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RESEARCH PAPER

MULTIPLE POSITIVE SOLUTIONS FOR A BOUNDARY VALUE PROBLEM WITH NONLINEAR NONLOCAL RIEMANN-STIELTJES INTEGRAL BOUNDARY CONDITIONS

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Abstract

In this paper, we study the existence of positive solutions to the fractional boundary value problem

$$D_{0+}^{\alpha}x(t) + q(t)f(t, x(t)) = 0,\ 0 < t < 1,$$

together with the boundary conditions

$$x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, D_{0+}^{\beta}x(1) = \int_0^1 h(s, x(s))\, dA(s),$$

where $n > 2$, $n - 1 < \alpha \leq n$, $\beta \in [1, \alpha - 1]$, and D_{0+}^{α} and D_{0+}^{β} are the standard Riemann-Liouville fractional derivatives of order α and β, respectively. We consider two different cases: f, $h : [0, 1] \times R \to R$, and f, $h : [0, 1] \times [0, \infty) \to [0, \infty)$. In the first case, we prove the existence and uniqueness of the solutions of the above problem, and in the second case, we obtain sufficient conditions for the existence of positive solutions of the above problem. We apply a number of different techniques to obtain our results including Schauder's fixed point theorem, the Leray-Schauder alternative, Krasnosel'skii's cone expansion and compression theorem, and the Avery-Peterson fixed point theorem. The generality of the Riemann-Stieltjes boundary condition includes many problems studied in the literature. Examples are included to illustrate our findings.

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1. **Introduction**

In this paper, we consider the fractional differential equation

$$D^{\alpha}_{0+}x(t) + q(t)f(t, x(t)) = 0, \ \ 0 < t < 1, \tag{1.1}$$

together with the boundary conditions

$$x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, \tag{1.2}$$

$$D^{\beta}_{0+}x(1) = \int_0^1 h(s, x(s))\, dA(s), \tag{1.3}$$

where $n - 1 < \alpha \leq n,\ n > 2,\ \beta \in [1, \alpha - 1]$ is fixed, $q : (0, 1) \to [0,\infty)$ is a continuous function, $f, h : (0, 1)\times[0,\infty) \to [0,\infty)$ are continuous functions. The nonlinear boundary condition in (1.3) is a Riemann-Stieltjes integral with A being nondecreasing and of bounded variation. Here, D^{α}_{0+} and D^{β}_{0+} are Riemann-Liouville fractional derivatives of order α and β, respectively.

Fractional differential equations have drawn the attention of many researchers during last two decades due to their applications in biology and engineering; see the monographs [12, 18, 22] and the references cited there in. Due to the importance of positive solutions in real world applications, researchers have devoted much of their interest to obtaining sufficient conditions for the existence of positive solution of various fractional differential equations. This is evident from the book by Henderson and Luca [12].

In the following, we describe some of the works that attracted us to study the problem (1.1)–(1.3).

In [6], Cabada and Wang obtained a sufficient condition for the existence of a positive solution to the fractional boundary value problem

$$^{c}D^{\alpha}u(t) + f(t, u(t)) = 0, \ 0 < t < 1, \ 2 < \alpha < 3,$$

$$u(0) = u''(0) = 0, \ u(1) = \lambda \int_0^1 u(s)\, ds, \ \ 0 < \lambda < 2,$$

where $^{c}D^{\alpha}$ is the Caputo fractional derivative.

Inspired by the work in [6], Sun and Wang [24] studied the existence of multiple positive solutions to the nonlinear fractional differential equation with integral boundary conditions

$$^{c}D^{\alpha}u(t) + f(t, u(t), u'(t)) = 0, \ 0 \leq t \leq 1, \ 2 < \alpha < 3,$$

$$u(0) = u''(0) = 0, \ u(1) = \lambda \int_0^1 u(s)\, ds, \ \ 0 < \lambda < 2,$$

where ${}^cD^\alpha$ is Caputo's fractional derivative.

In [15], Jankowski used a fixed point theorem due to Avery and Peterson to study the existence of three positive solutions to the fractional boundary value problems with the Riemann-Stieltjes boundary conditions

$$D_1^\alpha x(t) + f(t, x(t),\ x'\beta(t)) = 0,\ 0 < t < 1,\ 2 < \alpha \le 3,$$
$$x(1) = 0,\ x'(1) = 0,\ x(0) = \int_0^1 x(t)\, d\Lambda(t),$$

and

$$D_1^\alpha x(t) + g(t, x(\mu(t)), x'\beta(t)) = 0,\ 0 < t < 1,\ 2 < \alpha \le 3,$$
$$x(1) = 0,\ x'(1) = 0,\ x'(0) + \int_0^1 x(t)\, d\Lambda(t) = 0,$$

where $\int_0^1 x(t)\, d\Lambda(t)$ is a Stieltjes type integral with a suitable function Λ of bounded variation.

Benmezai and Saadi [4] studied existence of positive solutions to the three point fractional boundary value problem

$$D_{0+}^\alpha u(t) + \lambda f(t, u(t)) = 0,\ 0 < t < 1,\ 2 < \alpha \le 3,$$
$$u(0) = u'(0) = 0,\ u'(1) - \mu u'(\eta) = \int_0^1 g(u)u'(s)\, ds,$$

where μ and λ are parameters with $\mu \ge 0$, $\lambda > 0$, $\eta \in (0, 1)$, and D_{0+}^α is the standard Riemann-Liouville fractional derivative of order α.

Wang [26] studied the existence of at least one positive solution of the fractional differential equation involving Riemann-Stieltjes integral conditions

$$D_{0+}^\alpha x(t) + \lambda a(t) f(t, x(t)) = 0,\ 0 < t < 1,\ n - 1 < \alpha \le n,\ n \ge 3,$$
$$x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0,\ x(1) = \mu \int_0^1 h(x(t))\, dA(t),$$

where D_{0+}^α is again the standard Riemann-Liouville derivative and λ and $\mu > 0$ are parameters.

Tan et al. [25] studied the existence and uniqueness of positive solutions to the fractional differential equation with nonlocal boundary conditions

$$-D_{0+}^\alpha x(t) = f(t, x(t), x'(t)) + g(t, x(t)),\ 0 < t < 1,\ n - 1 < \alpha \le n,\ n \ge 3,$$
$$x^k(0) = 0,\ 0 \le k \le n - 2,\ x(1) = \int_0^1 x(s)\, dA(s),$$

where D_{0+}^α is the Riemann-Liouville fractional derivative, $\int_0^1 x(s)\, dA(s)$ denotes the Riemann-Stieltjes integral of x with respect to A, $A : [0, 1] \to R$ is a function of bounded variation, and dA can be a signed measure.

Cabada et al. [5] used a fixed point theorem due to Krasnosel'skii to obtain a positive solution of the nonlinear fractional boundary value problem

$$^{c}D^{\alpha}u(t)+f(t,u(t))=0,\ 0<t<1,\ 2<\alpha\leq 3,$$
$$u'(0)=u''(0)=0,\ u(1)=\lambda\int_0^1 u(s)\,ds,\ \ 0\leq\lambda<1,$$

where $^{c}D^{\alpha}$ is Caputo's fractional differential operator of order α.

Guezane-Lakoud and Ashryalev [9] used the Guo-Krasnosel'skii fixed point theorem to establish a positive solution to the system

$$^{c}D^{\alpha}_{0+}u(t)=g(t)f(u(t)),\ 0<t<1,\ 1<\alpha<2,$$
$$u'(0)=0,\ Eu(0)-Bu(1)=\int_0^1 h(u(s))\,ds,$$

where $f,h:R^n\to R^n$ and $g:[0,1]\to R$ are given functions, $u:[0,1]\to R^n$ is the unknown function, and $^{c}D^{\alpha}_{0+}$ is the Caputo fractional derivative.

Ahmad et al. [1] used the Leray-Schauder alternative theorem and Banach contraction principle to obtain sufficient condition for the existence of positive solutions to the coupled system of nonlinear fractional differential equations

$$D^{\alpha}x(t)=f(t,x(t),y(t),D^{\gamma}y(t)),\ 0\leq t\leq T,\ 1<\alpha\leq 2,\ 0<\gamma<1$$
$$D^{\beta}y(t)=g(t,x(t),D^{\delta}x(t),y(t)),\ 0\leq t\leq T,\ 1<\beta\leq 2,\ 0<\delta<1$$

together with the coupled nonlocal integral boundary conditions

$$x(0)=h(y),\ \ \int_0^T y(s)\,ds=\mu_1 x(\eta),$$
$$y(0)=\phi(x),\ \ \int_0^T x(s)\,ds=\mu_2 y(\zeta),\ \ \eta,\,\zeta\in(0,T),$$

where $^{c}D^{i},\ i=\alpha,\ \beta,\ \gamma,$ and δ are Caputo fractional derivatives of order α, β, γ and δ, respectively.

Luca and Tudorache [20] used a nonlinear alternative of Leray-Schauder type to show the existence of at least one positive solution of the system of nonlinear fractional differential equations

$$\begin{cases} D^{\alpha}_{0+}u(t)+\lambda f(t,u(t),v(t))=0,\ \ 0<t<1,\ n-1<\alpha\leq n,\\ D^{\beta}_{0+}v(t)+\lambda g(t,u(t),v(t))=0,\ \ 0<t<1,\ m-1<\beta\leq m,\end{cases}\tag{1.4}$$

with the integral boundary conditions

$$u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0,\ u(1)=\int_0^1 u(s)\,dH(s)$$

$$v(0) = v'(0) = \cdots = v^{(m-2)}(0) = 0, \ v(1) = \int_0^1 v(s)\, dK(s),$$

where $m, n \in N$, $n, m \geq 3$, D_{0+}^{α} and D_{0+}^{β} are the Riemann-Liouville derivatives of order α and β respectively, and the integrals in the boundary conditions are Riemann-Stieltjes integrals.

In another work, Henderson and Luca [11] studied the nonexistence of positive solutions of the system (1.3) with the coupled integral boundary conditions

$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \ u(1) = \int_0^1 v(s)\, dH(s)$$
$$v(0) = v'(0) = \cdots = v^{(m-2)}(0) = 0, \ v(1) = \int_0^1 u(s)\, dK(s).$$

Very recently, Henderson and Luca [13] used the Guo-Krasnosel'skii fixed point theorem and nonlinear alternative of Leray-Schauder type to find the existence of at least one positive solution to the system (1.4) with the coupled boundary conditions

$$\begin{cases} u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \ \ u'(1) = \int_0^1 v(s)\, dH(s) \\ v(0) = v'(0) = \cdots = v^{(m-2)}(0) = 0, \ \ v'(1) = \int_0^1 u(s)\, dK(s), \end{cases} \tag{1.5}$$

where f and g are sign changing continuous functions. Henderson et al. [14] used Guo-Krasnosel'skii fixed point theorem to find the existence of at least one positive solution to the system (1.4)–(1.5), where f and g are non negative functions.

The motivation for the present work has come from a recent work by Qiao and Zhou [23], who used the Krasnosel'skii fixed point theorem to find the existence of at least one positive solution to the multi-point singular fractional boundary value problem

$$D_{0+}^{\alpha} x(t) + q(t) f(t, x(t)) = 0, \ 0 < t < 1, \ n > 2, \ n-1 < \alpha \leq n,$$
$$x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0,$$
$$D_{0+}^{\beta} x(1) = \sum_{i=1}^{\infty} \alpha_i x(\xi_i), \ \ \beta \in [1, \alpha - 1],$$
$$\alpha_i \geq 0, \ i = 1, 2, \cdots, \ 0 < \xi_1 < \xi_2 < \cdots < \xi_{i-1} < \xi i < \cdots < 1,$$
$$\frac{\Gamma(\alpha)}{\Gamma(\alpha - \beta)} - \sum_{i=1}^{\infty} \alpha_i \xi_i^{\alpha - 1} > 0,$$

where D_{0+}^{α} and D_{0+}^{β} are Riemann-Liouville fractional derivatives. In particular, if $h(t, x) \equiv x$, then the Riemann-Stieltjes integral $\lambda[x] = \int_0^1 x(s)\, dA(s)$ covers a variety of nonlocal boundary conditions including the cases

$$\lambda[x] = \gamma x(\delta), \;\; \gamma \geq 0, \;\; \delta \in (0,1),$$
$$\lambda[x] = \sum_{i=1}^{l} \gamma_j x(\delta_i), \;\; \gamma_i \in R, \;\; i = 1, 2, \cdots, l, \;\; 0 < \delta_1 < \delta_2 < \cdots < \delta_l < 1,$$
$$\lambda[x] = \int_0^1 x(t) g(t)\, dt, \;\; g \in C((0,1), R).$$

From the above cited references and the available literature, it appears that no work has been done on the existence of the positive solution of the fractional boundary value problem (FBVP for short) (1.1)–(1.3). This motivates us to study the existence of at least one positive solution of the FBVP (1.1)–(1.3). Our existence theorems are based on the Krasnosel'skii's cone expansion and compression fixed point theorem and the Leray-Schauder fixed point theorem.

In order to establish our results, we assume that the following conditions are satisfied:

$(A1)$: $f,\, h \in C((0,1) \times (0,\infty), [0,\infty))$;

$(A2)$: $q \in C((0,1),[0,\infty))$ and q does not vanish identically on any subinterval of $(0,1)$;

$(A3)$: for any positive numbers r_1 and r_2 with $r_1 < r_2$, there exist continuous functions p_f and $p_h : (0,1) \to [0,\infty)$ such that $\int_0^1 G(1,s)q(s)p_f(s)\, ds < \infty$ and $\int_0^1 p_h\, dA(s) < \infty$, where

$$f(t,x) \leq p_f(t) \text{ and } h(t,x) \leq p_h(t), \;\; 0 < t < 1, \;\; t^{\alpha-1} r_1 \leq x \leq r_2.$$

The remainder of this paper is divided into three sections. Basic notations and preliminaries are given in the Section 2. Section 3 contains results on the existence and uniqueness of a nontrivial solution of the FBVP (1.1)–(1.3) with both f and $h : [0,1] \times R \to R$. In Section 4, we consider the FBVP (1.1)–(1.3) with f and h as considered in $(A1)$. Examples are given to illustrate our theorems.

2. Preliminaries

In this section, we provide some lemmas that are needed to prove the main results in this paper. For the basic definition and results on the Riemann-Liouville fractional derivative of order α for a function $x(t)$, and the fractional integral of $x(t)$, we refer the reader to [12, 18, 22].

Lemma 2.1. ([18]) *The general solution to $D^{\alpha} x(t) = 0$ with $\alpha \in (n-1, n]$ and $n > 1$ is the function*

$$x(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},\ c_i \in R,\ i = 1, 2, \cdots, n.$$

Lemma 2.2. ([18]) *Let $\alpha > 0$. Then the following equality holds for $x(t)$:*

$$D_{0+}^{-\alpha} D_{0+}^{\alpha} x(t) = x(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

where $c_i \in R$, $i = 1, 2, \cdots, n$, and n is the smallest integer greater than or equal to α.

Lemma 2.3. *For $y \in C[0, 1]$, the unique solution to the problem*

$$\begin{cases} D_{0+}^{\alpha} x(t) + y(t) = 0, \\ x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, \\ D_{0+}^{\beta} x(1) = \int_0^1 V(s)\, dA(s), \end{cases} \tag{2.1}$$

is given by

$$x(t) = \int_0^1 G(t, s) q(s) y(s)\, ds + \frac{\Gamma(\alpha - \beta)}{\Gamma(\alpha)} t^{\alpha-1} \int_0^1 V(s)\, dA(s), \tag{2.2}$$

where $G(t, s)$ is the Green's function defined by

$$G(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1},\ 0 \le s \le t \le 1, \\ t^{\alpha-1}(1-s)^{\alpha-\beta-1},\ 0 \le t \le s \le 1. \end{cases} \tag{2.3}$$

Proof. In view of Lemmas 2.1 and 2.2, a general solution of the fractional equation $D_{0+}^{\alpha} x(t) + y(t) = 0$ is given by

$$x(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n} - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds \tag{2.4}$$

Since $x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0$, from (2.4) we obtain that $c_2 = c_3 = \cdots = c_n = 0$. Hence, (2.4) becomes

$$x(t) = c_1 t^{\alpha-1} - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds. \tag{2.5}$$

Taking the β-th fractional derivative on the both sides of (2.5), we have

$$D_{0+}^{\beta} x(t) = c_1 (\alpha - \beta + n - 1)(\alpha - \beta + n - 2) \cdots (\alpha - \beta) t^{\alpha-\beta-1} \frac{\Gamma(\alpha)}{\Gamma(n + \alpha - \beta)}$$
$$- \frac{1}{\Gamma(\alpha - \beta)} \int_0^t (t-s)^{\alpha-\beta-1} y(s)\, ds. \tag{2.6}$$

Using the integral boundary condition $D_{0+}^{\beta} x(1) = \int_0^1 V(s)\, dA(s)$ in (2.6), we obtain

$$c_1 = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\beta-1} y(s)\, ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} \int_0^1 V(s) dA(s).$$

This together with the (2.5) yields (2.2), which is a solution of (2.1). This completes the proof of the lemma. □

LEMMA 2.4. *For any s, $t \in [0,1]$, the Green's function $G(t,s)$ given in (2.3) satisfies*

$$t^{\alpha-1} G(1,s) = t^{\alpha-1} \max_{0 \le t \le 1} G(t,s) \le G(t,s) \le \max_{0 \le t \le 1} G(t,s) = G(1,s). \quad (2.7)$$

P r o o f. It is clear that $G(t,s)$ is continuous for all s, $t \in [0,1]$. Furthermore, $(1-s)^{\alpha-\beta-1} \ge (1-s)^{\alpha-1}$ for all $s \in [0,1]$ and $\beta \in [1, \alpha-1]$ imply that $G(t,s) \ge 0$ for all s, $t \in [0,1]$.

Set $g_1(t,s) = t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1}$ and $g_2(t,s) = t^{\alpha-1}(1-s)^{\alpha-\beta-1}$; then, for any s, $t \in [0,1]$, we have $\frac{\partial g_2}{\partial t} = (\alpha-1)t^{\alpha-2}(1-s)^{\alpha-\beta-1} \ge 0$, which implies that $g_2(t,s)$ is a nondecreasing function of t. For $0 \le s \le t \le 1$, we have

$$\frac{\partial g_1(t,s)}{\partial t} \ge (\alpha-1)t^{\alpha-2}(1-s)^{\alpha-\beta-1}[1-(1-s)^{\beta}] \ge 0,$$

which implies that $g_1(t,s)$ is a nondecreasing function of t. Consequently, $G(t,s)$ is a nondecreasing function of t. Hence,

$$\max_{0 \le t \le 1} G(t,s) = G(1,s) = (1-s)^{\alpha-\beta-1}[1-(1-s)^{\beta}].$$

In order to complete the proof of the lemma, it is enough to show that $g_1(t,s) \ge t^{\alpha-1} g_1(1,s)$ for $0 \le s \le t \le 1$, which follows from the following calculations:

$$\begin{aligned} g_1(t,s) &= t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1} \\ &= t^{\alpha-1}(1-s)^{\alpha-\beta-1} - t^{\alpha-1}\left(1-\frac{s}{t}\right)^{\alpha-1} \\ &\ge t^{\alpha-1}[(1-s)^{\alpha-\beta-1} - (1-s)^{\alpha-1}]. \end{aligned}$$

This proves the lemma. □

We shall use the following fixed point theorems to prove our main results in Section 3.

THEOREM 2.1. (Schauder's fixed point theorem [8]) *Let B be a nonempty, bounded, closed subset of a Banach space X and $T: B \to B$ be a completely continuous operator. Then, T has at least one fixed point in B.*

Theorem 2.2. (Leray-Schauder theorem [8]) *Let B be a convex subset of a Banach space X, $0 \in B$, and $T : B \to X$ be a completely continuous operator. Then, either*:

(i): *T has at least one fixed point in B; or*
(ii): *The set $\{x \in B : x = \mu Tx, 0 < \mu < 1\}$ is unbounded.*

Next, we present some concepts about cones in a Banach space.

Definition 2.1. Let X be a real Banach space. A nonempty convex closed set $P \subset X$ is said to be a cone provided that

(i): $ku \in P$ for all $u \in P$ and all $k \geq 0$, and
(ii): $u, -u \in P$ implies $u = 0$.

Theorem 2.3. (Krasnosel'skii's fixed point theorem [8]) *Let X be a real Banach space and $K \subset X$ be a cone. Assume that K_1 and K_2 are bounded open subsets of X with $\theta \in K_1$, $\overline{K}_1 \subset K_2$, and $T : K \cap (\overline{K_2} \backslash K_1) \to K$ is a completely continuous operator such that either*:

(i): $\|Tx\| \leq \|x\|,\ x \in K \cap \partial K_1$, *and* $\|Tx\| \geq \|x\|,\ x \in K \cap \partial K_2$; *or*
(ii): $\|Tx\| \geq \|x\|,\ x \in K \cap \partial K_1$, *and* $\|Tx\| \leq \|x\|,\ x \in K \cap \partial K_2$.

Then T has a fixed point in $K \cap (\overline{K_2} \setminus K_1)$.

Definition 2.2. A map Φ is said to be a nonnegative continuous concave functional on a cone P of a real Banach space X if $\Phi : P \to R_+$ is continuous and

$$\Phi(tx + (1-t)y) \geq t\Phi(x) + (1-t)\Phi(y)$$

for all x, $y \in P$ and $t \in [0, 1]$.

Similarly, we say the map ϕ is a nonnegative continuous convex functional on a cone P of a real Banach space X if $\phi : P \to R_+$ is continuous and

$$\phi(tx + (1-t)y) \leq t\phi(x) + (1-t)\phi(y)$$

for all x, $y \in P$ and $t \in [0, 1]$.

An operator is called *completely continuous* if it is continuous and maps bounded sets into precompact sets.

We will use the following notations as introduced by Avery and Peterson [3]. Let ϕ and Θ be nonnegative convex functionals on P, let Φ be a nonnegative continuous concave functional on P, and let ψ be a nonnegative continuous functional on P. Then, for positive numbers c_1, c_2, c_3, and c_4, we define the following sets:

$$\begin{cases} P(\phi, c_4) = \{x \in P : \phi(x) < c_4\}; \\ \overline{P(\phi, c_4)} = \{x \in P : \phi(x) \leq c_4\}; \\ P(\phi, \Phi, c_2, c_4) = \{x \in P : c_2 \leq \Phi(x), \phi(x) \leq c_4\}; \\ P(\phi, \Theta, \Phi, c_2, c_3, c_4) = \{x \in P : c_2 \leq \Phi(x), \Theta(x) \leq c_3, \phi(x) \leq c_4\}; \\ R(\phi, \psi, c_1, c_4) = \{x \in P : c_1 \leq \psi(x), \phi(x) \leq c_4\}. \end{cases} \tag{2.8}$$

The following fixed point theorem due to Avery and Peterson [3] will be used to establish the existence of multiple positive solutions to FBVP (1.1)–(1.3).

Theorem 2.4. (Avery and Peterson [3]) *Let P be a cone in a real Banach space X. Let ϕ and Θ be nonnegative continuous convex functionals on P, let Φ be a nonnegative continuous concave functional on P, and let ψ be a nonnegative continuous functional on P satisfying $\psi(kx) \leq k\psi(x)$ for $0 \leq k \leq 1$, such that for some positive numbers $\bar{M}$ and c_4*

$$\Phi(x) \leq \psi(x) \text{ and } \|x\| \leq \bar{M}\phi(x)$$

for all $x \in \overline{P(\phi, c_4)}$. Suppose

$$T : \overline{P(\phi, c_4)} \to \overline{P(\phi, c_4)}$$

is a completely continuous operator and there exist constants c_1, c_2, and c_3 with $c_1 < c_2$ such that

- **(S1):** *$\{x \in P(\phi, \Theta, \Phi, c_2, c_3, c_4) : \Phi(x) > c_2\}$ is nonempty and $\Phi(Tx) > c_2$ for $x \in P(\phi, \Theta, \Phi, c_2, c_3, c_4)$;*
- **(S2):** *$\Phi(Tx) > c_2$ for $x \in P(\phi, \Phi, c_2, c_4)$ with $\Theta(Tx) > c_3$;*
- **(S3):** *$0 \notin R(\phi, \psi, c_1, c_4)$ and $\psi(Tx) < c_1$ for $x \in R(\phi, \psi, c_1, c_4)$ with $\psi(x) = c_1$.*

Then T has at least three fixed points x_1, x_2, $x_3 \in \overline{P(\phi, c_4)}$, such that $\phi(x_i) \leq c_4$, $i = 1, 2, 3$, $c_2 < \Phi(x_1)$, $c_1 < \psi(x_2)$, $\Phi(x_2) < c_2$, and $\psi(x_3) < c_1$.

In this paper, we take $X = C[0,1]$ to be the Banach space endowed with the norm $\|x\| = \max\limits_{0 \leq t \leq 1} |x(t)|$. Define a cone K on X by

$$K = \{x \in X : x(t) \geq 0, \, x(t) \geq t^{\alpha-1}\|x\|, \, 0 \leq t \leq 1\}, \tag{2.9}$$

and an operator $T : K \to X$ by

$$(Tx)(t) = \int_0^1 G(t,s)q(s)f(s,x(s))\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} t^{\alpha-1} \int_0^1 h(s,x(s))\,dA(s). \tag{2.10}$$

In view of the Lemma 2.3, it is easy to verify that $x(t)$ is a positive solution of FBVP (1.1)–(1.3) if and only if $x(t)$ is a fixed point of the operator T on the cone K.

Lemma 2.5. $T(K) \subset K$.

P r o o f. Let $x \in K$; then $x(t) \geq 0$ and $x(t) \geq t^{\alpha-1}\|x\|$ for $t \in [0,1]$. Since $f(t,x) \geq 0$, $h(t,x) \geq 0$, $q(t) \geq 0$, A is non decreasing, and $G(t,s) \geq 0$ for all $s, t \in [0,1]$, we have that $(Tx)(t) \geq 0$ for all $t \in [0,1]$. For $s, t \in [0,1]$,

$$\begin{aligned}
\|Tx\| &= \max_{0\leq t\leq 1}\Big|\int_0^1 G(t,s)q(s)f(s,x(s))\,ds \\
&\quad + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}t^{\alpha-1}\int_0^1 h(s,x(s))\,dA(s)\Big| \\
&\leq \max_{0\leq t\leq 1} t^{\alpha-1}\int_0^1 G(1,s)q(s)f(s,x(s))\,ds \\
&\quad + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\max_{0\leq t\leq 1} t^{\alpha-1}\int_0^1 h(s,x(s))\,dA(s) \\
&\leq \int_0^1 G(1,s)q(s)f(s,x(s))\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\int_0^1 h(s,x(s))\,dA(s),
\end{aligned}$$

and

$$\begin{aligned}
(Tx)(t) &= \int_0^1 G(t,s)q(s)f(s,x(s))\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}t^{\alpha-1}\int_0^1 h(s,x(s))\,dA(s) \\
&\geq t^{\alpha-1}\int_0^1 G(1,s)q(s)f(s,x(s))\,ds \\
&\quad + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}t^{\alpha-1}\int_0^1 h(s,x(s))\,dA(s),
\end{aligned}$$

which implies that $(Tx)(t) \geq t^{\alpha-1}\|Tx\|$, That is, $T(K) \subset K$, and this proves the lemma. □

Let us denote $K(r) = \{x \in K : \|x\| < r\}$ and $\partial K(r) = \{x \in K : \|x\| = r\}$.

Lemma 2.6. *Suppose that conditions* $(A1)$–$(A2)$ *hold and there exist constants* r_1 *and* r_2 *with* $0 < r_1 < r_2$ *such that condition* $(A3)$ *is satisfied. Then the operator* $T : \overline{K(r_2)} \setminus K(r_1) \to K$ *is completely continuous.*

P r o o f. From the continuity of $G(t,s)$, and $(A1)$–$(A3)$, it follows that T is continuous on $\overline{K(r_2)} \setminus K(r_1)$. For any $x \in \overline{K(r_2)} \setminus K(r_1)$, we have

$$|Tx| = \left|\int_0^1 G(t,s)q(s)f(s,x(s))\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}t^{\alpha-1}\int_0^1 h(s,x(s))\,dA(s)\right|$$

$$\leq \int_0^1 G(1,s)q(s)p_f(s)\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\int_0^1 p_h(s)\,dA(s),$$

for $t \in [0,1]$, which implies that T is uniformly bounded.

Since $G(t,s)$ is continuous on $[0,1]\times[0,1]$, it is uniformly continuous on $[0,1]\times[0,1]$. Hence, for every $\epsilon > 0$ there exists $\delta \in \left(0, \frac{\epsilon}{\alpha-1}\right)$ such that,

$$|G(t_1,s) - G(t_2,s)| < \epsilon$$

for $|t_1 - t_2| < \delta$, and (t_1,s), $(t_2,s) \in [0,1]\times[0,1]$. Then, for any $x \in \overline{K(r_2)} \setminus K(r_1)$ and t_1, $t_2 \in [0,1]$ with $|t_1 - t_2| < \delta$, we have

$$\begin{aligned}
&|(Tx)(t_1) - (Tx)(t_2)| \\
&\quad \leq \int_0^1 |G(t_1,s) - G(t_2,s)|\, q(s)f(s,x(s))ds \\
&\qquad + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}|t_1^{\alpha-1} - t_2^{\alpha-1}| \int_0^1 h(s,x(s))\,dA(s) \\
&\quad \leq \epsilon \int_0^1 q(s)p_f(s)\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}|t_1 - t_2| \sum_{i=0}^{\alpha-2} t_1^{\alpha-2-i}\, t_2^i \int_0^1 p_h(s)\,dA(s) \\
&\quad \leq \epsilon \int_0^1 q(s)p_f(s)\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\delta(\alpha-1)\int_0^1 p_h(s)\,dA(s) \\
&\quad \leq \epsilon \left[\int_0^1 q(s)p_f(s)\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\int_0^1 p_h(s)\,dA(s)\right].
\end{aligned}$$

Hence, T is equicontinuous. Consequently, T is relatively compact on $\overline{K(r_2)} \setminus K(r_1)$, and hence compact on $\overline{K(r_2)} \setminus K(r_1)$. Thus, the operator $T : \overline{K(r_2)} \setminus K(r_1) \to K$ is completely continuous. This completes the proof of the lemma. □

Throughout the remainder of this paper, we set

$$\lambda = \int_0^1 G(1,s)q(s)\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\int_0^1 dA(s). \tag{2.11}$$

3. **Existence and uniqueness of solutions**

In this section, we consider the FBVP (1.1)–(1.3) with f and h : $[0,1]\times R \to R$. We define the operator $T : X \to X$ by (2.10). Then, an application of the Arzelà-Ascoli theorem shows that the mapping T is completely continuous.

THEOREM 3.1. *Assume that the functions f and h satisfies the Lipschitz conditions*

$$|f(t,x_1) - f(t,x_2)| \leq L|x_1 - x_2|$$

and

$$|h(t,x_1)-f(t,x_2)|\le M|x_1-x_2|$$

for (t,x_1), $(t,x_2)\in[0,1]\times R$ *with*

$$0<L\int_0^1 G(1,s)q(s)\,ds+\frac{M\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\int_0^1 dA(s)<1. \tag{3.1}$$

Then FBVP (1.1)–(1.3) has a unique solution.

P r o o f. Since $x(t)$ is a solution of FBVP (1.1)–(1.3) if and only if $x(t)$ is a fixed point of the operator T on X, it is enough to show that T has a fixed point in X. For any x_1, $x_2\in X$ with $t\in[0,1]$, we have

$$\begin{aligned}|(Tx_1-Tx_2)(t)| &\le \int_0^1 G(1,s)q(s)|f(s,x_1(s))-f(s,x_2(s))|\,ds\\ &\quad+\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\int_0^1 |h(s,x_1(s))-h(s,x_2(s))|\,dA(s)\\ &\le\left[L\int_0^1 G(1,s)q(s)\,ds+\frac{M\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\int_0^1 dA(s)\right]\|x_1-x_2\|.\end{aligned}$$

Hence, in view of (3.1), T is a contraction mapping. By the Banach contraction principle, T has a unique fixed point. Hence, FBVP (1.1)–(1.3) has a unique solution, and this completes the proof of the theorem. □

Remark 3.1. Suppose, in addition to the assumptions of Theorem 3.1, that $f(t,0)\equiv 0$ and $h(t,0)\equiv 0$ on $[0,1]$; then $x(t)\equiv 0$ is a solution of the FBVP (1.1)–(1.3). In this case, by uniqueness, FBVP (1.1)–(1.3) has no nontrivial solutions.

Theorem 3.2. *Assume that*

$$\lim_{|x|\to\infty}\max_{0\le t\le 1}\frac{|f(t,x)|}{|x|}=0 \text{ and } \lim_{|x|\to\infty}\max_{0\le t\le 1}\frac{|h(t,x)|}{|x|}=0 \tag{3.2}$$

hold with $f(t,0)\not\equiv 0$ *and* $h(t,0)\not\equiv 0$ *on* $[0,1]$. *Then FBVP (1.1)–(1.3) has at least one nontrivial solution.*

P r o o f. Choose constants l and m, such that

$$l\int_0^1 G(1,s)q(s)\,ds+\frac{m\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\int_0^1 dA(s)\le 1.$$

By (3.2), there exists $c_1>0$ such that $|f(t,x)|\le l|x|$ and $|h(t,x)|\le m|x|$ for any $t\in[0,1]$ and $|x|\ge c_1$. Now, f, $h\in C([0,1]\times R,R)$ implies that there exist constants $l_1 > 0$ and $m_1 > 0$ such that $|f(t,x)| \le l_1$ and $|h(t,x)| \le m_1$ on $[0,1] \times [-c_1, c_1]$.

Setting $c_2 = \max\{c_1, l_1/l\}$ and $c_3 = \max\{c_2, m_1/m\}$, we see that

$$|f(t,x)| \le lc_2 \text{ on } [0,1] \times [-c_2, c_2] \tag{3.3}$$

and

$$|h(t,x)| \le mc_3 \text{ on } [0,1] \times [-c_3, c_3]. \tag{3.4}$$

Set $c_4 = \max\{c_2, c_3\}$ and define a bounded, closed, and convex set $B \subset X$ by

$$B = \{x \in X : \|x\| \le c_4\}.$$

Then, for any $x \in B$, we have $|x(t)| \le c_4$ on $[0,1]$. Hence, from (2.7), (2.10), (3.3), and (3.4), we have

$$\begin{aligned}|(Tx)(t)| &= \left|\int_0^1 G(t,s)q(s)f(s,x(s))\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}t^{\alpha-1}\int_0^1 h(s,x(s))\,dA(s)\right| \\ &\le c_4\left[l\int_0^1 G(1,s)q(s)\,ds + \frac{m\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\int_0^1 dA(s)\right] \le c_4.\end{aligned}$$

Hence $\|Tx\| \le c_4$, that is, $T(B) \subset B$. By the Schauder fixed point theorem, Theorem 2.1 above, T has at least one fixed point in B. Clearly, $x(t) \equiv 0$ is not a fixed point because $f(t,0) \not\equiv 0$ and $h(t,0) \not\equiv 0$ on $[0,1]$. Therefore, FBVP (1.1)–(1.3) has at least one nontrivial solution. This proves the theorem. □

Remark 3.2. The conditions (3.2) in Theorem 3.2 can be satisfied by a broad range of functions. For example, condition (3.2) is satisfied by all bounded functions. In the case of unbounded functions, many functions, such as $p(t)x^\lambda sgnx + 1$ and $p(t) + x^\lambda sgnx \ln(x^2+1) + e^{sinx}$, with $p \in C[0,1]$ and $\lambda \in (0,1)$, also satisfy condition (3.2).

Example 3.1. Consider the problem (1.1)–(1.3) with $f(t,x) = h(t,x) = x^{1/3} + \sin t$. The conditions of Theorem 3.2 are satisfied, so the problem (1.1)–(1.3) has at least one nontrivial solution. On the other hand, since f and h do not satisfy Lipschitz conditions in x near 0, the solution may not be unique.

Example 3.2. Consider (1.1)–(1.3) with $f(t,x) = h(t,x) = p\,tan^{-1}x + e^t$, where $0 < p < \frac{1}{\lambda}$. It is easy to see that $|f(t,x_1) - f(t,x_2)| \le p|x_1 - x_2|$ and $|h(t,x_1) - h(t,x_2)| \le p|x_1 - x_2|$ for any (t,x_1), $(t,x_2) \in [0,1] \times R$. Then, by Theorem 3.1, this problem has a unique solution. Moreover, since $f(t,0) \not\equiv 0$ and $h(t,0) \not\equiv 0$, the solution is a nontrivial one.

4. **Existence of Positive Solutions**

In this section, we shall use the Theorems 2.1–2.4 to prove our results. We assume, throughout this section, that f and h satisfies $(A1)$.

THEOREM 4.1. *Suppose that there exist continuous functions* h_1, $f_1 : (0,1)\times[0,\infty)\to[0,\infty)$ *and a constant* $c_1>0$ *such that*

$(H1)$**:** $f(t,x)\le f_1(t)c_1$ *and* $h(t,x)\le h_1(t)c_1$ *for* $0\le x(t)\le c_1$ *and* $0\le t\le 1$,

and

$$\int_0^1 G(1,s)q(s)f_1(s)\,ds+\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\int_0^1 h_1(s)\,dA(s)\le 1. \tag{4.1}$$

Then FBVP (1.1)–(1.3) has at least one non-negative solution.

P r o o f. We define a convex, closed, and bounded set $B\subset X$ by

$$B=\{x\in X: 0\le x(t)\le c_1,\ t\in[0,1]\}. \tag{4.2}$$

Then, for $x\in B$, we have

$$\begin{aligned}(Tx)(t)&=\int_0^1 G(t,s)q(s)f(s,x(s))\,ds+\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}t^{\alpha-1}\int_0^1 h(s,x(s))\,dA(s)\\ &\le c_1\left[\int_0^1 G(1,s)q(s)f_1(s)\,ds+\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\int_0^1 h_1(s)\,dA(s)\right]\le c_1.\end{aligned}$$

Hence $Tx\in B$, that is, $T(B)\subset B$. Clearly, $T:B\to B$ is completely continuous. By Theorem 2.1, T has a fixed point x in B, which in turn is a non-negative solution of FBVP (1.1)–(1.3). This proves the theorem. □

REMARK 4.1. In the proof of Theorem 4.1, we used Schauder's fixed point theorem to obtain a fixed point of the operator T in B. We can also use Theorem 2.2 to prove this existence. In fact, we may observe that $x(t)=\mu(Tx)(t)\le\mu c_1<\infty$ for $x\in B$ implies that the set $\{x\in B: x=\mu Tx,\ 0<\mu<1\}$ is bounded. Hence, by Theorem 2.2, the operator T has a fixed point in B, which corresponds to a non-negative solution of the problem (1.1)–(1.3).

Although the conditions of Theorem 4.1 appear to be simple and easily verifiable, other nice conditions for the existence of a non-negative solution of the problem (1.1)–(1.3) are given in the following result.

THEOREM 4.2. *Suppose that*

$(H2)$**:** $\limsup_{x\to 0}\max_{0\le t\le 1}\frac{f(t,x)}{x}=0$ *and* $\limsup_{x\to 0}\max_{0\le t\le 1}\frac{h(t,x)}{x}=0.$

Then the problem (1.1)–(1.3) has at least one non-negative solution.

P r o o f. By $(H2)$, there exists a constant c_1 such that

$$f(t,x) \le \epsilon x(t) \text{ for } 0 \le x(t) \le c_1 \text{ and } 0 \le t \le 1$$

and

$$h(t,x) \le \epsilon x(t) \text{ for } 0 \le x(t) \le c_1 \text{ and } 0 \le t \le 1$$

hold, where $\epsilon > 0$ is chosen so that it satisfies $\epsilon\lambda \le 1$, where λ is given in (2.11). For the above choice of c_1, we consider the closed, convex, and bounded subset B of X given in (4.2). Then for $x \in B$, we have

$$\begin{aligned}(Tx)(t) &= \int_0^1 G(t,s)q(s)f(s,x(s))\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}t^{\alpha-1}\int_0^1 h(s,x(s))\,dA(s)\\ &\le \int_0^1 G(1,s)q(s)\epsilon x(s)\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\int_0^1 \epsilon x(s)\,dA(s)\\ &\le c_1\epsilon\left[\int_0^1 G(1,s)q(s)\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\int_0^1 dA(s)\right] \le c_1.\end{aligned}$$

This proves that $T(B) \subset B$. The conclusion then follows as before. □

THEOREM 4.3. *Assume that there exist continuous functions h_2 and $f_2 : [0,1] \to [0,\infty)$ and a constant $c^* > 0$ such that*

$(H3)$**:** *$f(t,x) \le f_2(t)x(t)$ and $h(t,x) \le h_2(t)x(t)$ for $x(t) \ge c^*$ and $0 \le t \le 1$, and*

$$\int_0^1 G(1,s)q(s)f_2(s)\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\int_0^1 h_2(s)\,dA(s) < 1.$$

Then FBVP (1.1)–(1.3) has at least one non-negative solution.

P r o o f. From the continuity of $h(t,x)$ and $f(t,x)$, we can find constants γ_f and γ_h such that

$$\gamma_f = \max_{\substack{0\le t\le 1\\ 0\le x\le c^*}} f(t,x) \text{ and } \gamma_h = \max_{\substack{0\le t\le 1\\ 0\le x\le c^*}} h(t,x). \tag{4.3}$$

Hence, from $(H3)$, we have

$$f(t,x) \le f_2(t)x(t) + \gamma_f \text{ for } x \ge 0 \text{ and } 0 \le t \le 1$$

and

$$h(t,x) \le h_2(t)x(t) + \gamma_h \text{ for } x \ge 0 \text{ and } 0 \le t \le 1.$$

Choose a constant $c_4 > 0$ such that

$$c_4 \geq \frac{\gamma_f \int_0^1 G(1,s)q(s)\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\gamma_h \int_0^1 dA(s)}{1 - \left[\int_0^1 G(1,s)q(s)f_2(s)\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\int_0^1 h_2(s)\,dA(s)\right]}$$

and define a bounded, closed, and convex subset B of X by

$$B = \{x \in X : 0 \leq x(t) \leq c_4,\ 0 \leq t \leq 1\}. \tag{4.4}$$

Then for $x \in B$, we have

$$\begin{aligned}
(Tx)(t) &= \int_0^1 G(t,s)q(s)f(s,x(s))\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}t^{\alpha-1}\int_0^1 h(s,x(s))\,dA(s) \\
&\leq \int_0^1 G(1,s)q(s)(f_2(s)x(s)+\gamma_f)\,ds \\
&\quad + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\int_0^1 (h_2(s)x(s)+\gamma_h)\,dA(s) \\
&\leq c_4\left[\int_0^1 G(1,s)q(s)f_2(s)\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\int_0^1 h_2(s)\,dA(s)\right] \\
&\quad + \gamma_f \int_0^1 G(1,s)q(s)\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\gamma_h \int_0^1 dA(s) \leq c_4,
\end{aligned}$$

which implies that $T(B) \subset B$. Clearly, $T : B \to B$ is completely continuous, so by Theorem 2.1, T has a fixed point $x(t)$ in B that is a non-negative solution of (1.1)–(1.3). □

As described in Remark 4.1, Theorem 2.2 could also be applied to find the existence of a fixed point of T in B.

THEOREM 4.4. *If*

$(H4)$: $\limsup_{x\to\infty} \max_{0\leq t\leq 1} \frac{f(t,x)}{x} = 0$ *and* $\limsup_{x\to\infty} \max_{0\leq t\leq 1} \frac{h(t,x)}{x} = 0,$

then the problem (1.1)–(1.3) has at least one non-negative solution.

P r o o f. By $(H4)$, we can find a constant $c^* > 0$ such that

$$f(t,x) \leq \epsilon x(t) \ \text{ for } \ x(t) \geq c^* \ \text{ and } \ 0 \leq t \leq 1$$

and

$$h(t,x) \leq \epsilon x(t) \ \text{ for } \ x(t) \geq c^* \ \text{ and } \ 0 \leq t \leq 1,$$

where ϵ satisfies $\epsilon\lambda < 1$. For the above choice of c^*, we consider the constants γ_f and γ_h as in (4.3). Then, we have

$$f(t,x(t)) \leq \epsilon x(t) + \gamma_f \ \text{ for } \ x(t) \geq 0 \ \text{ and } \ 0 \leq t \leq 1$$

and

$$h(t,x(t)) \leq \epsilon x(t) + \gamma_h \ \text{ for } \ x(t) \geq 0 \ \text{ and } \ 0 \leq t \leq 1.$$

Choose a constant $c_4 > 0$ such that

$$c_4 \geq \frac{\gamma_f \int_0^1 G(1,s)q(s)\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\gamma_h \int_0^1 dA(s)}{1-\epsilon\left[\int_0^1 G(1,s)q(s)\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\int_0^1 dA(s)\right]}.$$

For the above choice of c_4, we consider the bounded, closed, and convex subset B of X in (4.4). Then, for $x \in B$, we have

$$\begin{aligned}(Tx)(t) &= \int_0^1 G(t,s)q(s)f(s,x(s))\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}t^{\alpha-1}\int_0^1 h(s,x(s))\,dA(s)\\ &\leq \int_0^1 G(1,s)q(s)(\epsilon x(s)+\gamma_f)\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\int_0^1 (\epsilon x(s)+\gamma_h)\,dA(s)\\ &\leq c_4\epsilon\left[\int_0^1 G(1,s)q(s)\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\int_0^1 dA(s)\right]\\ &\quad + \gamma_f\int_0^1 G(1,s)q(s)\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\gamma_h\int_0^1 dA(s) \leq c_4,\end{aligned}$$

which implies that $T(B) \subset B$. The conclusion again follows as before. □

In Theorems 4.1–4.4, the solution whose existence is guaranteed is a non-negative one that may be the zero solution. In the following, we apply the well known Krasnosel'skii fixed point theorem, Theorem 2.3 above, to obtain the existence of a positive solution of (1.1)–(1.3).

We introduce the following "height" functions to control the growth of the nonlinear terms $f(t,x)$ and $h(t,x)$. For any $r > 0$, let

$$\begin{aligned}f_1(t,r) &= \min\{f(t,x);\ t^{\alpha-1}r \leq x \leq r\},\ \ 0 < t < 1;\\ f_2(t,r) &= \max\{f(t,x);\ t^{\alpha-1}r \leq x \leq r\},\ \ 0 < t < 1;\\ h_1(t,r) &= \min\{h(t,x);\ t^{\alpha-1}r \leq x \leq r\},\ \ 0 < t < 1;\ \text{ and}\\ h_2(t,r) &= \max\{h(t,x);\ t^{\alpha-1}r \leq x \leq r\},\ \ 0 < t < 1.\end{aligned}$$

THEOREM 4.5. *Let (A1)–(A2) hold and assume that there exist constants r_1 and r_2 with $0 < r_1 < r_2$ such that (A3) holds. In addition, assume that one of the following conditions is satisfied:*

(H5): $r_1 \leq \int_0^1 G(1,s)q(s)f_1(s,r_1)\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\int_0^1 h_1(s,r_1)\,dA(s) < \infty$, *and*

$$\int_0^1 G(1,s)q(s)f_2(s,r_2)\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\int_0^1 h_2(s,r_2)\,dA(s) \leq r_2;$$

(H6): $\int_0^1 G(1,s)q(s)f_2(s,r_1)\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\int_0^1 h_2(s,r_1)\,dA(s) \le r_1$, *and*

$$r_2 \le \int_0^1 G(1,s)q(s)f_1(s,r_2)\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\int_0^1 h_1(s,r_2)\,dA(s) < \infty.$$

Then FBVP (1.1)–(1.3) has at least one positive solution $x^*(t)$ *that is strictly increasing,* $x^* \in K$, *and* $r_1 \le x^*(t) \le r_2$ *for* $0 \le t \le 1$.

P r o o f. We shall prove the existence of a positive solution of (1.1)–(1.3) if $(H5)$ holds. The proof if $(H6)$ holds is similar. By Lemma 2.6, $T : K \cap (\overline{K(r_2)} \setminus K(r_1)) \to K$ is completely continuous, where K is the cone in X defined in (2.9).

By $(H5)$, if $x \in \partial K(r_1)$, then $\|x\| = r_1$ and $t^{\alpha-1}r_1 \le x(t) \le r_1$ holds for $0 \le t \le 1$. Hence, by the definitions of $f_1(t,r_1)$ and $h_1(t,r_1)$, we have

$$f(t,x(t)) \ge f_1(t,r_1) \ \text{ for } \ t^{\alpha-1}r_1 \le x(t) \le r_1, \ \ 0 < t < 1$$

and

$$h(t,x(t)) \ge h_1(t,r_1) \ \text{ for } \ t^{\alpha-1}r_1 \le x(t) \le r_1, \ \ 0 < t < 1.$$

Then, from (2.7) and (2.10), we have

$$\begin{aligned}
\|Tx\| &= \max_{0\le t\le 1}\bigg|\int_0^1 G(t,s)q(s)f(s,x(s))\,ds \\
&\qquad + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}t^{\alpha-1}\int_0^1 h(s,x(s))\,dA(s)\bigg| \\
&\ge \max_{0\le t\le 1}\bigg|t^{\alpha-1}\int_0^1 G(1,s)q(s)f_1(s,r_1)\,ds \\
&\qquad + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}t^{\alpha-1}\int_0^1 h_1(s,r_1)\,dA(s)\bigg| \\
&\ge \int_0^1 G(1,s)q(s)f_1(s,r_1)\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\int_0^1 h_1(s,r_1)\,dA(s) \\
&\ge r_1 = \|x\|,
\end{aligned}$$

which implies that $\|Tx\| \ge \|x\|$ for $x \in K \cap \partial K(r_1)$.

Next, let $x \in K \cap \partial K(r_2)$. Then, $\|x\| = r_2$ with $t^{\alpha-1}r_2 \le x(t) \le r_2$ for $0 \le t \le 1$. Hence, by the definitions of $f_2(t,r_2)$ and $h_2(t,r_2)$, we have

$$f(t,x(t)) \le f_2(t,r_2), \ \text{ for } \ t^{\alpha-1}r_2 \le x(t) \le r_2, \ \ 0 < t < 1$$

and

$$h(t,x(t)) \le h_2(t,r_2), \ \text{ for } \ t^{\alpha-1}r_2 \le x(t) \le r_2, \ \ 0 < t < 1.$$

Consequently, from (2.10), we have

$$\begin{aligned}
\|Tx\| &= \max_{0\le t\le 1}\left|\int_0^1 G(t,s)q(s)f(s,x(s))\,ds \right.\\
&\quad \left. + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}t^{\alpha-1}\int_0^1 h(s,x(s))\,dA(s)\right|\\
&\le \int_0^1 G(1,s)q(s)f(s,x(s))\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\int_0^1 h(s,x(s)\,dA(s)\\
&\le \int G(1,s)q(s)f_2(s,r_2)\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\int_0^1 h_2(s,r_2)\,dA(s)\\
&\le r_2 = \|x\|,
\end{aligned}$$

that is, $\|Tx\| \le \|x\|$ holds for $x \in K \cap \partial K(r_2)$. By Theorem 2.3, T has a fixed point $x^*(t)$ in $K \cap (\overline{K(r_2)} \setminus K(r_1))$, and $r_1 \le \|x^*\| \le r_2$, which implies x^* is a positive solution of FBVP (1.1)–(1.3). Moreover,

$$\begin{aligned}
x'^*(t) = (Tx^*)'(t) = \int_0^1 \frac{\partial G(t,s)}{\partial t}q(s)f(s,x(s))ds\\
+ (\alpha-1)t^{\alpha-2}\int_0^1 h(s,x(s))dA(s) > 0
\end{aligned}$$

implies that $x^*(t)$ is an increasing solution of (1.1)–(1.3). This completes the proof of the theorem. □

Example 4.1. Consider the boundary value problem (1.1)–(1.3) with $\alpha = \frac{7}{2}$, $\beta = \frac{3}{2}$, $q(t) = \frac{1}{4\sqrt{1-t}}$, $f(t,x) = x^5 + \frac{1}{2x^{\frac{1}{3}}}$, $h(t,x) = t(x^5 + \frac{1}{2x^{\frac{1}{3}}})$, and

$$A(t) = \begin{cases} t, & \text{if } t \in [0,4/9) \cup [5/9,8/9),\\ 4/9, & \text{if } t \in [4/9,5/9),\\ 8/9, & \text{if } t \in [8/9,1]. \end{cases} \tag{4.5}$$

Clearly, $\int_0^1 dA(t) = \frac{8}{9}$, $\Gamma(\alpha-\beta) = \Gamma(2) = 1$, $\Gamma(\alpha) = \Gamma(\frac{7}{2}) = 3.3233$,

$$\begin{aligned}
f_1(t,r) = \frac{1}{t}h_1(t,r) &= \min\{x^5 + \frac{1}{2x^{\frac{1}{3}}} : t^{\frac{5}{2}}r \le x \le r\}\\
&\ge t^{\frac{25}{2}}r^5 + \frac{1}{2}r^{\frac{-1}{3}}
\end{aligned}$$

and

$$\begin{aligned}
f_2(t,r) = \frac{1}{t}h_2(t,r) &= \max\{x^5 + \frac{1}{2x^{\frac{1}{3}}} : t^{\frac{5}{2}}r \le x \le r\}\\
&\le r^5 + \frac{1}{2}t^{\frac{-5}{6}}r^{\frac{-1}{3}}.
\end{aligned}$$

Choose $r_1 = \frac{1}{100}$ and $r_2 = 1$. For this choice of r_1 and r_2, we can find $p_f(t) = r_2^5 + \frac{1}{2}t^{\frac{-5}{6}}r_1^{\frac{-1}{3}}$ and $p_h(t) = t(r_2^5 + \frac{1}{2}t^{\frac{-5}{6}}r_1^{\frac{-1}{3}})$ such that $(A3)$ holds. For $r_1 = \frac{1}{100}$, we have

$$\begin{aligned}
&\int_0^1 G(1,s)q(s)f_1(s,r_1)\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\int_0^1 h_1(s,r_1)\,dA(s)\\
&> \frac{1}{4\Gamma(\frac{7}{2})}\int_0^1 [(1-s)-(1-s)^{\frac{5}{2}}]\frac{1}{(1-s)^{\frac{1}{2}}}\left[\left(\frac{1}{100}\right)^5 s^{\frac{25}{2}} + \frac{1}{2}\left(\frac{1}{100}\right)^{\frac{-1}{3}}\right] ds\\
&> \frac{1}{100} = r_1.
\end{aligned}$$

Next, for $r_2 = 1$, we have

$$\begin{aligned}
&\int_0^1 G(1,s)q(s)f_2(s,r_2)\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\int_0^1 h_2(s,r_2)\,dA(s)\\
&\quad\le \frac{1}{4\Gamma(\frac{7}{2})}\int_0^1 [(1-s)-(1-s)^{\frac{5}{2}}]\frac{1}{(1-s)^{\frac{1}{2}}}\left(1+\frac{1}{2}s^{\frac{-5}{6}}\right) ds\\
&\qquad + \frac{3}{2\Gamma(\frac{7}{2})}\int_0^1 dA(s)\\
&\quad\le \frac{1}{4\Gamma(\frac{7}{2})}\int_0^1 (1-s)^{\frac{1}{2}}\left(1+\frac{1}{2}s^{\frac{-5}{6}}\right) ds + \frac{4}{3\Gamma(\frac{7}{2})}\\
&\quad= \frac{1}{4\Gamma(\frac{7}{2})}\left[\int_0^1 (1-s)^{\frac{1}{2}}\,ds + \frac{1}{2}\int_0^1 (1-s)^{\frac{1}{2}}s^{\frac{-5}{6}}ds\right] + \frac{4}{3\Gamma(\frac{7}{2})}\\
&\quad= \frac{1}{4\Gamma(\frac{7}{2})}\left[\frac{2}{3} + \frac{1}{2}\beta\left(\frac{1}{6},\frac{3}{2}\right)\right] + \frac{4}{3\Gamma(\frac{7}{2})}\\
&\quad= \frac{1}{4\Gamma(\frac{7}{2})}\left[\frac{2}{3} + \frac{1}{2}\frac{\Gamma(\frac{1}{6})\Gamma(\frac{3}{2})}{\Gamma(\frac{5}{3})}\right] + \frac{4}{3\Gamma(\frac{7}{2})}\\
&\quad= \frac{1}{4\Gamma(\frac{7}{2})}\left[\frac{2}{3} + \frac{1}{2}\frac{(5.566)(0.886)}{(0.9027)}\right] + \frac{4}{3\Gamma(\frac{7}{2})}\\
&\quad= 0.6568 < 1 = r_2.
\end{aligned}$$

Hence, by $(H5)$ of Theorem 4.5, the problem (1.1)–(1.3) has at least one positive increasing solution $x^*(t)$ with $\frac{1}{100} \le x^*(t) \le 1$ for $0 \le t \le 1$.

Consider the functions $g(s) = G(1,s)$, with $\int_{1/2}^1 g(s)\,ds > 0$, and $c(t) = t^{\alpha-1}$. Then by (2.7), we have

$$c(t)g(s) \le G(t,s) \le g(s) \text{ for } 0 \le t,\ s \le 1. \tag{4.6}$$

Since (4.6) is valid, we can take the subinterval $[\frac{1}{2}, 1] \subset [0, 1]$ for which the inequality

$$\mu G(1, s) \le G(t, s) \le G(1, s) \tag{4.7}$$

replaces (2.7), where

$$\mu = \frac{1}{2^{\alpha-1}} = \min_{t\in[\frac{1}{2},1]} c(t) = \min_{t\in[\frac{1}{2},1]} \frac{1}{t^{\alpha-1}}. \tag{4.8}$$

In this case, the operator T, defined in (2.10), maps the cone

$$N = \{x \in C[0, 1] : x(t) \ge 0,\ t \in [0, 1]\}$$

into the sub-cone P, where

$$P = \{x \in C[0, 1] : \min_{t\in[1/2,1]} x(t) \ge \mu\|x\|\}, \tag{4.9}$$

where μ is given in (4.8).

Now, we shall use Theorem 2.4 to find sufficient conditions for the existence of three positive solutions of the FBVP (1.1)–(1.3).

THEOREM 4.6. *Assume that there exist continuous functions f_1, h_1 : $[0, 1] \times [0, \infty) \to [0, \infty)$ such that (4.1) holds and that there exist constants c_1, c_2, c_3, and c_4 with*

$$0 < c_1 < c_2 < \frac{c_2}{\mu} = c_3 \le c_4$$

such that

- *(H7)*: $f(t, x) < f_1(t)c_1$ *and* $h(t, x) < h_1(t)c_1$ *for* $0 \le x(t) \le c_1$ *and* $0 \le t \le 1$;
- *(H8)*: $f(t, x) > \frac{c_2}{\mu\lambda}$ *and* $h(t, x) > \frac{c_2}{\mu\lambda}$ *for* $c_2 \le x(t) \le \frac{c_2}{\mu}$ *and* $1/2 \le t \le 1$;
- *(H9)*: $f(t, x) \le f_1(t)c_4$ *and* $h(t, x) \le h_1(t)c_4$ *for* $0 \le x(t) \le c_4$ *and* $0 \le t \le 1$.

Then the FBVP (1.1)–(1.3) has at least three positive solutions x_i with $\|x_i\| \le c_4$, $i = 1, 2, 3$.

P r o o f. Consider the cone P given in (4.9). We define a nonnegative continuous concave functional Φ on P by

$$\Phi(x) = \min_{t\in[1/2,1]} |x(t)|$$

so that $\Phi(x) \le \|x\|$. We consider two nonnegative continuous convex functionals ϕ and Θ on P given by

$$\Theta(x) = \phi(x) = \|x\|,$$

and a nonnegative continuous function ψ on P given by

$$\psi(x) = \|x\|.$$

Then,

$$\psi(kx) = \|kx\| \leq |k|\|x\| \leq |k|\psi(x) = k\psi(x), \ \ 0 \leq k \leq 1,$$
$$\Phi(x) = \min_{t\in[1/2,1]} |x(t)| \leq \|x\| = \psi(x),$$

and we can find $\overline{M} \geq 1$ such that

$$\|x\| = \phi(x) \leq \overline{M}\phi(x) \ \text{ for every } \ x \in \overline{P(\phi, c_4)}.$$

We consider the operator $T : P \to X$ defined by (2.10). It is clear that $x(t)$ is a solution of the FBVP (1.1)–(1.3) if and only if it is a fixed point of T in the cone P. Also, conditions $(A1)$ and $(A2)$ imply that $(Tx)(t) \geq 0$ for $t \in [0,1]$. Proceeding along the lines of the proof of Lemma 2.5, we can show that $T(P) \subset P$. If $x \in \overline{P(\phi, c_4)}$, then $\phi(x) = \|x\| \leq c_4$ for $0 \leq x \leq c_4$ and $0 \leq t \leq 1$. Then, by $(H7)$, we have

$$\begin{aligned}
\phi(Tx) = \|Tx\| &= \max_{t\in[0,1]} |(Tx)(t)| \\
&= \max_{t\in[0,1]} \left| \int_0^1 G(t,s)q(s)f(s,x(s))\,ds \right. \\
&\quad \left. + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} t^{\alpha-1} \int_0^1 h(s,x(s))\,dA(s) \right| \\
&\leq c_4 \left[\int_0^1 G(1,s)q(s)f_1(s)\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} \int_0^1 h_1(s)\,dA(s) \right] \leq c_4.
\end{aligned}$$

Hence, $T : \overline{P(\phi, c_4)} \to \overline{P(\phi, c_4)}$.

Next, we prove that $T : \overline{P(\phi, c_4)} \to \overline{P(\phi, c_4)}$ is completely continuous. From the continuity of $G(t,s)$, $f(t,x)$, and $h(t,x)$ for $(t,s) \in [0,1] \times [0,1]$, it follows that T is continuous on P. For the given $c_4 > 0$, we consider the set

$$\overline{P_{c_4}} = \{x \in P : \|x\| \leq c_4\}.$$

Setting

$$M_1 = \max_{\substack{t\in[0,1]\\ x\in[0,c_4]}} f(t,x) \text{ and } M_2 = \max_{\substack{t\in[0,1]\\ x\in[0,c_4]}} h(t,x),$$

we have

$$\begin{aligned}
|(Tx)(t)| &= \left| \int_0^1 G(t,s)q(s)f(s,x(s))\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} t^{\alpha-1} \int_0^1 h(s,x(s))\,dA(s) \right| \\
&\leq M_1 \int_0^1 G(1,s)q(s)\,ds + \frac{M_2\Gamma(\alpha-\beta)}{\Gamma(\alpha)} \int_0^1 dA(s),
\end{aligned}$$

which implies that T is uniformly bounded on $\overline{P_{c_4}}$. Since $G(t,s)$ is continuous on $[0,1] \times [0,1]$, it is uniformly continuous there, so for every $\epsilon > 0$, there exists $\delta \in \left(0, \frac{\epsilon}{\alpha-1}\right)$ such that $|G(t_1,s)-G(t_2,s)| < \epsilon$ for $|t_1-t_2| \le \delta$ and (t_1,s), $(t_2,s) \in [0,1]\times[0,1]$. Consequently, for any $x \in \overline{P_{c_4}}$ and t_1, $t_2 \in [0,1]$ with $|t_1-t_2| < \delta$, we have

$$\begin{aligned}
|(Tx)(t_1)-(Tx)(t_2)| &\le \int_0^1 |G(t_1,s)-G(t_2,s)|q(s)f(s,x(s))\,ds \\
&\quad + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}|t_1^{\alpha-1}-t_2^{\alpha-1}| \int_0^1 h(s,x(s))\,dA(s) \\
&\le \epsilon M_1 \int_0^1 q(s)\,ds \\
&\quad + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}|t_1-t_2| M_2 \left(\sum_{i=0}^{\alpha-2} t_1^{\alpha-i-2}t_2^i\right) \int_0^1 dA(s) \\
&\le \epsilon M_1 \int_0^1 q(s)\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} M_2 \delta(\alpha-1) \int_0^1 dA(s) \\
&\le \epsilon \left(M_1 \int_0^1 q(s)\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} M_2 \int_0^1 dA(s)\right).
\end{aligned}$$

Hence, $T(\overline{P_{c_4}})$ is equicontinuous, and so the set $T(\overline{P_{c_4}})$ is relatively compact. Thus, $T : \overline{P_{c_4}} \to \overline{P_{c_4}}$ is completely continuous by the Arzelà Ascoli theorem. Thus, for the convex function $\phi(x) = \|x\|$ on P, the mapping $T : \overline{P(\phi, c_4)} \to \overline{P(\phi, c_4)}$ is completely continuous.

Set $x_0(t) = \frac{c_2+c_3}{2} = \frac{c_2}{2\mu}(\mu+1)$ for any $t \in [0,1]$; then $x_0(t) > 0$. Moreover,

$$\Theta(x_0) = \|x_0\| = \frac{c_2+c_3}{2} = \frac{c_2}{2\mu}(\mu+1) < \frac{c_2}{\mu} = c_3,$$

$$\Phi(x_0) = \min_{t\in[1/2,1]} |x_0| = \frac{c_2+c_3}{2} > \frac{2c_2}{2} = c_2,$$

and

$$\phi(x_0) = \frac{c_2+c_3}{2} = \frac{c_2}{2\mu}(\mu+1) < \frac{c_2}{\mu} = c_3 \le c_4,$$

imply that the set $\{x \in P(\phi, \Theta, \Phi, c_2, c_3, c_4) : \Phi(x) > c_2\}$ is nonempty.

Now, we consider the interval $c_2 \le x(t) \le c_3 = \frac{c_2}{\mu}$ for $t \in [1/2, 1]$. Then we have

$$\begin{aligned}
\Phi(Tx) = \min_{t\in[1/2,1]} \Bigg[&\int_0^1 G(t,s)q(s)f(s,x(s))\,ds \\
&+ \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} t^{\alpha-1} \int_0^1 h(s,x(s))\,dA(s)\Bigg]
\end{aligned}$$

$$\geq \mu\left[\int_0^1 G(1,s)q(s)f(s,x(s))\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\int_0^1 h(s,x(s))\,dA(s)\right]$$
$$> \mu\left[\frac{c_2}{\mu\lambda}\int_0^1 G(t,s)q(s)\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\frac{c_2}{\mu\lambda}\int_0^1 dA(s)\right] = c_2$$

for $x \in P(\phi,\Theta,\Phi,c_2,c_3,c_4)$. Hence, condition $(S1)$ of Theorem 2.4 is satisfied.

Next, assume that $x \in P(\phi,\Phi,c_2,c_4)$ with $\Theta(Tx) > c_3$. Then we have

$$\Phi(Tx) = \min_{t\in[1/2,1]}(Tx)(t) \geq \mu\|Tx\| = \mu\Theta(Tx) > \mu c_3 = c_2,$$

which proves $(S2)$ holds.

Clearly $\phi(0) = 0 < c_1$ implies that $\phi \in R(\phi,\psi,c_1,c_4)$. Let $x \in \phi \in R(\phi,\psi,c_1,c_4)$ with $\psi(x) = \|x\| = c_1$. Then,

$$\psi(Tx) = \max_{t\in[0,1]}\left|\int_0^1 G(t,s)q(s)f(s,x(s))\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}t^{\alpha-1}\int_0^1 h(s,x(s))\,dA(s)\right|$$
$$< c_1\left[\int_0^1 G(1,s)q(s)f_1(s)\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\int_0^1 h_1(s)\,dA(s)\right] \leq c_1.$$

by (4.1). Hence, $(S3)$ is satisfied. Therefore, by Theorem 2.4, the FBVP (1.1)–(1.3) has at least three positive solutions x_1, x_2, and x_3 with $\|x_i\| \leq c_4$, $i = 1,2,3$. The location of the solutions x_i, $i = 1,2,3$, are $c_2 \leq \min_{t\in[1/2,1]} x_1(t)$, $c_3 \leq \|x_2\|$ with $\min_{t\in[1/2,1]} x_2(t) < c_3$, and $\|x_3\| < c_3$. The proof of the theorem is complete. □

Remark 4.2. The use of conditions $(H7)$ and $(H9)$ in Theorem 4.6 forces us to assume that (4.1) holds, which is required to prove $T : \overline{P(\phi,c_4)} \to \overline{P(\phi,c_4)}$ and $(S3)$ holds.

Theorem 4.7. *Assume that there is a constant $c_2 > 0$ such that $(H8)$ holds and there are continuous functions f_1 and $h_1 : [0,1]\times[0,\infty) \to [0,\infty)$ such that*

$(H10)$: $\limsup_{x\to\infty} \max_{0\leq t\leq 1} \frac{f(t,x)}{f_1(t)x} = 0$ *and* $\limsup_{x\to\infty} \max_{0\leq t\leq 1} \frac{h(t,x)}{h_1(t)x} = 0$

and

$(H11)$: $\limsup_{x\to 0} \max_{0\leq t\leq 1} \frac{f(t,x)}{f_1(t)x} = 0$ *and* $\limsup_{x\to 0} \max_{0\leq t\leq 1} \frac{h(t,x)}{h_1(t)x} = 0.$

Then the FBVP (1.1)–(1.3) has at least three positive solutions.

P r o o f. To prove this theorem, we shall obtain the constants c_4 and c_1 from the assumptions $(H10)$ and $(H11)$. Then the remainder of the proof is similar to the proof of Theorem 4.6.

By $(H10)$, there exist $\epsilon > 0$ with

$$0 < \epsilon < \frac{1}{\int_0^1 G(1,s)q(s)f_1(s)\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\int_0^1 h_1(s)\,dA(s)} \tag{4.10}$$

and $\eta > 0$ such that $f(t,x) \leq \epsilon f_1(t)x(t)$ and $h(t,x) \leq \epsilon h_1(t)x(t)$ for $x(t) \geq \eta$ and $0 \leq t \leq 1$. Set $M_f = \max_{\substack{0\leq x(t)\leq\eta \\ 0\leq t\leq 1}} f(t,x)$ and $M_h = \max_{\substack{0\leq x(t)\leq\eta \\ 0\leq t\leq 1}} h(t,x)$; then we have $f(t,x) \leq \epsilon f_1(t)x(t) + M_f$ and $h(t,x) \leq \epsilon h_1(t)x(t) + M_h$ for $x(t) \geq 0$ and $0 \leq t \leq 1$. Now, we choose a constant $c_4 > 0$ such that

$$c_4 \geq \max\left\{\frac{c_2}{\mu}, \frac{M_f\int_0^1 G(1,s)q(s)\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}M_h\int_0^1 \,dA(s)}{1-\epsilon\left[\int_0^1 G(1,s)q(s)f_1(s)\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\int_0^1 h_1(s)\,dA(s)\right]}\right\}.$$

Consider the non-negative continuous convex functional ϕ on the cone P defined by $\phi(x) = \|x\|$. Then, for $x \in \overline{P(\phi, c_4)}$, we have

$$\begin{aligned}
\phi(Tx) = \|Tx\| &= \max_{t\in[0,1]} |(Tx)(t)| \\
&= \max_{t\in[0,1]}\left|\int_0^1 G(t,s)q(s)f(s,x(s))\,ds \right.\\
&\quad \left. + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}t^{\alpha-1}\int_0^1 h(s,x(s))\,dA(s)\right| \\
&\leq \int_0^1 G(1,s)q(s)(\epsilon f_1(s)x(s) + M_f)\,ds \\
&\quad + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\int_0^1 (\epsilon h_1(s)x(s) + M_h)\,dA(s) \\
&\leq \int_0^1 G(1,s)q(s)(\epsilon f_1(s)\|x\| + M_f)\,ds \\
&\quad + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\int_0^1 (\epsilon h_1(s)\|x\| + M_h)\,dA(s) \\
&\leq \epsilon c_4\left[\int_0^1 G(1,s)q(s)f_1(s)\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\int_0^1 h_1(s)\,dA(s)\right] \\
&\quad + M_f\int_0^1 G(1,s)q(s)\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}M_h\int_0^1 \,dA(s) \leq c_4.
\end{aligned}$$

Hence, $T : \overline{P(\phi, c_4)} \to \overline{P(\phi, c_4)}$. By choosing the functions Φ, Θ, and ψ as in Theorem 4.6, and using $(H8)$, we can prove that conditions $(S1)$ and $(S2)$ of Theorem 2.4 are satisfied. To complete the proof of the theorem, it is sufficient to find a constant c_1, which we can obtain from condition $(H11)$. Indeed, by $(H11)$, there exist constants $\epsilon > 0$ satisfying (4.10) and c_1 with $0 < c_1 < c_2$ such that $f(t,x) \leq \epsilon f_1(t)x$ and $h(t,x) \leq \epsilon h_1(t)x$ for $0 \leq x \leq c_1$ and $0 \leq t \leq 1$. Hence, for $0 \leq x \leq c_1$ and the continuous functional $\psi(x) = \|x\|$ on the cone P, we have

$$\begin{aligned}\psi(Tx) = \max_{t\in[0,1]} \Bigg| &\int_0^1 G(t,s)q(s)f(s,x(s))\,ds \\ &+ \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} t^{\alpha-1} \int_0^1 h(s,x(s))\,dA(s)\Bigg| \\ &\leq \epsilon \left[\int_0^1 G(1,s)q(s)f_1(s)\|x\|\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} \int_0^1 h_1(s)\|x\|\,dA(s)\right] \\ &\leq \epsilon c_1 \left[\int_0^1 G(1,s)q(s)f_1(s)\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} \int_0^1 h_1(s)\,dA(s)\right] < c_1.\end{aligned}$$

This completes the proof of the theorem. □

Example 4.2. Consider the boundary value problem (1.1)–(1.3) with $\alpha = \frac{7}{2}$, $\beta = \frac{3}{2}$, $f(t,x) = h(t,x) = e^{29}x^2e^{-x}$, $A(t)$ as in (4.5), and $q(t) = \frac{1}{(1-t)-(1-t)^{5/2}}$, $0 < t < 1$. Hence, $\lambda = 1.26747$ and $\mu = 2^{-5/2} < 1$. Assuming $f_1(t) = h_1(t) = 1$, we see that $(H10)$ and $(H11)$ are satisfied. Also, condition $(H8)$ is satisfied with $c_2 = 5$. Hence, by Theorem 4.7, this problem has at least three positive solutions.

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