s_1 \quad s + \frac{1}{2} \right) = \overline{\ }\, 2^{1+2(s_1+s)} \quad (\quad 2s_1 \quad 2s),$$

it becomes

$$(z) = c_1 \quad \overline{\ }\, c^{s_1} z^{\frac{\alpha}{2}\left(\frac{1}{\alpha} - \frac{b}{c} + \frac{1}{2}\right)} \quad \left( \frac{2z^{-\frac{\alpha}{2}}}{\overline{c}}; \quad \frac{}{2}, \frac{}{2}\left(\frac{3}{\ } \quad \frac{b}{c} + \frac{1}{2}\right)\right).$$

Thus, the first assertion is proved.

The second assertion can also be proved by applying the duplication formula of gamma function to the solution given in the second assertion of the Proposition 3.1 □

To this point, we have presented several exact solutions of (1.3) and (1.4). These solutions are classified according to the kind of special functions used to express them. Now, we discuss the second advantage of rewriting the right-hand sides of (1.3) and (1.4) in factorized differential operator form. Specifically, we show that utilizing this form, we are able to generalize our treatment of (1.3) and (1.4) to the cases of (1.1) and (1.2) with arbitrary integers $m$, $m_1$ and $m_2$. As a result, we find that the solutions to (3.3) and (3.4) of Proposition 3.1 and the solutions to (3.7) and (3.8) of Proposition 3.2 can all be represented in a unified manner by a single general formula.

## 4. **Construction of exact solutions in the general case**

As in the beginning of Section 3, by virtue of (3.2) we represent the right-hand side of (1.1) by $P(\ )$ as follows:

$$P(\ ) = a_m \prod_{i=1}^{m} \left( \frac{1}{\ } z \frac{d}{dz} \quad s_i \right) \quad .$$

Here, $s_1, s_2, \ldots, s_m$ are the roots of the characteristic polynomial

$$P(s) = a_0 + \sum_{i=1}^{m} a_i \prod_{j=0}^{i-1} \left( s \quad \frac{j}{\ } \right).$$

Now, generalizing the results of the previous section, we formulate the following theorem.

THEOREM 4.1. *(1) If* $0 < \quad < m$ *and* $a_m > 0$, *then for* $z > 0$ (1.1) *has the following as a solution:*

$$(z) = c_1 H^{m,0}_{1,m} \left[ \frac{z^{-\alpha}}{a_m} \middle| \begin{matrix} (1, \ ) \\ ( \quad s_j, 1)_{1,m} \end{matrix} \right].$$

*(2) If* $\quad > m$, *then* (1.1) *has as the following as a solution*

$$(z) = \sum_{k=1}^{n} c_k z^{\alpha - k} {}_{m+1} \quad {}_1 \left[ a_m z^{\alpha} \middle| \begin{matrix} (1 \quad \frac{k}{\alpha} \quad s_i, 1)_{1,m}, (1,1) \\ (1 + \quad k, \ ) \end{matrix} \right],$$

*where* $c_k$ $(k = 1, \ldots, n)$ *are arbitrary constants.*

In a similar manner, we can rewrite the right hand sides of the equations in (1.2) as

$$\sum_{i=0}^{m_1} \frac{a_i}{i} z^i \frac{d^i}{dz^i} = 2^{m_1} a_{m_1} \prod_{i=1}^{m_1} \left( \frac{1}{2} z \frac{d}{dz} \quad \frac{s_i}{2} \right) \quad (z)$$

and

$$\sum_{i=0}^{m_2} \frac{b_i}{i} z^i \frac{d^i}{dz^i} = 2^{m_2} b_{m_2} \prod_{i=m_1+1}^{m_1+m_2} \left( \frac{1}{2} z \frac{d}{dz} \quad \frac{s_i}{2} \right) \quad (z).$$

Here, $s_1, s_2, \ldots, s_{m_1}$ and $s_{m_1+1}, s_{m_1+2}, \ldots, s_{m_1+m_2}$, are, respectively, the roots of the characteristic polynomials

$$P_1(s) = a_0 + \sum_{i=1}^{m_1} a_i \prod_{j=0}^{i-1} \left( s \quad \frac{j}{\ } \right), \qquad P_2(s) = b_0 + \sum_{i=1}^{m_2} b_i \prod_{j=0}^{i-1} \left( s \quad \frac{j}{\ } \right).$$

The following result concerns solutions of (1.2).

THEOREM 4.2. *Here, we use $m$ to represent $m_1+m_2$ and $A$ to represent $2^{m_1+m_2}a_{m_1}b_{m_2}$.*

(1) *If $0 < \quad < \frac{m}{2}$ and $a_{m_1}b_{m_2} > 0$, then for $z > 0$ the system (1.2) has the following as a solution:*

$$(z) = c_1 \operatorname{sgn}(b_{m_2}) H^{m,0}_{1,m}\left[\frac{z^{-2\alpha}}{A}\,\middle|\, \begin{array}{c} (1, 2 \quad ) \\ \left(\quad \frac{s_i}{2}+\frac{1}{2}, 1\right)_{1,m_1}, \left(\quad \frac{s_i}{2}, 1\right)_{m_1+1,m} \end{array}\right],$$

$$(z) = c_1 2^{\frac{m_2-m_1}{2}}\sqrt{\frac{b_{m_2}}{a_{m_1}}} H^{m,0}_{1,m}\left[\frac{z^{-2\alpha}}{A}\,\middle|\, \begin{array}{c} (1, 2 \quad ) \\ \left(\quad \frac{s_i}{2}, 1\right)_{1,m_1}, \left(\quad \frac{s_i}{2}+\frac{1}{2}, 1\right)_{m_1+1,m} \end{array}\right].$$

(2) *If $\quad > \frac{m}{2}$, then the system (1.2) has the following as a solution:*

$$\begin{aligned} (z) &= \sum_{k=1}^{n} c_{k,1} z^{\alpha-k} \quad {}_{k1}(Az^{2\alpha}) + 2^{m_1} a_{m_1} \sum_{k=1}^{n} c_{k,2} z^{2\alpha-k} \quad {}_{k2}(Az^{2\alpha}), \\ (z) &= 2^{m_2} b_{m_2} \sum_{k=1}^{n} c_{k,1} z^{2\alpha-k} \quad {}_{k1}(Az^{2\alpha}) + \sum_{k=1}^{n} c_{k,2} z^{\alpha-k} \quad {}_{k2}(Az^{2\alpha}), \end{aligned}$$

*where*

$$\begin{aligned}
{}_{k1}(z) &= {}_{m+1} \quad {}_1\left[z\,\middle|\, \begin{array}{c} \left(1 \quad \frac{k}{2\alpha} \quad \frac{s_i}{2}, 1\right)_{1,m_1}, \left(\frac{1}{2} \quad \frac{k}{2\alpha} \quad \frac{s_i}{2}, 1\right)_{m_1+1,m}, (1,1) \\ (1 + \quad k, 2 \quad ) \end{array}\right], \\
{}_{k2}(z) &= {}_{m+1} \quad {}_1\left[z\,\middle|\, \begin{array}{c} \left(\frac{3}{2} \quad \frac{k}{2\alpha} \quad \frac{s_i}{2}, 1\right)_{1,m_1}, \left(1 \quad \frac{k}{2\alpha} \quad \frac{s_i}{2}, 1\right)_{m_1+1,m}, (1,1) \\ (1 + 2 \quad k, 2 \quad ) \end{array}\right], \\
{}_{k1}(z) &= {}_{m+1} \quad {}_1\left[z\,\middle|\, \begin{array}{c} \left(1 \quad \frac{k}{2\alpha} \quad \frac{s_i}{2}, 1\right)_{1,m_1}, \left(\frac{3}{2} \quad \frac{k}{2\alpha} \quad \frac{s_i}{2}, 1\right)_{m_1+1,m}, (1,1) \\ (1 + 2 \quad k, 2 \quad ) \end{array}\right], \\
{}_{k2}(z) &= {}_{m+1} \quad {}_1\left[z\,\middle|\, \begin{array}{c} \left(\frac{1}{2} \quad \frac{k}{2\alpha} \quad \frac{s_i}{2}, 1\right)_{1,m_1}, \left(1 \quad \frac{k}{2\alpha} \quad \frac{s_i}{2}, 1\right)_{m_1+1,m}, (1,1) \\ (1 + \quad k, 2 \quad ) \end{array}\right].
\end{aligned}$$

*Here $c_1$, $c_{k,1}$, $c_{k,2}$ $(k = 1, \ldots, n)$ are constants.*

The proofs can be carried out analogously to the proofs of Proposition 3.1 and Proposition 3.2.

## 5. Solutions to a class of fractional linear partial di erential equations and systems thereof

In this section, we demonstrate the application of the propositions presented in this paper by providing exact solutions of generalizations of (1.5) and (1.6). The solutions obtained in this section reduce to a previously known solution in a particular case.

5.1. **Solutions to a fractional linear evolution equation.** Let us consider the following linear fractional evolution equation with variable coe - cients:

$$\frac{\ ^{\alpha}u(t,x)}{\ t^{\alpha}} = a_m(x+b)^p \frac{\ ^{m}u(t,x)}{\ x^m} + a_{m-1}(x+b)^{p-1}\frac{\ ^{m-1}u(t,x)}{\ x^{m-1}}$$
$$+ \cdots + a_0(x+b)^{p-m}u(t,x), \qquad > 0, \quad x > \quad b, \quad t > 0. \quad (5.1)$$

Here $a_i$ $(i = 0, \ldots, m)$, $b$ and $p$ are real numbers and $a_m > 0$. The infinitesimal scaling symmetries of (5.1) are

$$X_1 = u\frac{}{\ u}, \qquad X_2 = (x+b)\frac{}{\ x} + \frac{m \quad p}{}t\frac{}{\ t}.$$

The invariant solution corresponding to the generator $X = aX_1 + X_2$ ($a$ $\mathbb{R}$) is

$$u = (x+b)^a \quad (z), \text{ with } z = t(x+b)^{\frac{p-m}{\alpha}}, \qquad (5.2)$$

where $(z)$ solves the following reduced FODE:

$$\frac{d^\alpha}{dz^\alpha} = \frac{a_m\,(m \quad p)^m}{m} z^m \frac{d^m}{dz^m} + \frac{\bar{a}_{m-1}}{m-1} z^{m-1}\frac{d^{m-1}}{dz^{m-1}} + \cdots + \frac{\bar{a}_1}{} z\frac{d}{dz} + \bar{a}_0, \quad z > 0.$$

Here, the parameters $\bar{a}_0, \bar{a}_1, \ldots, \bar{a}_{m-1}$ depend on $a_1, a_2, \ldots, a_m,$ , $m$ and $p$.

Now, let us take a closer look at the case $m = 2$:

$$\frac{\ ^{\alpha}u}{\ t^{\alpha}} = a_2(x+b)^p u_{xx} + a_1(x+b)^{p-1}u_x + a_0(x+b)^{p-2}u \qquad (5.3)$$

for $> 0$, $x > \quad b$, $t > 0$, with $a_2 > 0$. For the case of $a_0 = 0$, solutions of (5.3) were found in [11] through Laplace transformation method and were expressed by Fox-$H$ functions. Now, generalizing the result of [11], let us find solutions of (5.3) for any values of $a_0$.

Applying the transformation (5.2) with $m = 2$, (5.3) reduced to the following FODE

$$\frac{d^\alpha}{dz^\alpha} = \bar{a} \quad + \frac{\bar{b}}{}z\frac{d}{dz} + \frac{\bar{c}}{\ ^2}z^2\frac{d^2}{dz^2}, \quad z > 0, \qquad (5.4)$$

where $\bar{a} = a(a \quad 1)a_2 + aa_1 + a_0$, $\bar{b} = (p \quad 2)\left(\frac{p-2}{\alpha} + 2a \quad 1 + \frac{a_1}{a_2}\right)a_2$ and $\bar{c} = (p \quad 2)^2 a_2$. From Proposition 3.1 and Proposition 3.2, we have the following solutions of (5.4) for $p = 2$ :

(1) For $0 < \quad < 2$, $\quad (z) = c_1 H^{2,0}_{1,2}\left[\frac{z^{-\alpha}}{A}\middle| \begin{array}{c} (1, \quad) \\ (\quad s_1, 1), (\quad s_2, 1) \end{array}\right].$

(2) For $> 2$,

$$(z) = \sum_{k=1}^{n} c_i z^{\alpha-k}\ _3 \quad _1\left[Az^\alpha \middle| \begin{array}{c} \left(1 \quad \frac{k}{\alpha} \quad s_1, 1\right), \left(1 \quad \frac{k}{\alpha} \quad s_2, 1\right), (1,1) \\ (1 + \quad k, \quad) \end{array}\right],$$

where $A = (p \quad 2)^2 a_2$ and $s_{1,2} = \frac{1}{2(p-2)}\left(1 \quad 2a \quad \frac{a_1}{a_2} \pm \sqrt{\left(1 \quad \frac{a_1}{a_2}\right)^2 \quad \frac{4a_0}{a_2}}\right)$.

If $a_0 = \frac{a_2}{4}\left(\frac{a_1}{a_2} \quad \frac{p}{2}\right)\left(\frac{a_1}{a_2} + \frac{p}{2} \quad 2\right)$ in (5.3), then by Corollary 3.1 we obtain the following solutions of (5.4):

(1) For $0 < \quad < 2$, $\quad (z) = c_1 z^{\frac{\alpha}{2}s} \quad \left( \frac{2z^{-\frac{\alpha}{2}}}{|p-2|\sqrt{a_2}}; \quad \frac{\alpha}{2}, 1 + \frac{\alpha}{2}s\right)$.

(2) For $\quad > 2$, $\quad (z) = \sum_{k=1}^{n} c_k z^{\alpha-k} {}_2 \quad {}_1\left[\frac{(p-2)^2 a_2 z^\alpha}{4} \,\middle|\, \begin{matrix} \left(2 \quad \frac{2k}{\alpha} \quad s, 2\right), (1,1) \\ (1 + \quad k, \quad ) \end{matrix}\right]$,

where $s = \frac{1}{(p-2)}\left(\frac{p}{2} \quad 2a \quad \frac{a_1}{a_2}\right)$.

If we set $a_0 = a_1 = p = 0$ in the above solutions, then they correspond to the solutions obtained in [1] and [8].

If $p = 2$, then (5.4) becomes

$$\frac{d^\alpha}{dz^\alpha} = (a(a \quad 1)a_2 + aa_1 + a_0) \quad , \quad z > 0,$$

and the solution given in Proposition 3.1 is reduced to

$$(z) = \sum_{k=1}^{n} c_i z^{\alpha-k} E_{\alpha,1+\alpha-k}\left((a(a \quad 1)a_2 + aa_1 + a_0)z^\alpha\right),$$

where $E_{\alpha,\beta}(z)$ is a Mittag-Le er function defined by

$$E_{\alpha,\beta}(z) = \sum_{i=0}^{\infty} \frac{z^i}{(\quad i + \quad)}.$$

Finally, we can obtain the invariant solutions of (5.3) by substituting these solutions into (5.2).

5.2. **Solutions to a system of fractional linear equations.** Let us consider the following system:

$$\begin{cases} \frac{\partial^\alpha u}{\partial t^\alpha} = a_1(x+c)^{m_1} v_x + b_1(x+c)^{m_1-1} v, \\ \frac{\partial^\alpha v}{\partial t^\alpha} = a_2(x+c)^{m_2} u_x + b_2(x+c)^{m_2-1} u, \end{cases} \quad > 0, \quad x > \quad c, \quad t > 0, \tag{5.5}$$

where $a_1, a_2, b_1, b_2, m_1, m_2, c \quad \mathbb{R}$ and $a_1 a_2 > 0$. Then, with the substitution

$$\begin{cases} u(x,t) = (x+c)^{d+\frac{m_1}{2}} \quad (z), \\ v(x,t) = (x+c)^{d+\frac{m_2}{2}} \quad (z), \end{cases} \quad \text{with} \quad z = t(x+c)^{\frac{m_1+m_2-2}{2\alpha}} \quad \text{and} \quad d \quad \mathbb{R},$$

we obtain the system of FODEs

$$\begin{cases} \frac{d^\alpha \varphi}{dz^\alpha} = \bar{a}_1 \quad + \frac{\bar{b}_1}{\alpha} z \frac{d\psi}{dz}, \\ \frac{d^\alpha \psi}{dz^\alpha} = \bar{a}_2 \quad + \frac{\bar{b}_2}{\alpha} z \frac{d\varphi}{dz}, \end{cases} \quad z > 0, \tag{5.6}$$

where $\bar{a}_1 = \left(d + \frac{m_2}{2}\right) a_1 + b_1$, $\bar{a}_2 = \left(d + \frac{m_1}{2}\right) a_2 + b_2$, $\bar{b}_1 = \frac{(m_1+m_2-2)a_1}{2}$ and $\bar{b}_2 = \frac{(m_1+m_2-2)a_2}{2}$.

We obtain the following results by virtue of Proposition 3.1 and Proposition 3.2.

**Case 1.** Let us consider the case $m = m_1 + m_2 = 2$. Then, for $0 < \quad < 1$, the solution to (5.6) given in Proposition 3.2 is

$$\begin{aligned} (z) &= c_1 \operatorname{sgn}((m \quad 2)a_1)\, H^{2,0}_{1,2}\left[\frac{z^{-2\alpha}}{(m \quad 2)^2 a_1 a_2}\middle|\begin{matrix} (1, 2 \quad) \\ \left(\tfrac{1}{2} \quad \tfrac{s_1}{2}, 1\right), \left( \quad \tfrac{s_2}{2}, 1\right)\end{matrix}\right], \\ (z) &= c_1\sqrt{\frac{a_2}{a_1}} H^{2,0}_{1,2}\left[\frac{z^{-2\alpha}}{(m \quad 2)^2 a_1 a_2}\middle|\begin{matrix} (1, 2 \quad) \\ \left( \quad \tfrac{s_1}{2}, 1\right), \left(\tfrac{1}{2} \quad \tfrac{s_2}{2}, 1\right)\end{matrix}\right], \end{aligned}$$

and for $\quad > 1$, the solution given in the third assertion of Proposition 3.1 is

$$\begin{aligned} (z) = \sum_{k=1}^{n} c_{k,1} z^{\alpha-k}\, {}_3\ {}_1\left[\frac{z^{2\alpha}}{M^2}\middle|\begin{matrix}\left(1 \quad \tfrac{k}{2\alpha} \quad \tfrac{s_1}{2}, 1\right), \left(\tfrac{1}{2} \quad \tfrac{k}{2\alpha} \quad \tfrac{s_2}{2}, 1\right), (1,1) \\ (1 + \quad k, 2 \quad)\end{matrix}\right] \\ + \frac{a_1}{M}\sum_{k=1}^{n} c_{k,2} z^{2\alpha-k}\, {}_3\ {}_1\left[\frac{z^{2\alpha}}{M^2}\middle|\begin{matrix}\left(\tfrac{3}{2} \quad \tfrac{k}{2\alpha} \quad \tfrac{s_1}{2}, 1\right), \left(1 \quad \tfrac{k}{2\alpha} + \tfrac{s_2}{2}, 1\right), (1,1) \\ (1 + 2 \quad k, 2 \quad)\end{matrix}\right], \end{aligned}$$

$$\begin{aligned} &(z) \\ &= \frac{a_2}{M}\sum_{k=1}^{n} c_{k,1} z^{2\alpha-k}\, {}_3\ {}_1\left[\frac{z^{2\alpha}}{M^2}\middle|\begin{matrix}\left(1 \quad \tfrac{k}{2\alpha} + \tfrac{s_1}{2}, 1\right), \left(\tfrac{3}{2} \quad \tfrac{k}{2\alpha} \quad \tfrac{s_2}{2}, 1\right), (1,1) \\ (1 + 2 \quad k, 2 \quad)\end{matrix}\right] \\ &+ \sum_{k=1}^{n} c_{k,2} z^{\alpha-k}\, {}_3\ {}_1\left[\frac{z^{2\alpha}}{M^2}\middle|\begin{matrix}\left(\tfrac{1}{2} \quad \tfrac{k}{2\alpha} \quad \tfrac{s_1}{2}, 1\right), \left(1 \quad \tfrac{k}{2\alpha} \quad \tfrac{s_2}{2}, 1\right), (1,1) \\ (1 + \quad k, 2 \quad)\end{matrix}\right], \end{aligned}$$

where $M = \frac{1}{m-2}$, $s_1 = \quad \frac{(2d+m_2)a_1+2b_1}{(m-2)a_1}$ and $s_2 = \quad \frac{(2d+m_1)a_2+2b_2}{(m-2)a_2}$.

**Case 2.** Next, let us consider the case $m_1 + m_2 = 2$. In this case, we can rewrite (5.6) as

$$\begin{cases} \frac{d^\alpha \varphi}{dz^\alpha} = \bar{a}_1 \quad , \\ \frac{d^\alpha \psi}{dz^\alpha} = \bar{a}_2 \quad , \end{cases} \qquad z > 0,$$

where $\bar{a}_1 = (d + \frac{m_2}{2})a_1 + b_1$, $\bar{a}_2 = (d + \frac{m_1}{2})a_2 + b_2$. Then, the solution given in Proposition 3.1 is reduced to

$$\begin{aligned} (z) &= \sum_{k=1}^{n} c_{k,1} z^{\alpha-k}\ {}_1(\bar{a}_1\bar{a}_2 z^{2\alpha}) + \bar{a}_1 \sum_{k=1}^{n} c_{k,2} z^{2\alpha-k}\ {}_2(\bar{a}_1\bar{a}_2 z^{2\alpha}), \\ (z) &= \bar{a}_2 \sum_{k=1}^{n} c_{k,1} z^{\alpha-k}\ {}_2(\bar{a}_1\bar{a}_2 z^{2\alpha}) + \sum_{k=1}^{n} c_{k,2} z^{2\alpha-k}\ {}_1(\bar{a}_1\bar{a}_2 z^{2\alpha}), \end{aligned}$$

where ${}_1(z) = E_{2\alpha,1+\alpha-k}(z)$ and ${}_2(z) = E_{2\alpha,1+2\alpha-k}(z)$. Similarly to the previous cases, using the solutions of the reduced system, we are able to obtain invariant solutions of (5.5).

The readers interested in more applications of the results obtained in this work are referred to work [2].

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