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RESEARCH PAPER

LYAPUNOV-TYPE INEQUALITIES FOR NONLINEAR FRACTIONAL DIFFERENTIAL EQUATION WITH HILFER FRACTIONAL DERIVATIVE UNDER MULTI-POINT BOUNDARY CONDITIONS

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Abstract

In this work, we establish Lyapunov-type inequalities for the fractional boundary value problems with Hilfer fractional derivative under multi-point boundary conditions, the results are new and generalize and improve some early results in the literature.

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Key Words and Phrases: Lyapunov inequality, fractional differential equation, Hilfer fractional derivative, multi-point boundary value problem, Green's function

1. Introduction

The well-known result of Lyapunov [8] states that if $u(t)$ is a nontrivial solution of the differential system

$$\begin{aligned} &u''(t) + r(t)u(t) = 0, \qquad t \in (a,b), \\ &u(a) = 0 = u(b), \end{aligned} \tag{1.1}$$

where $r(t)$ is a continuous function defined in $[a,b]$, then

$$\int_a^b |r(t)|dt > \frac{4}{b-a}, \tag{1.2}$$

and the constant 4 cannot be replaced by a larger number.

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Lyapunov inequality (1.2) is a useful tool in various branches of mathematics including disconjugacy, oscillation theory, and eigenvalue problems. Many improvements and generalizations of the inequality (1.2) have appeared in the literature. A thorough literature review of continuous and discrete Lyapunov-type inequalities and their applications can be found in the survey articles by Cheng [2], Brown and Hinton [1] and Tiryaki [9].

The study of Lyapunov-type inequalities for the differential equation depends on a fractional differential operator was initiated by Ferreira [3]. He first obtained a Lyapunov-type inequality when the differential equation depends on the Riemann-Liouville fractional derivative, the main result is as follows.

Theorem 1.1. *If the following fractional boundary value problem*

$$(D^{\alpha}_{a^+}u)(t) + q(t)u(t) = 0, \quad a < t < b,\ 1 < \alpha \leq 2, \tag{1.3}$$

$$u(a) = 0 = u(b), \tag{1.4}$$

has a nontrivial solution, where q is a real and continuous function, then

$$\int_a^b |q(s)|ds > \Gamma(\alpha)\left(\frac{4}{b-a}\right)^{\alpha-1}. \tag{1.5}$$

One year later, Ferreira [4] obtained a Lyapunov-type inequality when the differential equation depends on the Caputo fractional derivative.

Theorem 1.2. *If a nontrivial continuous solution of the fractional boundary value problem*

$$({}^{C}D^{\alpha}_{a^+}u)(t) + q(t)u(t) = 0, \quad a < t < b,\ 1 < \alpha \leq 2, \tag{1.6}$$

$$u(a) = 0 = u(b), \tag{1.7}$$

exists, where q is a real and continuous function, then

$$\int_a^b |q(s)|ds > \frac{\Gamma(\alpha)\alpha^{\alpha}}{[(\alpha-1)(b-a)]^{\alpha-1}}. \tag{1.8}$$

Many other generalizations and extensions of inequality (1.2) exist in the literature, see for instance [11] – [16] and references therein.

Motivated by the above works, in this paper, we establish Lyapunov-type inequalities for the fractional boundary value problems with Hilfer fractional derivative under a multi-point boundary condition,

$$(D^{\alpha,\beta}_{a^+}u)(t) + q(t)u(t) = 0, \quad a < t < b,\ 1 < \alpha \leq 2,\ 0 \leq \beta \leq 1, \tag{1.9}$$

$$u(a) = 0,\ u(b) = \sum_{i=1}^{m-2} \beta_i u(\xi_i), \tag{1.10}$$

where $D_{a^+}^{\alpha,\beta}$ denotes the Hilfer fractional derivative of order order α and type $0 \le \beta \le 1$.

In this paper, we assume that $a < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < b$, $\beta_i \ge 0\,(i = 1, 2, \ldots, m-2)$, $0 \le \sum_{i=1}^{m-2} \beta_i(\xi_i - a)^{1-(2-\alpha)(1-\beta)} < (b-a)^{1-(2-\alpha)(1-\beta)}$ and denote

$$T(t) = \frac{(t-a)^{1-(2-\alpha)(1-\beta)}}{(b-a)^{1-(2-\alpha)(1-\beta)} - \sum_{i=1}^{m-2} \beta_i(\xi_i - a)^{1-(2-\alpha)(1-\beta)}}, \quad a \le t \le b,$$

$$L = \frac{(\alpha-1)^{\alpha-1}(\alpha - 1 + 2\beta - \alpha\beta)^{\alpha-1+2\beta-\alpha\beta}}{(2\alpha - 2 + 2\beta - \alpha\beta)^{2\alpha-2+2\beta-\alpha\beta}}.$$

2. **Preliminaries**

In this section, we recall the concepts of the Riemann-Liouville fractional integral, the Riemann-Liouville fractional derivative, the Caputo fractional derivative of order $\alpha \ge 0$ and the Hilfer fractional derivative of order α $(n-1 < \alpha \le n, n \in N)$, and type $0 \le \beta \le 1$.

Let I be a certain interval in R. We denote by $AC(I;R)$ the space of real valued and absolutely continuous functions on I. For $n = 1, 2, \ldots,$ we denote by $AC^n(I;R)$ the space of real valued functions $f(x)$ which have continuous derivatives up to order $n-1$ on I with $f^{(n-1)} \in AC(I;R)$, that is

$$AC^n(I;R) = \left\{ f : I \to R \text{ such that } D^{n-1}f \in AC(I;R) \left(D = \frac{d}{dx} \right) \right\}.$$

Clearly, we have $AC^1(I;R) = AC(I;R)$.

Definition 2.1. ([7]) Let $f \in L^1((a,b);R)$, where $(a,b) \in R^2, a < b$. The Riemann-Liouville fractional integral of order $\alpha > 0$ of f is defined by

$$(I_{a^+}^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, \qquad \text{a.e. } t \in [a,b].$$

Definition 2.2. ([7]) Let $\alpha > 0$ and m be the smallest integer greater or equal than α. The Riemann-Liouville fractional derivative of order α of a function $f : [a,b] \to R$, where $(a,b) \in R^2, a < b$, is defined by

$$\begin{aligned}(D_{a^+}^{\alpha} f)(t) &= (D^m I_{a^+}^{m-\alpha} f)(t)\\ &= \frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{dt}\right)^m \int_a^t (t-s)^{m-\alpha-1} f(s) ds, \quad \text{a.e. } t \in [a,b].\end{aligned}$$

Definition 2.3. ([7]) Let $\alpha > 0$ and m be the smallest integer greater or equal than α. The Caputo fractional derivative of order α of a function $f \in AC^m[a,b]$ is defined by

$$({}^C D^{\alpha}_{a^+} f)(t) = (I^{m-\alpha}_{a^+} D^m f)(t)$$
$$= \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-\alpha-1} f^{(m)}(s)ds, \quad \text{a.e. } t \in [a,b].$$

Definition 2.4. ([5], [6]) The Hilfer fractional derivative or generalized Riemann-Liouville fractional derivative of order $\alpha(n-1 < \alpha \leq n, n \in N)$, and type $0 \leq \beta \leq 1$ with respect to t, is defined as

$$(D^{\alpha,\beta}_{a^+} f)(t) = \left(I^{\beta(n-\alpha)}_{a^+} \frac{d^n}{dt^n} \left(I^{(1-\beta)(n-\alpha)}_{a^+} f \right) \right)(t).$$

Remark 2.1. In the above definition, type β allows $D^{\alpha,\beta}_{a^+}$ to interpolate continuously between the classical Riemann-Liouville fractional derivative and the Caputo fractional derivative. As in the case $\beta = 0$, the definition reduces to the classical Riemann-Liouville fractional derivative and for $\beta = 1$, it gives the Caputo fractional derivative.

In [10], the compositional property of Riemann-Liouville fractional integral operator with the Hilfer fractional derivative operator is obtained.

Lemma 2.1. ([10]) *Let $f \in L^1(a,b)$, $n-1 < \alpha \leq n, n \in \mathbb{N}, 0 \leq \beta \leq 1$, $I^{(n-\alpha)(1-\beta)}_{a^+} f \in AC^k[a,b]$. Then the Riemann-Liouville fractional integral $I^{\alpha}_{a^+}$ and the Hilfer fractional derivative operator $D^{\alpha,\beta}_{a^+}$ are connected by the relation*

$$\left(I^{\alpha}_{a^+} D^{\alpha,\beta}_{a^+} f \right)(t) = f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{k-(n-\alpha)(1-\beta)}}{\Gamma(k-(n-\alpha)(1-\beta)+1)} \lim_{t \to a^+} \frac{d^k}{dt^k} \left(I^{(n-\alpha)(1-\beta)}_{a^+} f \right)(t).$$

3. Main results

We begin by writing problem (1.9)-(1.10) in its equivalent integral form.

Lemma 3.1. *If the function $u \in C[a,b]$ is a solution to the boundary value problem $(1.9)-(1.10)$, then u satisfies the integral equation*

$$u(t) = \int_a^b G(t,s)q(s)u(s)ds + T(t) \int_a^b \sum_{i=1}^{m-2} \beta_i G(\xi_i, s)q(s)u(s)ds,$$

where $G(t,s)$ is defined as

$$G(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases}\left(\frac{t-a}{b-a}\right)^{1-(2-\alpha)(1-\beta)}(b-s)^{\alpha-1}-(t-s)^{\alpha-1}, & a\le s\le t\le b,\\ \left(\frac{t-a}{b-a}\right)^{1-(2-\alpha)(1-\beta)}(b-s)^{\alpha-1}, & a\le t\le s\le b.\end{cases}$$

P r o o f. From Lemma 2.1, if $u \in C[a,b]$ is a solution to the boundary value problem (1.9)-(1.10), then we have

$$\begin{aligned}u(t) =&c_0\,\frac{(t-a)^{-(2-\alpha)(1-\beta)}}{\Gamma(1-(2-\alpha)(1-\beta))}+c_1\,\frac{(t-a)^{1-(2-\alpha)(1-\beta)}}{\Gamma(2-(2-\alpha)(1-\beta))}\\ &-\int_a^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}q(s)u(s)ds,\end{aligned}$$

where c_0 and c_1 are some real constants. Since $u(a)=0$, we get immediately that $c_0=0$, thus

$$u(t)=c_1\frac{(t-a)^{1-(2-\alpha)(1-\beta)}}{\Gamma(2-(2-\alpha)(1-\beta))}-\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}q(s)u(s)ds.$$

The boundary condition $u(b)=\sum_{i=1}^{m-2}\beta_i u(\xi_i)$ yields

$$\begin{aligned}&c_1\,\frac{(b-a)^{1-(2-\alpha)(1-\beta)}}{\Gamma(2-(2-\alpha)(1-\beta))}-\frac{1}{\Gamma(\alpha)}\int_a^b(b-s)^{\alpha-1}q(s)u(s)ds\\ =&\sum_{i=1}^{m-2}\beta_i\left[c_1\frac{(\xi_i-a)^{1-(2-\alpha)(1-\beta)}}{\Gamma(2-(2-\alpha)(1-\beta))}-\frac{1}{\Gamma(\alpha)}\int_a^{\xi_i}(\xi_i-s)^{\alpha-1}q(s)u(s)ds\right],\end{aligned}$$

so,

$$c_1=\frac{\Gamma(2-(2-\alpha)(1-\beta))\left[(I_{a^+}^{\alpha}qu)(b)-\sum_{i=1}^{m-2}\beta_i(I_{a^+}^{\alpha}qu)(\xi_i)\right]}{(b-a)^{1-(2-\alpha)(1-\beta)}-\sum_{i=1}^{m-2}\beta_i(\xi_i-a)^{1-(2-\alpha)(1-\beta)}}.$$

Hence

$$\begin{aligned}u(t)=&c_1\frac{(t-a)^{1-(2-\alpha)(1-\beta)}}{\Gamma(2-(2-\alpha)(1-\beta))}-\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}q(s)u(s)ds\\ =&\frac{(t-a)^{1-(2-\alpha)(1-\beta)}\left[(I_{a^+}^{\alpha}qu)(b)-\sum_{i=1}^{m-2}\beta_i(I_{a^+}^{\alpha}qu)(\xi_i)\right]}{(b-a)^{1-(2-\alpha)(1-\beta)}-\sum_{i=1}^{m-2}\beta_i(\xi_i-a)^{1-(2-\alpha)(1-\beta)}}\\ &-\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}q(s)u(s)ds,\end{aligned}$$

by the relation

$$\begin{aligned}&\frac{1}{(b-a)^{1-(2-\alpha)(1-\beta)}-\sum_{i=1}^{m-2}\beta_i(\xi_i-a)^{1-(2-\alpha)(1-\beta)}}=\frac{1}{(b-a)^{1-(2-\alpha)(1-\beta)}}\\ &+\frac{\sum_{i=1}^{m-2}\beta_i(\xi_i-a)^{1-(2-\alpha)(1-\beta)}}{(b-a)^{1-(2-\alpha)(1-\beta)}[(b-a)^{1-(2-\alpha)(1-\beta)}-\sum_{i=1}^{m-2}\beta_i(\xi_i-a)^{1-(2-\alpha)(1-\beta)}]},\end{aligned}$$

we obtain

$$
\begin{aligned}
u(t) &= \frac{1}{\Gamma(\alpha)}\int_a^b \frac{(t-a)^{1-(2-\alpha)(1-\beta)}(b-s)^{\alpha-1}}{(b-a)^{1-(2-\alpha)(1-\beta)}}q(s)u(s)ds \\
&\quad - \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}q(s)u(s)ds \\
&\quad + \frac{(t-a)^{1-(2-\alpha)(1-\beta)}\sum_{i=1}^{m-2}\beta_i\int_a^b \frac{(\xi_i-a)^{1-(2-\alpha)(1-\beta)}(b-s)^{\alpha-1}}{(b-a)^{1-(2-\alpha)(1-\beta)}}q(s)u(s)ds}{[(b-a)^{1-(2-\alpha)(1-\beta)}-\sum_{i=1}^{m-2}\beta_i(\xi_i-a)^{1-(2-\alpha)(1-\beta)}]\Gamma(\alpha)} \\
&\quad - \frac{(t-a)^{1-(2-\alpha)(1-\beta)}\sum_{i=1}^{m-2}\beta_i\int_a^{\xi_i}(\xi_i-s)^{\alpha-1}q(s)u(s)ds}{[(b-a)^{1-(2-\alpha)(1-\beta)}-\sum_{i=1}^{m-2}\beta_i(\xi_i-a)^{1-(2-\alpha)(1-\beta)}]\Gamma(\alpha)} \\
&= \int_a^b G(t,s)q(s)u(s)ds + T(t)\int_a^b \sum_{i=1}^{m-2}\beta_i G(\xi_i,s)q(s)u(s)ds.
\end{aligned}
$$

which concludes the proof. □

LEMMA 3.2. ([4]) *If* $1<\delta<2$, *then*

$$(2-\delta)(\delta-1)^{\frac{\delta-1}{2-\delta}} \le \frac{(\delta-1)^{\delta-1}}{\delta^\delta}.$$

LEMMA 3.3. *The function* G *defined in Lemma 3.1 satisfies the following property:*

$$|G(t,s)| \le \frac{(\alpha-1)^{\alpha-1}(\alpha-1+2\beta-\alpha\beta)^{\alpha-1+2\beta-\alpha\beta}}{(2\alpha-2+2\beta-\alpha\beta)^{2\alpha-2+2\beta-\alpha\beta}}\cdot\frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)},$$

where $(t,s)\in[a,b]\times[a,b]$.

P r o o f. We divide our proof in two parts.

Part I. Denote $\gamma-1 = 1-(2-\alpha)(1-\beta) = \alpha-1+2\beta-\alpha\beta$, the function $G(t,s)$ can be rewritten as the following form

$$(b-a)^{\gamma-1}\Gamma(\alpha)G(t,s) = \begin{cases} g_1(t,s), & a\le s\le t\le b,\\ g_2(t,s), & a\le t\le s\le b,\end{cases}$$

where

$$
\begin{aligned}
g_1(t,s) &= (t-a)^{\gamma-1}(b-s)^{\alpha-1}-(b-a)^{\gamma-1}(t-s)^{\alpha-1}, && a\le s\le t\le b,\\
g_2(t,s) &= (t-a)^{\gamma-1}(b-s)^{\alpha-1}, && a\le t\le s\le b.
\end{aligned}
$$

Obviously, $g_2(t,s)$ is an increasing function in t. And $0\le g_2(t,s)\le g_2(s,s)$. Now we turn our attention to the function $g_1(t,s)$. We start by fixing an arbitrary $t\in[a,b)$. Differentiating $g_1(t,s)$ with respect to s, and by the condition $0\le(\frac{t-s}{b-s})^{2-\alpha}\le 1, 0\le(\frac{t-a}{b-a})^{\gamma-1}\le 1$, we get

$$\begin{aligned}\frac{\partial g_1(t,s)}{\partial s} &= (\alpha-1)[(b-a)^{\gamma-1}(t-s)^{\alpha-2}-(t-a)^{\gamma-1}(b-s)^{\alpha-2}]\\ &= (\alpha-1)(b-a)^{\gamma-1}(t-s)^{\alpha-2}\left[1-\left(\frac{t-a}{b-a}\right)^{\gamma-1}\left(\frac{t-s}{b-s}\right)^{2-\alpha}\right]\geq 0.\end{aligned}$$

Hence, for a given t, $g_1(t,s)$ is an increasing function of $s\in[a,t]$. Therefore, we have

$$g_1(t,a)\leq g_1(t,s)\leq g_1(t,t).$$

Since

$$\begin{aligned} g_1(t,a) &= (t-a)^{\gamma-1}(b-a)^{\alpha-1}-(b-a)^{\gamma-1}(t-a)^{\alpha-1}\\ &= (t-a)^{\gamma-1}(b-a)^{\alpha-1}\left[1-\left(\tfrac{b-a}{t-a}\right)^{2\beta-\alpha\beta}\right]<0,\end{aligned}$$

therefore,

$$|g_1(t,s)|\leq \max\left\{\max_{t\in[a,b]} g_1(t,t), -\max_{t\in[a,b]} g_1(t,a)\right\}.$$

Let

$$f_1(t)=g_1(t,t)=(t-a)^{\gamma-1}(b-t)^{\alpha-1},\quad t\in[a,b].$$

Now, we differentiate $f_1(t)$ on (a,b), and we obtain

$$f_1'(t)=(t-a)^{\gamma-2}(b-t)^{\alpha-2}[(\gamma-1)(b-t)-(\alpha-1)(t-a)].$$

Observe that $f_1'(t)$ has a unique zero, attained at the point

$$t=t_1^*=a+\frac{\gamma-1}{\alpha+\gamma-2}(b-a).$$

Since, $f_1''(t_1^*)\leq 0$, we conclude that

$$\begin{aligned}&\max_{t\in[a,b]} f_1(t)=f_1(t_1^*)\\ &=\frac{(\alpha-1)^{\alpha-1}(\gamma-1)^{\gamma-1}}{(\alpha+\gamma-2)^{\alpha+\gamma-2}}(b-a)^{\alpha+\gamma-2}\\ &=\frac{(\alpha-1)^{\alpha-1}(\alpha-1+2\beta-\alpha\beta)^{\alpha-1+2\beta-\alpha\beta}}{(2\alpha-2+2\beta-\alpha\beta)^{2\alpha-2+2\beta-\alpha\beta}}(b-a)^{2\alpha-2+2\beta-\alpha\beta}.\end{aligned}$$

Let

$$f_2(t)=-g_1(t,a)=(b-a)^{\gamma-1}(t-a)^{\alpha-1}-(t-a)^{\gamma-1}(b-a)^{\alpha-1},\quad t\in[a,b].$$

If $\beta=0$ or $\alpha=2$, then $f_2(t)\equiv 0$, if $\beta(2-\alpha)\neq 0$, we differentiate $f_2(t)$ on (a,b), we obtain

$$f_2'(t)=(b-a)^{\alpha-1}(t-a)^{\gamma-2}[(\alpha-1)(b-a)^{\gamma-\alpha}-(\gamma-1)(t-a)^{\gamma-\alpha}].$$

Observe that $f_2'(t)$ has a unique zero, attained at the point

$$t=t_2^*=a+\left(\frac{\alpha-1}{\gamma-1}\right)^{\frac{1}{\beta(2-\alpha)}}(b-a).$$

Since $f''(t_2^*)\leq 0$, we conclude that

$$\max_{t\in[a,b]} f_2(t) = f_2(t_2^*)$$
$$= \frac{\gamma-\alpha}{\gamma-1}\left(\frac{\alpha-1}{\gamma-1}\right)^{\frac{\alpha-1}{\gamma-\alpha}}(b-a)^{\alpha+\gamma-2}$$
$$= \frac{2\beta-\alpha\beta}{\alpha-1+2\beta-\alpha\beta}\left(\frac{\alpha-1}{\alpha-1+2\beta-\alpha\beta}\right)^{\frac{\alpha-1}{\beta(2-\alpha)}}(b-a)^{2\alpha-2+2\beta-\alpha\beta}.$$

Part II. Now, we prove that $\max_{t\in[a,b]} f_2(t) \le \max_{t\in[a,b]} f_1(t)$. If $\beta = 0$ or $\alpha = 2$, then $f_2(t) \equiv 0$, the conclusion is obvious. If $0 < \beta < 1$ and $1 < \alpha < 2$, let $\delta = \frac{\alpha+\gamma-2}{\gamma-1}$, then $1 < \delta < 2$. Applying Lemma 3.2, we obtain

$$\max_{t\in[a,b]} f_2(t) = \frac{\gamma-\alpha}{\gamma-1}\left(\frac{\alpha-1}{\gamma-1}\right)^{\frac{\alpha-1}{\gamma-\alpha}}(b-a)^{\alpha+\gamma-2}$$
$$= (2-\delta)(\delta-1)^{\frac{\delta-1}{2-\delta}}(b-a)^{\alpha+\gamma-2} \le \frac{(\delta-1)^{\delta-1}}{\delta^{\delta}}(b-a)^{\alpha+\gamma-2}$$
$$= \left[\frac{(\alpha-1)^{\alpha-1}(\gamma-1)^{\gamma-1}}{(\alpha+\gamma-2)^{\alpha+\gamma-2}}\right]^{\frac{1}{\gamma-1}}(b-a)^{\alpha+\gamma-2}$$
$$< \frac{(\alpha-1)^{\alpha-1}(\gamma-1)^{\gamma-1}}{(\alpha+\gamma-2)^{\alpha+\gamma-2}}(b-a)^{\alpha+\gamma-2} = \max_{t\in[a,b]} f_1(t).$$

Therefore,

$$|g_1(t,s)| \le \max\left\{\max_{t\in[a,b]} g_1(t,t) - \max_{t\in[a,b]} g_1(t,a)\right\}$$
$$= \max\left\{\max_{t\in[a,b]} f_1(t), \max_{t\in[a,b]} f_2(t)\right\} = \max_{t\in[a,b]} f_1(t)$$
$$= \frac{(\alpha-1)^{\alpha-1}(\alpha-1+2\beta-\alpha\beta)^{\alpha-1+2\beta-\alpha\beta}}{(2\alpha-2+2\beta-\alpha\beta)^{2\alpha-2+2\beta-\alpha\beta}}(b-a)^{2\alpha-2+2\beta-\alpha\beta}.$$

Thus

$$|G(t,s)| \le \frac{1}{(b-a)^{\gamma-1}\Gamma(\alpha)}\max_{s\in[a,b]}|g_1(t,s)|$$
$$\le \frac{(\alpha-1)^{\alpha-1}(\alpha-1+2\beta-\alpha\beta)^{\alpha-1+2\beta-\alpha\beta}}{(2\alpha-2+2\beta-\alpha\beta)^{2\alpha-2+2\beta-\alpha\beta}}\cdot\frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)}.$$

The proof is complete. □

Now, we are ready to state and prove the main result of this paper.

Theorem 3.1. *If a nontrivial continuous solution of the fractional boundary value problem*

$$(D_{a^+}^{\alpha,\beta}u)(t) + q(t)u(t) = 0, \quad a < t < b,\ 1 < \alpha \le 2,\ 0 \le \beta \le 1,$$
$$u(a) = 0, \quad u(b) = \sum_{i=1}^{m-2} \beta_i u(\xi_i),$$

exists, where q is a real and continuous function in $[a,b]$, then

$$\int_a^b |q(s)|ds \ge \frac{\Gamma(\alpha)}{(b-a)^{\alpha-1}L} \cdot \frac{1}{1+\sum_{i=1}^{m-2}\beta_i T(b)}. \tag{3.1}$$

P r o o f. Let $B = C[a,b]$ be the set of real valued and continuous functions in $[a,b]$. Then B is a Banach space with respect to the Chebyshev norm $\|u\| = \sup_{t\in[a,b]} |u(t)|$. It follows from Lemma 3.1 that a solution u to the boundary value problem satisfies the integral equation

$$u(t) = \int_a^b G(t,s)q(s)u(s)ds + T(t)\int_a^b \sum_{i=1}^{m-2} \beta_i G(\xi_i,s)q(s)u(s)ds,$$

Now, an application Lemma 3.3 yields

$$\|u\| \le \frac{(b-a)^{\alpha-1}L}{\Gamma(\alpha)} \left(1 + \sum_{i=1}^{m-2}\beta_i T(b)\right) \int_a^b |q(s)|ds\|u\|,$$

which implies that (3.1) holds. □

Let $\beta = 0$ in Theorem 3.1, then we have the following result.

COROLLARY 3.1. *If a nontrival solution to the fractional boundary value problem*

$$(D_{a^+}^{\alpha}u)(t) + q(t)u(t) = 0, \quad a < t < b,\ 1 < \alpha \le 2,$$
$$u(a) = 0, \quad u(b) = \sum_{i=1}^{m-2} \beta_i u(\xi_i),$$

exists, where q is a real and continuous function in $[a,b]$, then

$$\int_a^b |q(s)|ds \ge \Gamma(\alpha)(\frac{4}{b-a})^{\alpha-1} \frac{(b-a)^{\alpha-1} - \sum_{i=1}^{m-2}\beta_i(\xi_i-a)^{\alpha-1}}{(1+\sum_{i=1}^{m-2}\beta_i)(b-a)^{\alpha-1} - \sum_{i=1}^{m-2}\beta_i(\xi_i-a)^{\alpha-1}}. \tag{3.2}$$

Let $\beta = 1$ in Theorem 3.1, we have the following result.

COROLLARY 3.2. *If a nontrival solution to the fractional boundary value problem*

$$\begin{aligned} &({}^C D_{a^+}^{\alpha} u)(t) + q(t)u(t) = 0, \quad a < t < b,\ 1 < \alpha \le 2, \\ &u(a) = 0, \quad u(b) = \sum_{i=1}^{m-2} \beta_i u(\xi_i), \end{aligned}$$

exists, where q is a real and continuous function in $[a, b]$, then

$$\int_a^b |q(s)|ds \ge \frac{\Gamma(\alpha)\alpha^{\alpha}}{[(\alpha-1)(b-a)]^{\alpha-1}} \frac{b - a - \sum_{i=1}^{m-2} \beta_i(\xi_i - a)}{(1 + \sum_{i=1}^{m-2} \beta_i)(b-a) - \sum_{i=1}^{m-2} \beta_i(\xi_i - a)}. \tag{3.3}$$

REMARK 3.1. Let $\beta_1 = \beta_2 = \cdots = \beta_{m-2} = 0$ in Corollary 3.1, then we obtain (1.5), let $\beta_1 = \beta_2 = \cdots = \beta_{m-2} = 0$ in Corollary 3.2, we get (1.8).

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