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Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME 22, NUMBER 1 (2019) (Print) ISSN 1311-0454
(Electronic) ISSN 1314-2224

RESEARCH PAPER

FRACTIONAL CALCULUS IN ECONOMIC GROWTH MODELLING OF THE GROUP OF SEVEN

Inés Tejado [1], Emiliano Pérez [1], Duarte Valério [2]

Abstract

This paper presents models of economic growth for all the countries of the Group of Seven (G7) in the 1973–2016 period. The models consist of differential equations, of both integer and fractional order, where the gross domestic product (GDP) is a function of the country's land area, arable land, population, school attendance, gross capital formation (GCF), exports of goods and services, general government final consumption expenditure (GGFCE), and broad money (M3). Results show that fractional models have a better performance, measured by several summary statistics, without increasing the number of parameters, or sacrificing the ability to predict GDP evolution in the short term. A standard validation procedure for economic growth models is presented for the assessment of future models.

MSC 2010: Primary 26A33; Secondary 91B62, 91B84, 34A08, 37N40, 62P20

Key Words and Phrases: fractional calculus, economic growth, modelling, Group of Seven

1. **Introduction**

This paper develops models of economic growth for the current members of the Group of Seven (G7), from the inception of this group in 1973 until 2016. Multi-input dynamic systems, using integer order derivatives and fractional order derivatives, describe the evolution of the gross domestic

pp. 139–157 , DOI: 10.1515/fca-2019-0009 DE GRUYTER

product (GDP) as a function of several factors. The quality of these models is assessed using several statistical tools, showing the advantage of using fractional order derivatives to this purpose. The ability of predicting the short-term evolution of the GDP is also assessed. Possible explanations are presented of why this is so, and of the mechanism behind the fractional order. The validation procedure shows the way how initial conditions are handled is reasonable, and is expected to provide a standard for testing future models of economic growth.

1.1. **The G7.** The G7 currently comprises the following countries in alphabetical order: Canada (CAN), France (FRA), Germany (DEU), Italy (ITA), Japan (JPN), United Kingdom (GBR), and United States of America (USA).

It traces its beginnings to informal meetings in 1973 between finance ministers of some countries, to respond to the oil crisis and the recent changes in the international monetary system (in particular, the collapse of the monetary system of the Bretton Woods agreements). Formal summits followed, with the number of participating countries increasing from four to eight, and then settling at seven in 2014, see [24]. The European Economic Community, succeeded from 1993 on by the European Union (EUU), is represented at the meetings since 1977, but never chairs them; four of the G7 member countries (FRA, DEU, ITA, GBR) have been member states of the EUU as well during the entire period of concern.

The G7 countries are the seven wealthiest advanced countries in the world. China, India and Brazil have economies of comparable size, but are not yet developed economies according to the criteria of the World Bank or of the United Nations. Consequently, studying the evolution of the GDP of these countries is interesting as it may identify which factors are relevant to their position in the world economy.

The paper presents seven separate models for each of the seven members of the G7, another model for the EUU as a whole, and a ninth model that can be applied to all of each these eight economies. Models were found for the entire temporal range of concern (even though not all were members from the beginning), which is the period from 1973 to 2016. It begins with the first meetings that originated the G7; another reason why this year is a good choice for a starting point is because statistical data for the variables needed is consistently available from then on (but not always before). Likewise, 2016 is the last year for which figures can be found.

1.2. **Fractional derivatives in economic modelling.** Fractional derivatives have long been used to develop financial and economic models. In what concerns financial models, these can be based upon Lévy models

[1, 2, 3, 7, 21], upon a power law [40], upon the diffusion equation [14], upon continuous time random walks [12, 20, 22, 23, 28, 29], upon differentiable manifolds [5], upon fractals [15], and upon chaos theory [8, 9, 46]. As to economic models, economic growth has been modelled using a state-space representation of fractional derivatives [13, 26, 31, 50] that may be of variable order [49]; the world economy was modelled using a pseudo-phase plane and state space analysis [41, 42]; fractional calculus can be subject to an economic interpretation [34], and was used as well, together with diffusion models, to attempt a prediction of economic crises [6] and to describe economic processes with a long memory [32, 33] (which is reasonable given the presence of diffusion processes, modelled with fractional derivatives in other areas of science [19]). Models similar to those in this paper were developed for Portugal, Spain [36], France, Italy [35], and all the EU member-states [37].

1.3. **Paper organization.** The remainder of this paper is organized as follows. Section 2 presents the methodology followed. Section 3 introduces the sources of the data employed. Section 4 gives the results obtained. A discussion and conclusions are given in Section 5.

2. Methodology

2.1. **GDP models.** The rationale behind the models considered is that GDP depends on both available resources and impacts on the economy. So the first model considered, an integer order differential equation, has the following form for each country of the G7:

$$\begin{aligned} y(t) = \quad & C_1 x_1(t) + C_2 x_2(t) + C_3 x_3(t) + C_4 x_4(t) + C_5 \int_{t_0}^{t} x_5(t)\mathrm{d}t \\ & + C_6 x_6(t) + C_7 x_7(t) + C_8 \frac{\mathrm{d}x_8(t)}{\mathrm{d}t} + C_9 \frac{\mathrm{d}x_9(t)}{\mathrm{d}t}\,. \end{aligned} \tag{2.1}$$

Here $y(t)$ is the GDP (in 2010 US$), C_k are constant weights for each of the variables, t_0 is the first year considered, and x_k are the variables on which the output depends on, i.e.:

x_1: land area (in km^2) ,
x_2: arable land (in km^2),
x_3: population,
x_4: school attendance (years),
x_5: gross capital formation (GCF) (in 2010 US$) in the model, this variable is introduced accumulated,
x_6: exports of goods and services (in 2010 US$),
x_7: general government final consumption expenditure (GGFCE) (in 2010 US$),

x_8: broad money, M3 (in 2010 US$) in the model, the variation of this variable is considered,
x_9: variation of GCF x_5 (in 2010 US$).

Notice that:

land area x_1 is used as a measure of the available natural resources;
arable land x_2 is used as a measure of the quality of the natural resources;
population x_3 is used as a measure of the available human resources;
school attendance x_4 is used as a measure of the quality of human resources;
the accumulated GCF $\int_{t_0}^{t} x_5(t)\mathrm{d}t$ is used as a measure of manufactured resources;
exports x_6 are used as a measure of external impacts in the economy;
GGFCE x_7 is used as a measure of budgetary impacts in the economy;
the variation of M3 $\frac{\mathrm{d}x_8(t)}{\mathrm{d}t}$ is used as a measure of monetary impacts in the economy;
the variation of GCF $\frac{\mathrm{d}x_9(t)}{\mathrm{d}t} = \frac{\mathrm{d}x_5(t)}{\mathrm{d}t}$ is used as a measure of the impact of investment in the economy.

GCF appears twice in the model with different roles and so is given two different variables, x_5 and x_9, for clarity. These variables were chosen to represent both those of Keynesian models (short-term inputs with impacts in the economy) and those traditionally considered in growth accounting [10, 17, 18].

The importance of the variables in model (2.1) was determined for each country for the whole time period in order to propose other simpler models, as explained below in Section 2.2. As a result, a second integer order model with only six variables was also considered:

$$y(t) = \sum_{k=1,2,3,5,6,7} C_k x_k(t). \tag{2.2}$$

The effects of impacts in the economy, however, are not only instantaneous, but perdure with time. Consequently, model (2.2) was generalized to a third, fractional order model as

$$y(t) = \sum_{k=1,2,3} C_k x_k(t) + \sum_{k=5,6,7} C_k \mathrm{D}^{\alpha_k} x_k(t). \tag{2.3}$$

Notice that only variables x_5, x_6 and x_7 were considered to have fractional in" uence. This seems reasonable since these are the variables representing impacts. Furthermore, in this way, the fractional model has nine parameters, just as the original integer model (2.1) also has nine.

To implement the fractional differentiation operator D^{α_k}, the Caputo definition was chosen, because no initial conditions are needed in the frequency domain [45]. The operator was implemented as ${}_0\mathrm{D}_t^{\alpha_k} x_k(t)$, where the lower terminal 0 is the first year considered in each model. This means that the effects of the evolution of variables are considered only from that year on. Verifying whether this approximation (which reduces statistical data needed to develop models) is reasonable or not is one of the objectives of the present study.

2.2. **Optimizing and assessing performance.** To find the models (2.1)–(2.3) for each of the G7 countries, a fitting procedure was implemented in MATLAB. Nelder-Mead's simplex search method (implemented in function *fminsearch*) was used to minimize the mean square error (MSE):

$$\mathrm{MSE} = \frac{\sum_{j=1}^{N} (y_j \quad y_j)^2}{N}, \tag{2.4}$$

where N is the number of years (in this case, $N = 44$), and y_j and y_j are the real output and the model output, respectively. The MSE alone is not relied upon to evaluate the quality of the fit obtained by the resulting models: other performance indices were calculated as well. These were:

(1) The mean absolute deviation (MAD), given by

$$\mathrm{MAD} = \frac{\sum_{j=1}^{N} |y_j \quad y_j|}{N}. \tag{2.5}$$

(2) The coe cient of determination ($\mathrm{R}^2 \quad (0,1)$):

$$\mathrm{R}^2 = 1 \quad \frac{\sum_{j=1}^{N} (y_j \quad y_j)^2}{\sum_{j=1}^{N} (y_j \quad \bar{y})^2}, \tag{2.6}$$

where $\bar{y}$ is the mean of the GDP.

(3) The t values and p values for each variable.

These are calculated with MATLAB command *regstats*.

As will be seen from the results in Section 4, not all nine variables x_1, x_2, ..., and x_9 turned out to be necessary for every single model given by (2.1). This could be evaluated from the t and p values for each variable, by checking whether or not the performance indexes MAD and R^2 deteriorate significantly when removing one or more variables from the model, and also using the Akaike information criterion (AIC):

$$\mathrm{AIC} = N \log \frac{\sum_{j=1}^{N} (y_j \quad y_j)^2}{N} + 2K + \frac{2K(K+1)}{N \quad K \quad 1}\,, \tag{2.7}$$

being K the number of parameters of the model. The value of the AIC does not give information about the quality of a model. However, comparing the AIC values of different models, it can be seen which ones are more likely to be a good model for the data, as a lower value indicates a more likely model. Furthermore, if there are M models, the Akaike weight, given by

$$w_i = \frac{\exp\left(\dfrac{\mathrm{AIC}_i \quad \min\limits_{M} \mathrm{AIC}}{2} \right)}{\sum_{j=1}^{M} \exp\left(\dfrac{\mathrm{AIC}_j \quad \min\limits_{M} \mathrm{AIC}}{2} \right)}\,, \tag{2.8}$$

provides the probability of model i being the best of all the M models.

This is the rationale behind the development of models given by (2.2) and (2.3).

2.3. **Model time range and predictions of the future evolution of the GDP.** Models given by (2.1), (2.2) and (2.3) were obtained considering the entire 1973–2016 period. This allows making use of all the data at once, trying to obtain a long term fit. However:

It may lead to an overly rigid model, with an excessive in" uence of older data in parameters.

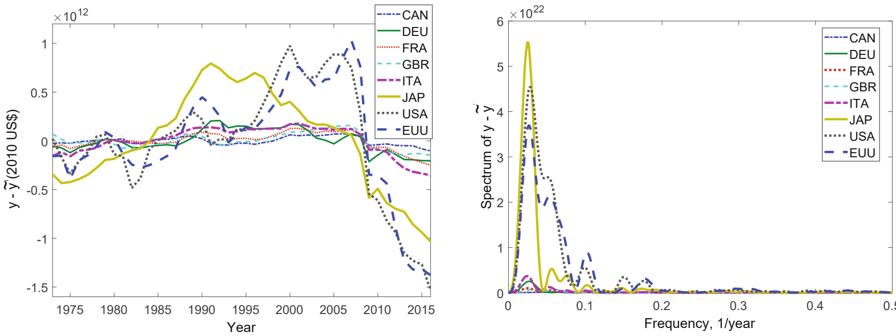
It does not allow verifying the capability of a model to predict the future evolution of the economy   another important performance assessment of a model for economic growth   since there is no further data with which this can be tested.

Consequently, models for a shorter time range were developed. The time range was chosen as follows:

The long term evolution of the GDP is roughly exponential. Consequently, exponentials were fit to $y(t)$, by least squares, for every country. Let $y(t)$ be the exponential curve that fits $y(t)$.

The spectral content of the oscillations $y(t) \quad y(t)$ was found using a Fast Fourier Transform. Figure 1 shows the values of the oscillations for all countries as well as their spectral content.

The first four peaks correspond to periods of 30, 19, 13, and 10 years. Given that the first is very large, and the next two are unclear or slightly misaligned in some countries, the period of 10



(a) (b)

FIGURE 1. Spectral content of the oscillations $y(t)$ $y(t)$ for all countries: (a) GDP oscillation around its exponential tendency (b) Corresponding spectral contents.

years was chosen, and, using (2.4), models obtained for $N = 10$ years were found, given by (2.1), (2.2) and (2.3).

Consequently, for each country, 34 models were found, for the periods 1973–1982, 1974–1983, 1975–1984, and so on, like a moving average. Each of those models can be used with the data of future years, to see how good is the prediction. This was assessed calculating MSE, R^2, MAD, AIC and w for each country.

3. Data sources

The full data used for our calculations is not tabulated in this paper, given its extension, but is available in [38]. It was collected as follows:

Variables for the EUU were obtained as a sum of the values for its member states in each year (save for x_4, addressed below).

Values for the GDP, x_1, x_2, x_3, x_5, x_6, and x_7 were obtained from [44].

As x_2 is only available until 2015, it was assumed that $x_2(2016) = x_2(2015)$. For Belgium and Luxemburg (states of the EUU), x_2 was assumed constant until 2000, the first year for which there is data. (This corresponds to an error of, at most, 1.9% in x_2 for the EUU in those years.)

In the case of DEU, x_1 and x_3 until 1990 were taken from [47]. The variable x_2 was reduced in proportion in the same period.

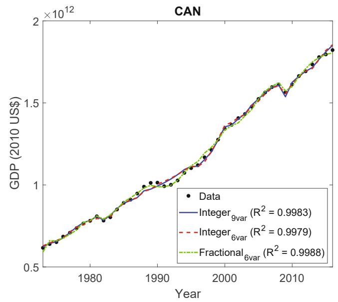
Values for x_4 were obtained from [16] (available with a 5–year period, and thus interpolated with a third-order spline) until 2010.

The figure for 2010 was then extended into the future, using the increase rate of a third-order spline interpolation of the values in [48].
As an exception to the above, x_4 values for Croatia, Estonia, Latvia, Lithuania, Slovenia and Slovak Republic (states of the EUU) are not given in [16]; so those of [44] were used instead in the same period. The value for x_4 in the EUU was then obtained as weighted average of the member states in each year, using each state's share in x_3 as the weight.
Values for x_8 for CAN were obtained from [44] until 2000. From then on the 2000 value was updated with the yearly growth rate of the index in [25].
Values for x_8 for GBR, JPN and USA were obtained from [44].
Values for x_8 for DEU, FRA, ITA and other states of the EUU were obtained from [11] until 2015, and converted to 2010 US$ with the price index in [44]. The value for 2016 is that of 2015 updated with the growth rate of [43].
As an exception to the above, values for x_8 for Luxembourg and Romania in [11] stopped in 2011 and 2013 respectively, and were updated with the growth rate of [44].

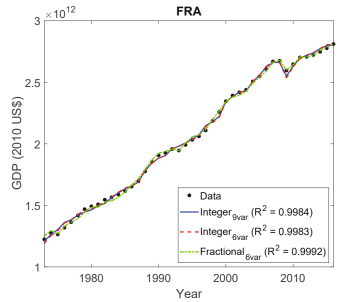
4. **Results**

This section presents an overview of the models and the predictions for the economies of the G7 in the period between 1973 to 2016. A full tabulation of results is again available in [39], due to its extension.

4.1. **Models for the entire period.** Figure 2 shows the results obtained by the models that cover the entire 1973–2016 period. The performance indices for the models are given in Table 1. In that table, t values that correspond to variables necessary for the model, assuming a 5% significance level, are given in bold (this information is also summarized in Table 2). As can be observed, the variables which are of importance for three or more countries are x_1, x_2, x_3, x_5, x_6, and x_7; this fact justifies the simplification of model (2.1) into (2.2). It is this later model that is then generalized for fractional orders. (Notice that the in"uence of x_9 was omitted as an independent variable; it was considered within x_5.)



(a)



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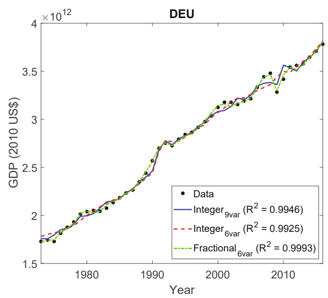

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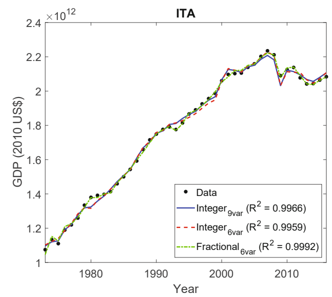

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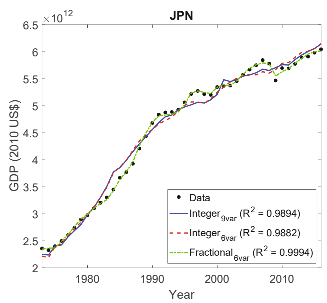

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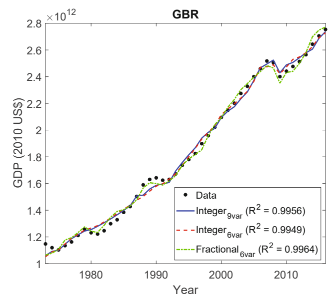

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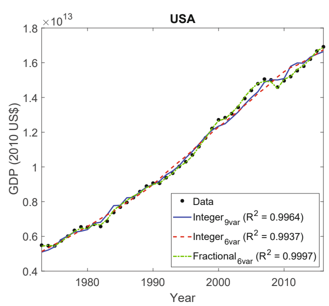
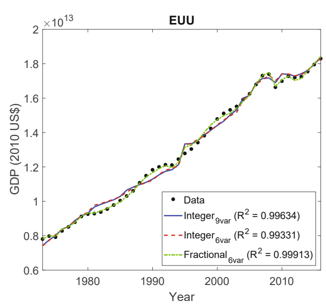


(g)



(h)

Figure 2. (... see on next page)

Figure 2. Fitting results for integer and fractional models for the G7 countries: (a) Canada (b) France (c) Germany (d) Italy (e) Japan (f) United Kingdom (g) United States of America (h) European Union. (Notice that the scale of y-axis is not the same for all countries).

Index / Statistic	Variable	**CAN** Integer (2.1)	**CAN** Integer (2.2)	**CAN** Fractional (2.3)	**FRA** Integer (2.1)	**FRA** Integer (2.2)	**FRA** Fractional (2.3)	**DEU** Integer (2.1)	**DEU** Integer (2.2)	**DEU** Fractional (2.3)	**ITA** Integer (2.1)	**ITA** Integer (2.2)	**ITA** Fractional (2.3)
MSE ($\times 10^{20}$)		2.302	2.824	1.623	4.025	4.410	2.130	21.295	29.824	2.800	4.245	5.114	1.049
R^2		0.998	0.9979	0.9988	0.9984	0.9983	0.9992	0.995	0.9925	0.9993	0.996	0.9959	0.9992
MAD ($\times 10^{10}$)		1.106	1.279	1.002	1.626	1.646	1.215	3.290	4.170	1.239	1.635	1.773	0.799
	x_1	0.012	**2.509**	**4.930**	**4.113**	**4.924**	0.787	**2.605**	**2.796**	0.256	**3.074**	**4.552**	**7.582**
	x_2	**2.924**	2.371	**3.751**	2.204	2.101	0.415	**4.362**	**4.076**	**3.199**	0.631	1.159	**5.029**
	x_3	**3.986**	**5.193**	**4.681**	**6.108**	**7.022**	0.571	2.131	**2.909**	**10.149**	**4.372**	**5.248**	**7.979**
	x_4	1.392			1.726			1.090			2.151		
t values	x_5	**4.793**	**5.767**	**8.855**	**8.078**	**10.301**	**14.931**	1.926	**5.324**	**15.600**	1.304	0.203	**19.827**
	x_6	**8.951**	**13.648**	**8.728**	**10.355**	**16.462**	**13.790**	1.433	1.723	**12.102**	**6.593**	**9.400**	**5.936**
	x_7	**5.963**	**8.537**	**6.759**	**4.718**	**7.663**	**8.703**	1.079	0.017	**32.024**	**11.354**	**20.341**	**40.507**
	x_8	0.264			0.368			9.869×10^{-2}			0.229		
	x_9	**2.403**			0.230			**3.580**			1.239		
AIC ($\times 10^3$)		2.086	2.086	2.062	2.111	2.106	2.074	2.184	2.190	2.086	2.113	2.112	2.043
w (%)		0	0	**100**	0	0	**100**	0	0	**100**	0	0	**100**

Index / Statistic	Variable	**JPN** Integer (2.1)	**JPN** Integer (2.2)	**JPN** Fractional (2.3)	**GBR** Integer (2.1)	**GBR** Integer (2.2)	**GBR** Fractional (2.3)	**USA** Integer (2.1)	**USA** Integer (2.2)	**USA** Fractional (2.3)	**EUU** Integer (2.1)	**EUU** Integer (2.2)	**EUU** Fractional (2.3)
MSE ($\times 10^{20}$)		163.6538	177.390	9.3223	12.402	14.324	9.940	62.731	85.929	4.353	759.967	799.042	129.762
R^2		0.989	0.9882	0.9994	0.995	0.9949	0.9964	0.996	0.9937	0.9997	0.9963	0.9933	0.9991
MAD ($\times 10^{10}$)		10.565	11.185	2.400	2.895	2.994	2.590	20.515	24.010	5.389	22.866	22.854	8.556
	x_1	**6.337**	**6.919**	**3.917**	**7.975**	**8.939**	**8.807**	**2.743**	**3.262**	**5.569**	**3.375**	**2.551**	**6.690**
	x_2	**5.241**	**5.563**	**5.193**	0.606	6.696×10^{-4}	1.818	1.853	0.592	0.504	1.335	**3.413**	**7.642**
	x_3	0.347	2.625	**3.077**	**5.451**	**6.205**	**7.524**	**3.486**	**3.947**	**9.225**	**2.971**	**2.645**	**7.440**
	x_4	0.715			1.467			**3.081**			**5.117**		
t values	x_5	0.384	1.542	**23.592**	**7.947**	**7.863**	**10.254**	1.062	0.054	**26.193**	**3.259**	**3.845**	**18.771**
	x_6	1.118	0.280	**8.784**	**3.137**	**4.679**	**6.471**	0.189	0.254	**7.217**	**3.022**	**3.319**	1.068
	x_7	**2.300**	**2.662**	**10.289**	0.999	0.229	**5.516**	1.348	0.891	**10.205**	0.446	**6.066**	**5.181**
	x_8	1.329			1.761			**3.340**			0.618		
	x_9	1.095			0.168			**2.552**			0.793		
AIC ($\times 10^3$)		2.274	2.268	2.139	2.160	2.158	2.142	2.333	2.338	2.207	2.341	2.335	2.255
w (%)		0	0	**100**	0	0.03	**99.97**	0	0	**100**	0	0	**100**

Table 1. Performance indices for integer and fractional models for G7

Country	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
CAN		✓	✓		✓	✓	✓		✓
FRA	✓		✓		✓	✓	✓		
DEU	✓	✓							✓
ITA	✓		✓			✓	✓		
JPN	✓	✓					✓		
GBR	✓		✓		✓	✓			
USA	✓		✓	✓				✓	✓
EUU	✓		✓	✓	✓	✓			

Table 2. Importance of the nine variables on model (2.1) for each country

4.2. **Models for 10 year periods.** Figure 3 shows how good the models for one of the nine cases considered (the one for all countries) are predicting the future evolution of the GDP. For comparison purposes, and since the GDP of different countries has different orders of magnitude, the performance of predictions obtained with each 10–year model is analyzed in terms of relative error with respect to the average value for the period as follows:

$$\frac{\text{index} \quad \text{index}_{ave}}{\text{index}_{ave}}. \tag{4.9}$$

Here, *index* refers to one of the indices MSE, R^2, and MAD.

Naturally, models developed for periods beginning in the 1970s can be used for very long predictions, while those developed for periods ending in this century can only be used for a few years. It is also to expect that predictions for many years into the future are poorer than those for just a few years. This is seen in the picture, that clearly shows the MSE and the MAD of the predictions deteriorating roughly exponentially with time, and the R^2 decreasing with time. It is also clear that relative errors begin with rather small values, showing that models are quite good at one-step ahead prediction (in this case, one-year prediction). The most striking characteristic of fractional models given by (2.3) is that, for the first two or three years, as seen in the inset of the figures, relative MSE and MAD are clearly more negative, and relative R^2 more positive. This means that fractional models have a clearly superior ability of short-term prediction: these signs correspond to prediction errors that are smaller than the average error for the period the model was actually designed for. On the other hand, fractional models have a clearly worst performance for long term predictions, but the errors of all models are by then so large that it is irrelevant what values are actually assumed.

The values of w for models obtained for periods of 10 years in one-step ahead prediction are given in Table 3 and Table 4. It can be seen that models obtained with (2.2) and (2.3) have comparable probabilities of being the best model, though with advantage for the 6–variable integer model (2.2).

5. Discussion and conclusions

The MSE, R^2 and MAD performance indications all show that models given by (2.3) clearly outperform the integer ones in what concerns the adjustment to the period of the data used to build each model. This is true for every single country and for the model that takes into account the data of all seven countries. The Akaike weight, summarized in Tables 3 and 4, shows conclusively that models given by (2.3) are in this respect the best of the three.

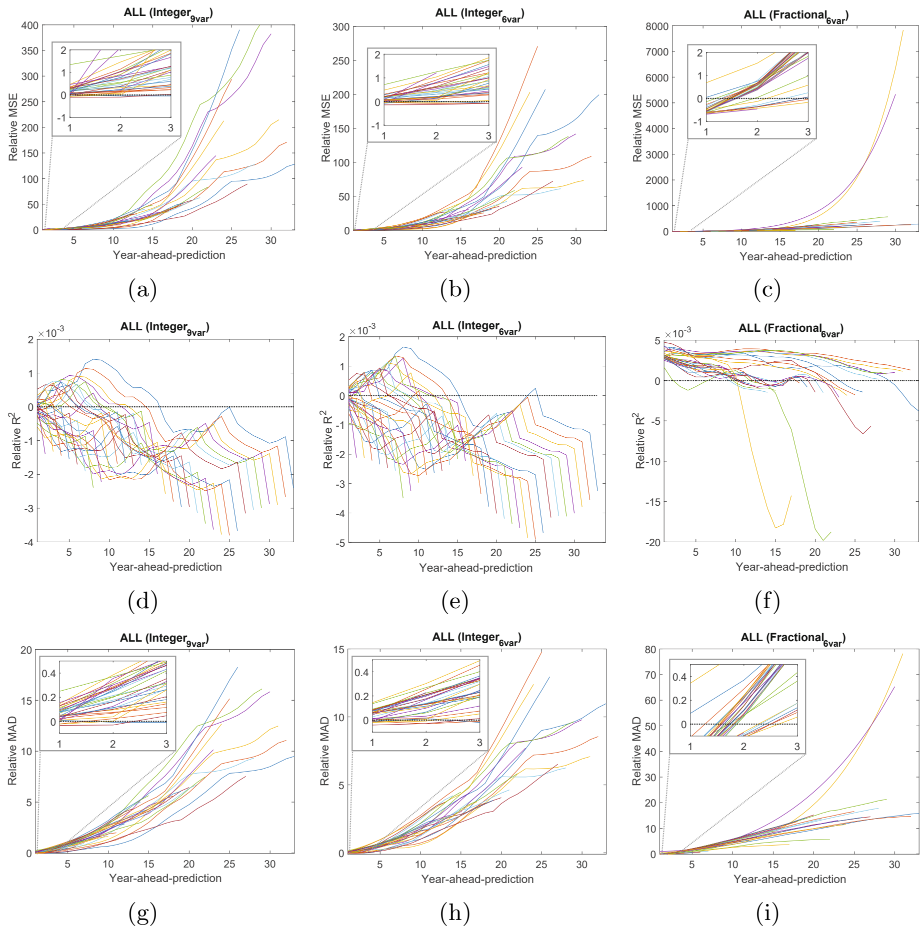


Figure 3. GDP predictions for the G7 countries for model (2.1) (left), model (2.2) (middle), and model (2.3) (right): relative MSE (top), relative R^2 (center) and relative MAD (bottom) (notice that the scale of y axis is not the same for all countries).

In the fractional models for the entire 44–year period, the order α_5 of x_5 is always 0, and so these fractional models have in fact one variable less than those given by (2.1). So this best performance comes along with a simplification of the model. Furthermore, all the variables kept are significant for the models, with the exception of only one variable (not always the same) for FRA, DEU, GBR and USA. This vindicates the choice of variables. That a good single model can be found for all countries indicates

Country		**CAN**			**DEU**			**FRA**	
Period	Model (2.1)	Model (2.2)	Model (2.3)	Model (2.1)	Model (2.2)	Model (2.3)	Model (2.1)	Model (2.2)	Model (2.3)
1974–1984	0	**50**	**50**	0	**50**	**50**	0	**50**	**50**
1975–1985	0	**50.49**	49.51	0	**50.49**	49.51	0	**50.49**	49.51
1976–1986	0	**51**	49	0	**51**	49	0	**51**	49
1977–1987	0	**51.51**	48.49	0	**51.51**	48.49	0	**51.51**	48.49
1978–1988	0	**52.04**	47.96	0	**52.04**	47.96	0	**52.04**	47.96
1979–1989	0	0	**100**	0	0	**100**	0	0	**100**
1980–1990	0	0	**100**	0	0	**100**	0	0	**100**
1981–1991	0	0	**100**	0	0	**100**	0	0.03	**99.97**
1982–1992	0	0	**100**	0	0	**100**	0	0	**100**
1983–1993	0	0	**100**	0	0	**100**	0	0	**100**
1984–1994	0	**100**	0	0	**100**	0	0	**100**	0
1985–1995	0	**100**	0	0	**100**	0	0	**100**	0
1986–1996	0	**79.16**	20.84	0	**100**	0	0	**77.94**	22.06
1987–1997	0	100	0	0	29	71	0	**68.50**	31.50
1988–1998	0	**100**	0	0	**58.75**	41.25	0	**100**	0
1989–1999	0	**100**	0	0	0	**100**	0	**100**	0
1990–2000	0	**100**	0	0	29.72	**70.28**	0	**100**	0
1991–2001	0	28	**72**	0	100	0	0	**100**	0
1992–2002	0	**99.26**	0.74	0	20.18	**79.82**	0	**100**	0
1993–2003	0	**100**	0	0	**100**	0	0	**100**	0
1994–2004	0	18.80	**81.20**	0	**100**	0	0	**100**	0
1995–2005	0	**100**	0	0	**98.43**	1.57	0	**83.30**	16.70
1996–2006	0	**98.58**	1.42	0	**95.85**	4.15	0	0.15	**99.85**
1997–2007	0	**100**	0	0	**100**	0	0	35.10	**64.90**
1998–2008	0	**78.80**	21.20	0	**99.86**	0.14	0	**100**	0
1999–2009	0	**100**	0	0	2.76	**97.20**	0	**100**	0
2000–2010	0	**100**	0	0	**100**	0	0	**100**	0
2001–2011	0	**99.99**	0.01	0	**100**	0	0	**100**	0
2002–2012	0	**100**	0	0	**100**	0	0	**52.10**	47.90
2003–2013	0	**100**	0	0	**88.46**	11.50	0	**100**	0
2004–2014	0	1.53	**98.47**	0	**100**	0	0	**100**	0
2005–2015	0	**99.96**	0.04	0	**100**	0	0	**100**	0
2006–2016	0	0	100	0	**99.97**	0.03	0	**100**	0

Country		**GBR**			**ITA**			**JPN**	
Period	Model (2.1)	Model (2.2)	Model (2.3)	Model (2.1)	Model (2.2)	Model (2.3)	Model (2.1)	Model (2.2)	Model (2.3)
1974–1984	0	**50**	**50**	0	**50**	**50**	0	**50**	**50**
1975–1985	0	**50.49**	49.50	0	**50.49**	49.50	0	**50.49**	49.50
1976–1986	0	**51**	49	0	**51**	49	0	**51**	49
1977–1987	0	**51.51**	48.48	0	**51.51**	48.48	0	**51.51**	48.48
1978–1988	0	**52.04**	47.96	0	**52.04**	47.96	0	**52.04**	47.96
1979–1989	0	0	**100**	0	0	**100**	0	0	**100**
1980–1990	0	0	**100**	0	0	**100**	0	0	**100**
1981–1991	0	0	**100**	0	0	**100**	0	0	**100**
1982–1992	0	0	**100**	0	0	**100**	0	0	**100**
1983–1993	0	0	**100**	0	0	**100**	0	0	**100**
1984–1994	0	**100**	0	0	**100**	0	0	23.77	**76.20**
1985–1995	0	35.42	**64.58**	0	**100**	0	0	15.53	**84.47**
1986–1996	0	**100**	0	0	**97.17**	2.83	0	**66.27**	33.73
1987–1997	0	**100**	0	0	**93.10**	6.90	0	**100**	0
1988–1998	0	**100**	0	0	**100**	0	0	34.06	**65.90**
1989–1999	0	**100**	0	0	**100**	0	0	**99.99**	0.01
1990–2000	0	**100**	0	0	**66.10**	33.90	0	**100**	0
1991–2001	0	**100**	0	0	**100**	0	0	**100**	0
1992–2002	0	**100**	0	0	**100**	0	0	**99.99**	0.01
1993–2003	0	**100**	0	0	**100**	0	0	**99.96**	0.04
1994–2004	0	25.87	**74.115**	0	**100**	0	0	**100**	0
1995–2005	0	**63.11**	36.90	0	**100**	0	0	**100**	0
1996–2006	0	0	**100**	0	**100**	0	0	**100**	0
1997–2007	0	0.89	**99.11**	0	8.06	**91.90**	0	**100**	0
1998–2008	0	**100**	0	0	**100**	0	0	**100**	0
1999–2009	0	**100**	0	0	**100**	0	0	**100**	0
2000–2010	0	**100**	0	0	**100**	0	0	**100**	0
2001–2011	0	**100**	0	0	**100**	0	0	**100**	0
2002–2012	0	**100**	0	0	**100**	0	0	**100**	0
2003–2013	0	**100**	0	0	**100**	0	0	**99.91**	0.09
2004–2014	0	45.78	**54.22**	0	**99.91**	0.09	0	**100**	0
2005–2015	0	**100**	0	0	**100**	0	0	**100**	0
2006–2016	0	**100**	0	0	**100**	0	0	**100**	0

TABLE 3. Values of w (in %) for predictions for all countries.

that the choice of variables is reasonable; again, the fractional version of this model outperforms those with integer orders.

Models for periods of 10 years prove to be equally superior, and also to have a good ability of prediction in the short term. This shows that the choice of the Caputo derivative is reasonable. As to the validation process

Country	USA			EUU			ALL		
Period	Model (2.1)	Model (2.2)	Model (2.3)	Model (2.1)	Model (2.2)	Model (2.3)	Model (2.1)	Model (2.2)	Model (2.3)
1974–1984	0	**50**	**50**	0	**50**	**50**	0	**54.90**	45.10
1975–1985	0	**50.49**	49.51	0	**50.49**	49.51	0	**54.46**	45.54
1976–1986	0	**51**	49	0	**51**	49	0	**54**	46
1977–1987	0	**51.52**	48.48	0	**51.52**	48.48	0	**52.52**	47.48
1978–1988	0	**52.04**	47.96	0	**52.04**	47.96	0	**52.04**	47.96
1979–1989	0	0	**100**	0	0	**100**	0	0	**100**
1980–1990	0	0	**100**	0	0	**100**	0	0	**100**
1981–1991	0	0	**100**	0	0	**100**	0	0	**100**
1982–1992	0	0	**100**	0	0	**100**	0	0	**100**
1983–1993	0	0	**100**	0	0	**100**	0	0	**100**
1984–1994	0	**100**	0	0	**100**	0	**72.55**	27.45	0
1985–1995	0	**100**	0	0	**100**	0	**71.41**	28.59	0
1986–1996	0	**100**	0	0	**100**	0	**100**	0	0
1987–1997	0	**100**	0	0	**81.20**	18.80	**100**	0	0
1988–1998	0	**100**	0	0	**99.89**	0.11	**100**	0	0
1989–1999	0	**99.88**	0.12	0	**100**	0	**100**	0	0
1990–2000	0	**100**	0	0	**55.36**	44.60	**100**	0	0
1991–2001	0	**100**	0	0	0	**100**	**74.06**	25.94	0
1992–2002	0	**100**	0	0	49.85	**50.15**	**52.02**	46.98	0
1993–2003	0	**100**	0	0	**100**	0	**50.01**	49.99	0
1994–2004	0	**100**	0	0	**100**	0	48.98	**51.02**	0
1995–2005	0	0.05	**99.95**	0	**100**	0	**99.05**	0.95	0
1996–2006	0	**100**	0	0	**100**	0	**100**	0	0
1997–2007	0	**100**	0	0	**100**	0	**100**	0	0
1998–2008	0	**100**	0	0	**100**	0	**100**	0	0
1999–2009	0	**100**	0	0	**99.93**	0.07	**100**	0	0
2000–2010	0	**100**	0	0	**100**	0	**100**	0	0
2001–2011	0	**100**	0	0	**100**	0	**100**	0	0
2002–2012	0	0	**100**	0	**99.94**	0.06	**100**	0	0
2003–2013	0	2.70	**97.30**	0	0.40	**99.60**	**100**	0	0
2004–2014	0	**92.68**	7.32	0	**100**	0	**100**	0	0
2005–2015	0	0	**100**	0	**76.67**	23.33	**50**	**50**	0
2006–2016	0	**99.77**	0.23	0	**100**	0	0	**100**	0

TABLE 4. Values of w (in %) for predictions for all countries (continued).

outlined in sections 2.2 and 2.3, it can serve to verify the performance of models of economic growth, in what concerns their adequacy to the period for which they were developed and to their prediction capability.

The main conclusion to take is that fractional derivatives prove to be a good option for the accurate modelling of economic growth. This is likely due to a better capture of the diffusion of some of the factors that condition economic growth, and of the diffusion of economic growth itself [4, 27, 30], since fractional derivatives model anomalous diffusion phenomena. Furthermore, fractional operators are non-local and possess a memory effect. This makes them more suitable for models for long series than models using integer derivatives and integrals alone. This is likely why fractional differential equations are able to describe economic growth over large time periods.

Acknowledgements

This work was partially supported by FEDER Funds (Programa Operativo FEDER de Extremadura 2014-2020) through the grant Ayuda a Grupos de Investigación (ref. GR15178) of the Junta de Extremadura and by FCT, through IDMEC, under LAETA, project UID/EMS/50022/2019. Inés Tejado would like to thank the Junta de Extremadura its support through the scholarship granted within the mobility program of 2017.

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[1] *Industrial Engineering School*
University of Extremadura
Avenida de Elvas, s/n 06006 Badajoz, SPAIN
e-mails: {itejbal,emilianoph}@unex.es
Received: October 1, 2018

[2] *IDMEC, Instituto Superior Técnico*
Universidade de Lisboa
Av. Rovisco Pais 1 1049-001 Lisbon, PORTUGAL
e-mail: duarte.valerio@tecnico.ulisboa.pt

Please cite to this paper as published in:

Fract. Calc. Appl. Anal., Vol. **22**, No 1 (2019), pp. 139–157,
DOI: 10.1515/fca-2019-0009; at https://www.degruyter.com/view/j/fca.