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RESEARCH PAPER

ON THE BEHAVIOR OF SOLUTIONS OF FRACTIONAL DIFFERENTIAL EQUATIONS ON TIME SCALE VIA HILFER FRACTIONAL DERIVATIVES

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Abstract

In this paper, we study the existence and stability of Hilfer-type fractional differential equations (dynamic equations) on time scales. We obtain sufficient conditions for existence and uniqueness of solutions by using classical fixed point theorems such as Schauder's fixed point theorem and Banach fixed point theorem. In addition, Ulam stability of the proposed problem is also discussed. As in application, we provide an example to illustrate our main results.

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1. **Introduction**

The analysis of dynamic equations on time scales, which goes back to its initiator Stefan Hilger, is an area of mathematics that has recently gained a lot of interest. It has been formed in order to unify the study of differential and difference equations. Many results involving differential equations carry over quite easily to corresponding results for difference equations, while other results seem to be totally different from their continuous counterparts. The study of dynamic equations on time scales shows such discrepancies,

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and helps avoid proving results twice-once for differential equations and once again for difference equations [7, 2].

The general idea is to prove a result for a dynamic equation where the domain of the unknown function is a so-called time scale, which may be an arbitrary closed subset of the reals. This way results not only related to the set of real numbers or set of integers but those pertaining to more general time scales are obtained. The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus [6, 1]. Dynamic equations on a time scale have enormous potential for applications such as in population dynamics. For example, it can model insect populations that are continuous while in season, die out in say winter, while their eggs are incubating or dormant, and then hatch in a new season, giving rise to a nonoverlapping population. Bohner and Peterson have expounded on various aspects of this new theory in the basic books by [4].

In recent years, the theory of fractional differential equations (FDEs) has played a very important role in a new branch of applied mathematics, which has been utilized for mathematical models in engineering, physics, chemistry, signal analysis, etc. There has been a tremendous development in the study of differential equations involving fractional derivatives (see [5, 17, 19, 24], and the references therein). Recently, a study of Hilfer-type of equation has received a significant amount of attention, we refer to [9, 10, 11, 13, 25] and the references therein. The objective of this paper is that existence and stability results devoted to dynamic equations on a time scale with Hilfer fractional derivative. Recently, Ahmadkhanlu et al.[2] investigated the existence and uniqueness results for FDEs on time scales. By Followed, Benkhettou et al. [6] studied the existence and uniqueness of solution for an initial value problem of FDEs on time scales involved Riemann-Lioville (R-L) derivative.

Motivated by the papers [2, 6], we consider the dynamic equation on time scales with Hilfer fractional derivative (HFD) of the form

$$\begin{cases} {}^{\mathbb{T}}\Delta_{0^+}^{\alpha,\beta}x(t) = f(t, x(t)), \quad t = [0, b] := J \subseteq \mathbb{T}, \\ {}^{\mathbb{T}}I_{0^+}^{1-\gamma}x(0) = x_0, \quad \gamma = \alpha + \beta - \alpha\beta, \end{cases} \tag{1.1}$$

where ${}^{\mathbb{T}}\Delta_{0^+}^{\alpha,\beta}$ is the HFD defined on $\mathbb{T}$, $0 < \alpha < 1$, $0 \leq \beta \leq 1$. Let $\mathbb{T}$ be a time scale, that is nonempty subset of Banach space. The function $f : J \times \mathbb{T} \to R$ is a right-dense continuous function.

The Ulam stability of functional equation, which was invented by Ulam on a talk given to a conference at Wisconsin University in 1940, is one of the essential subjects in the mathematical analysis area. The finding of Ulam stability plays a pivotal role in regard to this subject. For extensive study on the advance of Ulam type stability, one can refer to [3, 15, 16, 18] and the references therein. The credit of solving this problem partially goes to Hyers. To study Hyers-Ulam stability of FDEs, different researchers presented their works with different methods, see [14, 23, 24].

2. **Preliminaries**

In this section, we introduce definitions, notations and preliminary facts which are used in the sequel. An extensive study of the analysis on time scales can be found in [1].

2.1. **Time scales.**

By a time scale $\mathbb{T}$ we mean any closed subset of Banach space. Since a time scale $\mathbb{T}$ is not connected in generally, we need the concept of jump operators. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$, while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$. In this definition we put $inf\Theta = \sup \mathbb{T}$ and $\sup \Theta = \inf \mathbb{T}$. If $\sigma(t) > t$, we say t is left-scattered points. Points that are left-scattered point, while if $\rho(t) < t$, we say t is a left-scattered and left-scattered at the same time will be called isolated points. A point $t \in \mathbb{T}$ such that $t < \sup\{\mathbb{T}\}$ and $\sigma(t) = t$, is called a right-dense point. A point $t > \mathbb{T}$ such that $t \in \inf\{\mathbb{T}\}$ and $\rho(t) = t$, is called a left-dense point. Points that are right-dense and left-dense at the same time will be called dense points.

Definition 2.1. A function $f : \mathbb{T} \to R$ is called regulated if its right-sided limits exists (finite) at all right-dense points in $\mathbb{T}$, and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$. A function $f : \mathbb{T} \to R$ is called rd-continuous if it is continuous at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$.

Definition 2.2. Let $f : \mathbb{T} \to R$ and $t \in \mathbb{T}^k (= \mathbb{T})$. We define $f^\Delta \in R$ (provided it exists) with the property that for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\left| f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s] \right| \leq \epsilon \left| \sigma(t) - s \right|$$

for all $s \in U_{\mathbb{T}}(t, \delta)$. We recall $f^\Delta(t)$ the delta derivative (Δ-derivative for short) of f at t_0. Moreover, we say that f is delta differentiable (Δ-differentiable for short) on $\mathbb{T}^k$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^k$.

Proposition 2.1. *Suppose that $f : \mathbb{T} \to R$ is a continuous function on $\mathbb{T}$ and Δ-differentiable on $\mathbb{T}^k$. If $f^\Delta(t) \geq 0$ for all $t \in \mathbb{T}^k$, then f is* *nondecreasing at on* $\mathbb{T}$. *If* $f^{\Delta}(t_0) \leq 0$ *for all* $t \in \mathbb{T}^k$, *then* f *is nonincreasing on* $\mathbb{T}$.

Definition 2.3. Let $[a, b]$ be a closed bounded interval in $\mathbb{T}$. A function $F : [a, b] \to R$ is called a delta antiderivative of a function $f : [a, b) \to R$ provided that F is continuous on $[a, b]$ and Δ-differentiable on $[a, b)$; and $F^{\Delta}(t) = f(t)$ for all $t \in [a, b)$.Then we define the Δ-integral from a to b of f by

$$\int_a^b f(t)\Delta t = F(b) - F(a).$$

Proposition 2.2. *Suppose* $a, b \in \mathbb{T}$, $a < b$ *and* $f(t)$ *is continuous on* $[a, b]$, *then we have*

$$\int_a^b f(t)\Delta t = [\sigma(a) - a]\, f(a) + \int_{\sigma(a)}^b f(t)\Delta t.$$

Proposition 2.3. *Suppose* $\mathbb{T}$ *is a time scale and* $[a, b] \subset \mathbb{T}$, f *is increasing continuous function on* $[a, b]$. *If the extension of* f *is given in the following form:*

$$F(s) = \begin{cases} f(s); & s \in \mathbb{T} \\ f(t); & s \in (t, \sigma(t)) \notin \mathbb{T}. \end{cases}$$

Then we have

$$\int_a^b f(t)\Delta t \leq \int_a^b F(t)dt.$$

Definition 2.4. Let $\mathbb{T}$ be a time scale, $J \in \mathbb{T}$. The left-sided R-L fractional integral of order $\alpha \in R^+$ of function $f(t)$ is defined by

$$\left({}^{\mathbb{T}}I_{0^+}^{\alpha} f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} f(s)\Delta s, \quad (t > 0),$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.5. Suppose $\mathbb{T}$ is a time scale, $[0, b]$ is an interval of $\mathbb{T}$. The left-sided R-L fractional derivative of order $\alpha \in [n - 1, n)$, $n \in \mathbb{Z}^+$ of function $f(t)$ is defined by

$$\left({}^{\mathbb{T}}\Delta_{0^+}^{\alpha} f\right)(t) = \frac{1}{\Gamma(n - \alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t - s)^{n - \alpha - 1} f(s)\Delta s, \quad (t > 0).$$

Based on differentiating fractional integrals, a generalized definition called HFD can be proposed. The HFD is considered as an interpolator between the R-L and Caputo derivatives. For information on HFD, one can refer to [20, 21, 22].

Definition 2.6. The left-sided HFD of order $0 < \alpha < 1$ and $0 \leq \beta \leq 1$ of function $f(t)$ is defined by

$$^{\mathbb{T}}\Delta_{0^+}^{\alpha,\beta} f(t) = \left({}^{\mathbb{T}}I_{0^+}^{\beta(1-\alpha)}\ {}^{\mathbb{T}}\Delta({}^{\mathbb{T}}I_{0^+}^{(1-\beta)(1-\alpha)} f) \right)(t),$$

where ${}^{\mathbb{T}}\Delta := \frac{d}{dt}$.

Remark 2.1. [11, 13, 25]

(1) The operator ${}^{\mathbb{T}}\Delta_{0^+}^{\alpha,\beta}$ also can be written as

$$^{\mathbb{T}}\Delta_{0^+}^{\alpha,\beta} = {}^{\mathbb{T}}I_{0^+}^{\beta(1-\alpha)}\ {}^{\mathbb{T}}\Delta\ {}^{\mathbb{T}}I_{0^+}^{(1-\beta)(1-\alpha)} = {}^{\mathbb{T}}I_{0^+}^{\beta(1-\alpha)}\ {}^{\mathbb{T}}\Delta_{0^+}^{\gamma},\ \gamma = \alpha + \beta - \alpha\beta.$$

(2) Let $\beta = 0$, the left-sided R-L derivative can be presented as ${}^{\mathbb{T}}\Delta_{0^+}^{\alpha} := {}^{\mathbb{T}}\Delta_{0^+}^{\alpha,0}$.

(3) Let $\beta = 0$, left-sided Caputo fractional derivative can be presented as ${}_{c}^{\mathbb{T}}\Delta_{0^+}^{\alpha} := {}^{\mathbb{T}}I_{0^+}^{1-\alpha}\ {}^{\mathbb{T}}\Delta$.

Throughout this study, let $C[J, R]$ endowed with the norm $\|x\|_C = \max\{|x(t)| : t \in J\}$ and $L^1(J)$ is the space of Lebesgue-integrable functions $x : J \to R$ with the norm

$$\|x\|_1 = \int_1^b |x(s)|\, ds.$$

Definition 2.7. For $0 \leq \gamma < 1$, we denote the space $C_\gamma[J, R]$ as

$$C_\gamma[J, R] := \{f(t) : J \to R | t^\gamma f(t) \in C[J, R]\},$$

where $C_\gamma[J, R]$ is the weighted space of the continuous functions f on the finite interval J.

Obviously, $C_\gamma[J, R]$ is the Banach space with the norm

$$\|f\|_{C_\gamma} = \|t^\gamma f(t)\|_C.$$

Meanwhile, $C_\gamma^n[J, R] := \left\{ f \in C^{n-1}[J, R] : f^{(n)} \in C_\gamma[J, R] \right\}$ is the Banach space with the norm

$$\|f\|_{C_\gamma^n} = \sum_{i=0}^{n-1} \left\| f^{(k)} \right\|_C + \left\| f^{(n)} \right\|_{C_\gamma},\ n \in \mathbb{N}.$$

Moreover, $C^0_\gamma[J,R] := C_\gamma[J,R]$.

We introduce the spaces which are used to solve our problems.

$$C^{\alpha,\beta}_{1-\gamma} = \left\{ f \in C_{1-\gamma}[J,R], \Delta^{\alpha,\beta}_{0^+} f \in C_{1-\gamma}[J,R] \right\}$$

and

$$C^{\gamma}_{1-\gamma} = \left\{ f \in C_{1-\gamma}[J,R], \Delta^{\gamma}_{0^+} f \in C_{1-\gamma}[J,R] \right\}.$$

It is obvious that

$$C^{\gamma}_{1-\gamma}[J,R] \subset C^{\alpha,\beta}_{1-\gamma}[J,R].$$

Definition 2.8. The expression

$${}^{\mathbb{T}}\Delta^{\alpha}_{0^+} f(x) := {}^{\mathbb{T}}\Delta \; {}^{\mathbb{T}}I^{\alpha-1}_{0^+} f(t), \quad t > 0, \; 0 < \alpha < 1,$$

is called the left-sided R-L fractional derivative of order α of f provided the right-hand side exists.

Next, we review some lemmas which will be used to establish our existence results.

Lemma 2.1. [25] *If $\alpha > 0$ and $\beta > 0$, there exist*

$$\left[{}^{\mathbb{T}}I^{\alpha}_{0^+} s^{\beta-1} \right](t) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} t^{\beta+\alpha-1}$$

and

$$\left[{}^{\mathbb{T}}\Delta^{\alpha}_{0^+} s^{\alpha-1} \right](t) = 0, \quad 0 < \alpha < 1.$$

Lemma 2.2. *Let $\alpha > 0$ and $0 \leq \gamma \leq 1$. Then, ${}^{\mathbb{T}}I^{\alpha}_{0^+}$ is bounded from $C_\gamma[J,R]$ into $C_\gamma[J,R]$.*

Lemma 2.3. *Let $\alpha > 0$ and $0 \leq \gamma < 1$. If $\gamma \leq \alpha$, then ${}^{\mathbb{T}}I^{\alpha}_{0^+}$ is bounded from $C_\gamma[J,R]$ into $C[J,R]$.*

Lemma 2.4. *$f \in L^1(J)$ and ${}^{\mathbb{T}}\Delta^{\beta(1-\alpha)}_{0^+} f \in L^1(J)$ existed, then*

$${}^{\mathbb{T}}\Delta^{\alpha,\beta}_{0^+} \; {}^{\mathbb{T}}I^{\alpha}_{0^+} f = \; {}^{\mathbb{T}}I^{\beta(1-\alpha)}_{0^+} \; {}^{\mathbb{T}}\Delta^{\beta(1-\alpha)}_{0^+} f.$$

Lemma 2.5. *For $0 \leq \gamma < 1$ and $f \in C_\gamma[J,R]$, then*

$${}^{\mathbb{T}}I^{\alpha}_{0^+} f(0) := \lim_{t \to 0^+} {}^{\mathbb{T}}I^{\alpha}_{0^+} f(t) = 0, \; 0 \leq \gamma < \alpha.$$

Lemma 2.6. *Let $\alpha \geq 0$, $\beta \geq 0$ and $f \in L^1(J)$. Then*

$$ {}^{\mathbb{T}}I_{0^+}^{\alpha} \; {}^{\mathbb{T}}I_{0^+}^{\beta} f(t) \overset{a.e}{=} {}^{\mathbb{T}}I_{0^+}^{\alpha+\beta} f(t), \; t \in J. $$

In particular, if $f \in C_\gamma[J,R]$ or $f \in C[J,R]$, then equality holds at every $t \in J$.

Lemma 2.7. *Let $0 < \alpha < 1$, $0 \leq \gamma < 1$. If $f \in C_\gamma[J,R]$ and ${}^{\mathbb{T}}I_{0^+}^{1-\alpha} f \in C_\gamma^1[J,R]$, then*

$$ {}^{\mathbb{T}}I_{0^+}^{\alpha} \; {}^{\mathbb{T}}\Delta_{0^+}^{\alpha} f(t) = f(t) - \frac{\left({}^{\mathbb{T}}I_{0^+}^{1-\alpha} f\right)(0)}{\Gamma(\alpha)} t^{\alpha-1}, \quad \forall \; t \in J. $$

Definition 2.9. The right-sided fractional derivative operator of order $0 < \alpha < 1$ and $0 \leq \beta \leq 1$ is defined by

$$ {}^{\mathbb{T}}\Delta_{0^+}^{\alpha,\beta} = {}^{\mathbb{T}}I_{0^+}^{\beta(1-\alpha)} \; {}^{\mathbb{T}}\Delta \; {}^{\mathbb{T}}I_{0^+}^{(1-\beta)(1-\alpha)}. $$

Lemma 2.8. *Let $0 < \alpha < 1$, $0 \leq \beta \leq 1$ and $\gamma = \alpha + \beta - \alpha\beta$. If $f \in C_{1-\gamma}^{\gamma}[J,R]$, then*

$$ {}^{\mathbb{T}}I_{0^+}^{\gamma} \; {}^{\mathbb{T}}\Delta_{0^+}^{\gamma} f = {}^{\mathbb{T}}I_{0^+}^{\alpha} \; {}^{\mathbb{T}}\Delta_{0^+}^{\alpha,\beta} f, $$

and

$$ {}^{\mathbb{T}}\Delta_{0^+}^{\gamma} \; {}^{\mathbb{T}}I_{0^+}^{\alpha} f = {}^{\mathbb{T}}\Delta_{0^+}^{\beta(1-\alpha)} f. $$

Lemma 2.9. *Let $0 < \alpha < 1$, $0 \leq \beta \leq 1$ and $\gamma = \alpha + \beta - \alpha\beta$. If $f \in C_{1-\gamma}[J,R]$ and ${}^{\mathbb{T}}I_{0^+}^{1-\beta(1-\alpha)} f$ in $C_{1-\gamma}^1[J,R]$, then ${}^{\mathbb{T}}I_{0^+}^{\alpha} \; {}^{\mathbb{T}}\Delta_{0^+}^{\alpha,\beta} f$ exists in J and*

$$ {}^{\mathbb{T}}I_{0^+}^{\alpha} \; {}^{\mathbb{T}}\Delta_{0^+}^{\alpha,\beta} f(t) = f(t). $$

P r o o f. By Lemma 2.8, we have

$$ {}^{\mathbb{T}}I_{0^+}^{\alpha} \; {}^{\mathbb{T}}\Delta_{0^+}^{\alpha,\beta} f(t) = {}^{\mathbb{T}}I_{0^+}^{\gamma} \; {}^{\mathbb{T}}\Delta_{0^+}^{\gamma} f(t), $$

and applying Lemma 2.5 and Lemma 2.7

$$ {}^{\mathbb{T}}I_{0^+}^{\alpha} \; {}^{\mathbb{T}}\Delta_{0^+}^{\alpha,\beta} f(t) = f(t) - \frac{\left({}^{\mathbb{T}}I_{0^+}^{1-\gamma} f\right)(0)}{\Gamma(\gamma)} t^{\gamma-1}. $$

Finally, we get

$$ {}^{\mathbb{T}}I_{0^+}^{\alpha} \; {}^{\mathbb{T}}\Delta_{0^+}^{\alpha,\beta} f(t) = f(t). $$

$\square$

LEMMA 2.10. [5] *Let* $v : [0,b] \to [0,\infty)$ *be a real function and* $w(\cdot)$ *is a nonnegative, locally integrable function on* $[0,b]$ *and there are constants* $a > 0$ *and* $0 < \alpha < 1$ *such that*

$$v(t) \leq w(t) + a\int_0^t \frac{v(s)}{(t-s)^\alpha}ds.$$

Then there exists a constant $K = K(\alpha)$ *such that*

$$v(t) \leq w(t) + Ka\int_0^t \frac{w(s)}{(t-s)^\alpha}ds,$$

for every $t \in [0,b]$.

3. Existence results

This section is concerned to the existence of solution of the problem (1.1). We adopt the same idea as in [9]. We begin with the following lemma.

LEMMA 3.1. *Suppose* $J = [0,b] \subseteq \mathbb{T}$. *Let* $\gamma = \alpha + \beta - \alpha\beta$, *where* $0 < \alpha < 1$ *and* $0 \leq \beta \leq 1$. *Let* $f : J \times R \to R$ *be the function such that* $f(\cdot, x(\cdot)) \in C_{1-\gamma}[J,R]$ *for any* $x \in C_{1-\gamma}[J,R]$. *If* $x \in C^\gamma_{1-\gamma}[J,R]$, *then* x *satisfies* (1.1) *if and only if* x *satisfies the following integral equation*

$$x(t) = \frac{x_0}{\Gamma(\gamma)}t^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,x(s))\Delta s, \quad t > 0. \tag{3.1}$$

P r o o f. Let $x \in C^\gamma_{1-\gamma}[J,R]$ be a solution of (1.1). We want to prove that x is also a solution of the integral equation (3.1). By the definition of $C^\gamma_{1-\gamma}[J,R]$, Lemma 2.3 and Definition 2.6, we have ${}^{\mathbb{T}}I^{1-\gamma}_{0^+}x \in C[J,R]$ and ${}^{\mathbb{T}}\Delta^\gamma_{0^+}x = {}^{\mathbb{T}}\Delta({}^{\mathbb{T}}I^{1-\gamma}_{0^+}x) \in C_{1-\gamma}[J,R]$.

Thus by Definition 2.7 we have

$${}^{\mathbb{T}}\Delta^{1-\gamma}_{0^+}x \in C^1_{1-\gamma}[J,R].$$

Now we apply Lemma 2.7 to obtain

$${}^{\mathbb{T}}I^\gamma_{0^+}\ {}^{\mathbb{T}}\Delta^\gamma_{0^+}x(t) = x(t) - \frac{x_0}{\Gamma(\gamma)}t^{\gamma-1}, \quad t \in J, \tag{3.2}$$

Since by our hypothesis ${}^{\mathbb{T}}\Delta^\gamma_{0^+}x \in C_{1-\gamma}[J,R]$, Lemma 2.8 yields

$${}^{\mathbb{T}}I^\gamma_{0^+}\ {}^{\mathbb{T}}\Delta^\gamma_{0^+}x = {}^{\mathbb{T}}I^\alpha_{0^+}\ {}^{\mathbb{T}}\Delta^{\alpha,\beta}_{0^+}x = {}^{\mathbb{T}}I^\alpha_{0^+}f, \quad \text{in } J. \tag{3.3}$$

From (3.2)-(3.3) we obtain

$$x(t) = \frac{x_0}{\Gamma(\gamma)}t^{\gamma-1} + \left[{}^{\mathbb{T}}I^\gamma_{0^+}f(s,x(s))\right](t), \quad t \in J, \tag{3.4}$$

which is equation (3.1).

Now we prove the sufficiency. Let $x \in C^{\gamma}_{1-\gamma}[J,R]$ satisfy equation (3.1) which can be written as (3.4). Applying the operator ${}^{\mathbb{T}}\Delta^{\gamma}_{0^+}$ to both sides of (3.4), it follows from Lemma 2.1, Lemma 2.8 and Definition 2.8 that

$$ {}^{\mathbb{T}}\Delta^{\gamma}_{0^+}x = {}^{\mathbb{T}}\Delta^{\beta(1-\alpha)}_{0^+}f. \tag{3.5} $$

From (3.5) and the hypothesis ${}^{\mathbb{T}}\Delta^{\gamma}_{0^+}x \in C_{1-\gamma}[J,R]$, we have

$$ {}^{\mathbb{T}}\Delta\ {}^{\mathbb{T}}I^{1-\beta(1-\alpha)}_{0^+}f = {}^{\mathbb{T}}\Delta^{\beta(1-\alpha)}_{0^+}f \in C_{1-\gamma}[J,R]. \tag{3.6} $$

Also, since $f \in C_{1-\gamma}[J,R]$, by Lemma 2.2,

$$ {}^{\mathbb{T}}I^{1-\beta(1-\alpha)}_{0^+}f \in C_{1-\gamma}[J,R]. \tag{3.7} $$

It follows from (3.6) and (3.7) and the Definition 2.8 that

$$ {}^{\mathbb{T}}I^{1-\beta(1-\alpha)}_{0^+}f \in C^{1}_{1-\gamma}[J,R]. $$

Thus f and ${}^{\mathbb{T}}I^{1-\beta(1-\alpha)}f$ satisfy the conditions of Lemma 2.7.

Now by ${}^{\mathbb{T}}I^{\beta(1-\alpha)}_{0^+}$ to both sides of (3.5) and using Definition 2.9 and Lemma 2.7, we can write

$$ {}^{\mathbb{T}}\Delta^{\alpha,\beta}_{0^+}x(t) = f(t,x(t)) - \frac{\left[{}^{\mathbb{T}}I^{1-\beta(1-\alpha)}_{0^+}f(s,x(s))\right](0)}{\Gamma(\beta(1-\alpha))}t^{\beta(1-\alpha)-1}. \tag{3.8} $$

Since $1-\gamma < 1-\beta(1-\alpha)$, Lemma 2.5 implies that

$$ \left[{}^{\mathbb{T}}I^{1-\beta(1-\alpha)}_{0^+}f(s,x(s))\right](0) = 0. $$

Hence the relation (3.8) reduces to

$$ {}^{\mathbb{T}}\Delta^{\alpha,\beta}_{0^+}x(t) = f(t,x(t)), \quad t \in J. \tag{3.9} $$

Now, we show that the initial condition ${}^{\mathbb{T}}I^{1-\gamma}_{0^+}x(0) = x_0$ also holds. We apply ${}^{\mathbb{T}}I^{1-\gamma}_{0^+}$ to both sides of (3.4), then Lemma 2.1 and 2.6 imply that

$$ {}^{\mathbb{T}}I^{1-\gamma}_{0^+}x(t) = x_0 + \left[{}^{\mathbb{T}}I^{1-\beta(1-\alpha)}_{0^+}f(s,x(s))\right](t). \tag{3.10} $$

In (3.10), taking the limits as $t \to 0$, we obtain

$$ \begin{aligned} {}^{\mathbb{T}}I^{1-\gamma}_{0^+}x(0) &= x_0 + \left[{}^{\mathbb{T}}I^{1-\beta(1-\alpha)}_{0^+}f(s,x(s))\right](0) \\ &= x_0. \end{aligned} $$

This completes the proof. □

For further investigation, we give the following assumptions:

(H1) The function $f : J \times R \to R$ is a rd-continuous.

(H2) The function f is completely continuous and there exists a function $\mu \in L^1(J)$ such that

$$|f(t,x)| \leq \mu(t), \quad t \in J, \; x \in R.$$

(H3) Let f be a rd-continuous bounded function such that $f(\cdot, x(\cdot)) \in C_{1-\gamma}^{\beta(1-\alpha)}[J,R]$ for any $x \in C_{1-\gamma}[J,R]$ and there exists a positive constants $L > 0$ such that

$$|f(t,x) - f(t,y)| \leq L\,|x-y|\,.$$

THEOREM 3.1. *Assume that (H1)-(H2) are fulfilled. Then, equation* (1.1) *has at least one solution.*

P r o o f. We shall use Schauder's fixed point theorem [8]. The proof will be given in several steps as followed in [2, 6].

Consider the operator $N : C_{1-\gamma}[J,R] \to C_{1-\gamma}[J,R]$. The equivalent Volterra integral equation (3.1) which can be written in the operator form

$$x(t) = (Nx)(t), \tag{3.11}$$

where

$$(Nx)(t) = x_0(t) + \left[{}^{\mathbb{T}}I_{0+}^{\alpha} f(s,x(s))\right](t) \tag{3.12}$$

with

$$x_0(t) = \frac{x_0}{\Gamma(\gamma)} t^{\gamma-1}. \tag{3.13}$$

We shall show that the operator N is continuous and completely continuous.

Claim 1: N is continuous.

Let x_n be a sequence such that $x_n \to x$ in $C_{1-\gamma}[J,R]$. Then for each $t \in J$,

$$\begin{aligned}
&\left|t^{1-\gamma}((Nx_n)(t) - (Nx)(t))\right| \\
&\leq \frac{t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left|f(s,x_n(s)) - f(s,x(s))\right| \Delta s \\
&\leq \frac{t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sup_{s\in J} \left|f(s,x_n(s)) - f(s,x(s))\right| \Delta s \\
&\leq \frac{t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left|f(s,x_n(s)) - f(s,x(s))\right| ds, \quad \text{(by Proposition 2.3)} \\
&\leq \frac{b^{\alpha}}{\Gamma(\alpha)} B(\gamma,\alpha) \left\|f(s,x_n(\cdot)) - f(s,x(\cdot))\right\|_{C_{1-\gamma}},
\end{aligned}$$

where we use the formula

$$\int_a^t (t-s)^{\alpha-1} |x(s)|\, ds \le \left(\int_a^t (t-s)^{\alpha-1}(s-a)^{\gamma-1} ds\right) \|x\|_{C_{1-\gamma}}$$
$$= (t-a)^{\alpha+\gamma-1} B(\gamma,\alpha) \|x\|_{C_{1-\gamma}}.$$

Since f is continuous, Lebesgue dominated convergence theorem implies

$$\|Nx_n - Nx\|_{C_{1-\gamma}} \to 0 \quad \text{as} \quad n \to \infty.$$

Claim 2: N maps bounded sets into bounded sets in $C_{1-\gamma}[J,R]$.

Indeed, it is enough to show that for $q > 0$, there exists a positive constant l such that $x \in B_q = \left\{x \in C_{1-\gamma}[J,R] : \|x\|_{C_{1-\gamma}} \le q\right\}$, we have $\|N(x)\|_{C_{1-\gamma}} \le l$,

$$\begin{aligned}
\left|t^{1-\gamma}(Nx)(t)\right| &\le \frac{|x_0|}{\Gamma(\gamma)} + \frac{t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s,x(s))|\, \Delta s \\
&\le \frac{|x_0|}{\Gamma(\gamma)} + \frac{t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |\mu(s)|\, \Delta s \\
&\le \frac{|x_0|}{\Gamma(\gamma)} + \frac{b^{\alpha}}{\Gamma(\alpha)} B(\gamma,\alpha) \|\mu\|_{C_{\gamma-1}} \\
&:= l,
\end{aligned}$$

Claim 3: N maps bounded sets into equicontinuous set of $C_{1-\gamma}[J,R]$.

Let $t_1, t_2 \in J$, $t_1 < t_2$, B_q be a bounded set of $C_{1-\gamma}[J,R]$ as in claim 2, and $x \in B_q$. Then

$$\begin{aligned}
&\left|(Nx)(t_2)t_2^{1-\gamma} - (Nx)(t_1)t_1^{1-\gamma}\right| \\
&\le \left|\frac{t_2^{1-\gamma}}{\Gamma(\alpha)} \int_0^{t_2} (t_2-s)^{\alpha-1} f(s,x(s))\Delta s - \frac{t_1^{1-\gamma}}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} f(s,x(s))\Delta s\right| \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left|t_2^{1-\gamma}(t_2-s)^{\alpha-1} - t_1^{1-\gamma}(t_1-s)^{\alpha-1}\right| |\mu(s)|\, \Delta s \\
&\quad + \frac{t_2^{1-\gamma}}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} |\mu(s)|\, \Delta s \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left|t_2^{1-\gamma}(t_2-s)^{\alpha-1} - t_1^{1-\gamma}(t_1-s)^{\alpha-1}\right| |\mu(s)|\, ds \\
&\quad + \frac{t_2^{1-\gamma}}{\Gamma(\alpha)} (t_2-t_1)^{\alpha+\gamma-1} B(\gamma,\alpha) \|\mu\|_{C_{1-\gamma}}.
\end{aligned}$$

We see that the right-hand part of the above inequality tends to zero independently of $x \in B_q$ as $t_1 - t_2 \to 0$. Hence along with the Arzëla-Ascoli theorem and the results of Claims 1-3, it is concluded that $N : C_{1-\gamma}[J,R] \to C_{1-\gamma}[J,R]$ is continuous and completely continuous.

Claim 4: A priori bounds.

Now, it suffices to show that the set

$$\omega = \{x \in C_{1-\gamma}[J,R] : \ x = \delta(Nx), \ 0 < \delta < 1\}$$

is bounded set.

Let $x \in \omega$, $x = \delta(Nx)$ for some $0 < \delta < 1$. Thus for each $t \in J$, we have

$$x(t) = \delta \left[\frac{x_0}{\Gamma(\gamma)} t^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(t, x(s)) \Delta s \right].$$

In view of (H2), for each $t \in J$, we have

$$\begin{aligned} |x(t)| &= |(Nx)(t)| \\ &\leq \frac{|x_0|}{\Gamma(\gamma)} + \frac{b^\alpha}{\Gamma(\alpha)} B(\gamma, \alpha) \|\mu\|_{C_{1-\gamma}}. \end{aligned}$$

The above implies that the set ω is bounded. Thus Schauder's fixed point theorem helps to deduct the solution of the problem which states N has a fixed point which is a solution of problem (1.1). □

4. Stability analysis

Next, we shall give the definitions and the criteria of Ulam-Hyers (UH) stability and Ulam-Hyers-Rassias (UHR) stability for Hilfer-type dynamic equations on time scales. We use some ideas from [18].

DEFINITION 4.1. Equation (1.1) is UH stable if there exists a real number $c_f > 0$ such that for each $\epsilon > 0$ and for each solution $z \in C_{1-\gamma}^{\gamma}[J,R]$ of the inequality

$$\left| {}^{\mathbb{T}}\Delta_{0^+}^{\alpha,\beta} z(t) - f(t, z(t)) \right| \leq \epsilon, \quad t \in J, \tag{4.1}$$

there exists a solution $x \in C_{1-\gamma}^{\gamma}[J,R]$ of equation (1.1) with

$$|z(t) - x(t)| \leq c_f \epsilon, \quad t \in J.$$

DEFINITION 4.2. Equation (1.1) is generalized UH stable if there exists $\psi_f \in C(R^+, R^+)$, $\psi_f(0) = 0$ such that for each solution $z \in C_{1-\gamma}^{\gamma}[J,R]$ of the inequality (4.1), there exists a solution $x \in C_{1-\gamma}^{\gamma}[J,R]$ of equation (1.1) with

$$|z(t) - x(t)| \leq \psi_f \epsilon, \quad t \in J.$$

Definition 4.3. Equation (1.1) is UHR stable with respect to $\varphi \in C_{1-\gamma}[J,R]$ if there exists a real number $c_f > 0$ such that for each $\epsilon > 0$ and for each solution $z \in C^{\gamma}_{1-\gamma}[J,R]$ of the inequality

$$\left| {}^{\mathbb{T}}\Delta^{\alpha,\beta}_{0^+} z(t) - f(t,z(t)) \right| \leq \epsilon\varphi(t), \quad t \in J, \tag{4.2}$$

there exists a solution $x \in C^{\gamma}_{1-\gamma}[J,R]$ of the equation (1.1) with

$$|z(t) - x(t)| \leq c_f \epsilon \varphi(t), \quad t \in J.$$

Definition 4.4. Equation (1.1) is generalized UHR stable with respect to $\varphi \in C_{1-\gamma}[J,R]$ if there exists a real number $c_{f,\varphi} > 0$ such that for each solution $z \in C_{1-\gamma}[J,R]$ of the inequality

$$\left| {}^{\mathbb{T}}\Delta^{\alpha,\beta}_{0^+} z(t) - f(t,z(t)) \right| \leq \varphi(t), \quad t \in J,$$

there exists a solution $x \in C^{\gamma}_{1-\gamma}[J,R]$ of equation (1.1) with

$$|z(t) - x(t)| \leq c_{f,\varphi} \varphi(t), \quad t \in J.$$

Remark 4.1. A function $z \in C^{\gamma}_{1-\gamma}[J,R]$ is a solution of the inequality

$$\left| {}^{\mathbb{T}}\Delta^{\alpha,\beta}_{0^+} z(t) - f(t,z(t)) \right| \leq \epsilon, \quad t \in J,$$

if and only if there exists a function $g \in C^{\gamma}_{1-\gamma}[J,R]$ such that

(1) $|g(t)| \leq \epsilon, \quad t \in J.$
(2) ${}^{\mathbb{T}}\Delta^{\alpha,\beta}_{0^+} z(t) = f(t,z(t)) + g(t)$, $t \in J$.

Lemma 4.1. *If a function $z \in C^{\gamma}_{1-\gamma}[J,R]$ is a solution of the inequality* (4.1) *then with $z_0(t) = \frac{z_0}{\Gamma(\gamma)} t^{\gamma-1}$,*

$$\left| z(t) - z_0(t) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,z(s)) \Delta s \right| \leq \frac{\epsilon b^{\alpha}}{\Gamma(\alpha+1)}.$$

P r o o f. The proof directly follows from Remark 4.1 and Lemma 3.1. □

Remark 4.2. Clearly,

(1) Definition 4.1⇒ Definition 4.2,
(2) Definition 4.3⇒ Definition 4.4.

We ready to prove our stability results for problem (1.1). The arguments are based on the Banach contraction principle. We need the following hypothesis:

(H4) There exists an increasing function $\varphi \in C_{1-\gamma}[J, R]$ and there exists $\lambda_\varphi > 0$ such that for any $t \in J$,

$$^{\mathbb{T}}I^{\alpha}_{0^+}\varphi(t) \leq \lambda_\varphi \varphi(t).$$

Lemma 4.2. *Assume that (H1) and (H3) are fulfilled. If*

$$\left(\frac{Lb^\alpha}{\Gamma(\alpha)} B(\gamma, \alpha)\right) < 1, \tag{4.3}$$

then problem (1.1) *has a unique solution.*

P r o o f. Consider the operator $N : C_{1-\gamma}[J, R] \to C_{1-\gamma}[J, R]$.

$$(Nx)(t) = x_0(t) + \left[{}^{\mathbb{T}}I^{\alpha}_{0^+} f(s, x(s))\right](t) \tag{4.4}$$

with $x_0(t) = \frac{x_0}{\Gamma(\gamma)} t^{\gamma-1}$.

By Lemma 3.1, it is clear that the fixed points of N are solutions of problem (1.1).

Let $x_1, x_2 \in C_{1-\gamma}[J, R]$ and $t \in J$, then we have

$$\begin{aligned}
&\left|t^{1-\gamma}((Nx_1)(t) - (Nx_2)(t))\right| \\
&\leq \frac{t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, x_1(s)) - f(s, x_2(s))| \, \Delta s \\
&\leq \frac{t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sup_{s \in J} |f(s, x_1(s)) - f(s, x_2(s))| \, \Delta s \\
&\leq \frac{t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, x_1(s)) - f(s, x_2(s))| \, ds \\
&\leq \frac{Lt^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |x_1(s) - x_2(s)| \, ds \\
&\leq \frac{Lb^\alpha}{\Gamma(\alpha)} B(\gamma, \alpha) \|x_1 - x_2\|_{C_{1-\gamma}}.
\end{aligned}$$

Then,

$$\|Nx_1 - Nx_2\|_{C_{1-\gamma}} \leq \frac{Lb^\alpha}{\Gamma(\alpha)} B(\gamma, \alpha) \|x_1 - x_2\|_{C_{1-\gamma}}.$$

From (4.3), it follows that N has a unique fixed point which is solution of problem (1.1). □

THEOREM 4.1. *If the hypotheses (H1), (H3) and* (4.3) *are satisfied, then the problem* (1.1) *is UH stable.*

P r o o f. Let $\epsilon > 0$ and let $z \in C_{1-\gamma}^{\gamma}[J, R]$ be a function which satisfies the inequality (4.1) and let $x \in C_{1-\gamma}^{\gamma}[J, R]$ be the unique solution of the following Hilfer-type dynamic equation

$$\begin{aligned} {}^{\mathbb{T}}\Delta_{0^+}^{\alpha,\beta} x(t) &= f(t, x(t)), \quad t \in J := [0, b], \\ {}^{\mathbb{T}}I_{0^+}^{1-\gamma} x(0) &= {}^{\mathbb{T}}I_{0^+}^{1-\gamma} z(0) = x_0. \end{aligned}$$

By Lemma 3.1, we get

$$x(t) = x_0(t) + \left[{}^{\mathbb{T}}I_{0^+}^{\alpha} f(s, x(s))\right](t)$$

with $x_0(t) = \frac{x_0}{\Gamma(\gamma)} t^{\gamma-1}$.

Integrating (4.1) and Lemma 4.1, we obtain

$$\left| z(t) - z_0(t) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, z(s)) \Delta s \right| \leq \frac{\epsilon b^{\alpha}}{\Gamma(\alpha+1)}. \tag{4.5}$$

For any $t \in J$,

$$\begin{aligned} |z(t) - x(t)| &\leq \Bigg| z(t) - z_0(t) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, z(s)) \Delta s \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[f(s, z(s)) - f(s, x(s)) \right] \Delta s \Bigg| \\ &\leq \left| z(t) - z_0(t) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, z(s)) \Delta s \right| \\ &\quad + \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |z(s) - x(s)| \, ds. \end{aligned}$$

By using (4.5)

$$|z(t) - x(t)| \leq \frac{\epsilon b^{\alpha}}{\Gamma(\alpha+1)} + \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |z(s) - x(s)| \, ds,$$

and by Lemma 2.10, we obtain

$$\begin{aligned} |z(t) - x(t)| &\leq \frac{b^{\alpha}}{\Gamma(\alpha+1)} \left[1 + \frac{\nu L}{\Gamma(\alpha+1)} b^{\alpha} \right] \epsilon, \\ &:= c_f \epsilon, \end{aligned}$$

where $\nu = \nu(\alpha)$ is a constant, which completes the proof of the theorem. Moreover, if we set $\psi(\epsilon) = c_f \epsilon$; $\psi(0) = 0$, then the problem (1.1) is generalized UH stable. □

THEOREM 4.2. *Assume that (H1), (H3), (H4) and* (4.3) *are satisfied. Then, the problem* (1.1) *is UHR stable.*

P r o o f. Let $z \in C_{1-\gamma}^{\gamma}[J, R]$ be solution of the following inequality (4.2) and let $x \in C_{1-\gamma}^{\gamma}[J, R]$ be the unique solution of the Hilfer type dynamics equation (1.1). By Lemma 3.1,

$$x(t) = x_0(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s)) \Delta s.$$

By integration of (4.2) and applying Lemma 4.1 we obtain

$$\left| z(t) - z_0(t) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, z(s)) \Delta s \right| \leq \epsilon \lambda_\varphi \varphi(t). \tag{4.6}$$

On the other hand, we have

$$\begin{aligned} |z(t) - x(t)| &\leq \left| z(t) - z_0(t) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, z(s)) \Delta s \right| \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left| f(s, z(s)) - f(s, x(s)) \right| \Delta s \\ &\leq \left| z(t) - z_0(t) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, z(s)) \Delta s \right| \\ &\quad + \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left| z(s) - x(s) \right| ds \\ &\leq \epsilon \lambda_\varphi \varphi(t) + \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left| z(s) - x(s) \right| ds. \end{aligned}$$

By applying Lemma 2.10, we obtain

$$|z(t) - x(t)| \leq \left[(1 + \nu_1 L \lambda_\varphi) \lambda_\varphi \right] \epsilon \varphi(t),$$

where $\nu_1 = \nu_1(\alpha)$ is a constant,then for any $t \in J$:

$$|z(t) - x(t)| \leq c_f \epsilon \varphi(t),$$

which completes the proof of the theorem. □

In next section, we give an example to illustrate the theory.

5. **Application**

We consider the following Hilfer-type problem on time scales

$${}^{\mathbb{T}}\Delta_{0^+}^{\alpha,\beta} x(t) = \frac{e^{-t}}{(9 + e^t)} \left(\frac{|x(t)|}{1 + |x(t)|} \right), \quad t = [0, 1] := J \subseteq \mathbb{T}, \tag{5.1}$$

$${}^{\mathbb{T}}I_{0^+}^{1-\gamma} x(0) = 0. \tag{5.2}$$

Set $f(t,x) = \frac{e^{-t}x}{(9+e^t)(1+x)}$, $(t,x) \in \mathbb{T} \times [0,+\infty)$.

Let $x, y \in [0,+\infty)$ and $t \in J$. Then we have

$$|f(t,x) - f(t,y)| \leq \frac{e^{-t}}{9+e^t}|x-y| \leq \frac{1}{10}|x-y|.$$

It is obvioustat our assumptions in Theorem 3.1 holdwith $L = \frac{1}{10}$.

Denote $\alpha = \frac{2}{3}$, $\beta = \frac{1}{2}$ and choose $\gamma = \frac{5}{6}$ and $b = 1$.

From (4.3),

$$\left(\frac{Lb^\alpha}{\Gamma(\alpha)}B(\gamma,\alpha)\right) = 0.1274 < 1.$$

Now all the assumptions in Lemma 4.2 and Theorem 4.1 are satisfied. The problem (5.1)-(5.2) has a unique solution and it is UH stable.

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