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RESEARCH PAPER

NOETHER SYMMETRY AND CONSERVED QUANTITY FOR FRACTIONAL BIRKHOFFIAN MECHANICS AND ITS APPLICATIONS

Chuan-Jing Song [1,2], **Yi Zhang** [3*]

Abstract

Noether theorem is an important aspect to study in dynamical systems. Noether symmetry and conserved quantity for the fractional Birkhoffian system are investigated. Firstly, fractional Pfaff actions and fractional Birkhoff equations in terms of combined Riemann-Liouville derivative, Riesz-Riemann-Liouville derivative, combined Caputo derivative and Riesz-Caputo derivative are reviewed. Secondly, the criteria of Noether symmetry within combined Riemann-Liouville derivative, Riesz-Riemann-Liouville derivative, combined Caputo derivative and Riesz-Caputo derivative are presented for the fractional Birkhoffian system, respectively. Thirdly, four corresponding conserved quantities are obtained. The classical Noether identity and conserved quantity are special cases of this paper. Finally, four fractional models, such as the fractional Whittaker model, the fractional Lotka biochemical oscillator model, the fractional Hénon-Heiles model and the fractional Hojman-Urrutia model are discussed as examples to illustrate the results.

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Key Words and Phrases: Noether symmetry, conserved quantity, fractional Birkhoffian system, Riemann-Liouville derivative, Caputo derivative

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1. **Introduction**

The Riemann-Liouville derivative and Caputo derivative are two popular versions of the derivatives of fractional calculus (FC). The Riemann-Liouville derivative sounds some more convenient for theoretical studies in FC, however, its extraordinary singularity is limiting its applications in engineering and physical modeling. A fractional derivative with weak singularity, the so-called Caputo derivative, was introduced by Caputo in 1960s. This derivative can successfully be used for solving fractional initial value problems with more natural initial conditions and has been applied widely in the modeling process of many practical problems. Anyway, both of these definitions have their irreplaceable advantages.

In fact, fractional order dynamical systems can better describe phenomena from the engineering practice and can more truly reveal the natural world. Therefore, the fractional dynamics has made great progress in theories and applications, such as the fractional Lagrangian mechanics, the fractional Hamiltonian mechanics, the fractional generalized Hamiltonian mechanics, the fractional Birkhoffian mechanics and the fractional dynamics of the nonholonomic system [2, 25, 26, 28, 30, 31, 35, 36, 37, 38, 40, 42, 46], as well as their applications, see for example, [5, 7, 8, 9, 11, 23, 41].

The Noether symmetry and conserved quantity for the fractional dynamical system were first introduced by Frederico and Torres in 2007 by introducing a new fractional order operator [15]. Using this operator, some results have been obtained [13, 16, 17, 18, 19, 24, 29, 48]. However, Ferreira and Malinowska expressed their doubts about the validity of the Noether theorem got through this fractional operator, and they presented a counterexample to express their ideas, [12]. At the same time, they recommended Atanacković's definition of fractional conserved quantity. Actually, in 2009, based on the classical definition of the conserved quantity, Atanacković [4] gave a definition of fractional conserved quantity, which is different from the definition in [15]. Based on this definition, fractional conserved quantity for the non-conservative system [14], the Hamiltonian system [45], the generalized Hamiltonian system [27] and the Birkhoffian system [43, 44, 47] were achieved.

As we all know, Birkhoffian mechanics is considered as an important direction of modern analytical mechanics to study [20], and it is more general than Newtonian mechanics, Lagrangian mechanics and Hamiltonian mechanics, as well as general holonomic and nonholonomic mechanics [33, 34]. Besides, Birkhoffian mechanics can also apply to statistical mechanics, quantum mechanics, biological physics, hadron physics, atomic and molecular physics and so forth [39]. In order to better solve the fractional dimension problems in science and engineering, and further explore the internal properties and dynamical behaviors of fractional dynamical systems, we intend to study the fractional Noether symmetry and conserved quantity on the basis of Atanacković's definition for the fractional Birkhoffian mechanics, which was established by Luo [28] in 2014.

This paper is organized as follows. First, the some basics of the fractional Birkhoffian mechanics are reviewed. Secondly, the Noether theory for the fractional Birkhoffian system with combined Riemann-Liouville derivative, Riesz-Riemann-Liouville derivative, combined Caputo derivative and Riesz-Caputo derivative is established, respectively. And finally, some four applications of the results are presented.

2. **Fractional derivatives and their properties**

In this section, we list some definitions of fractional derivatives and their properties, including the Riemann-Liouville derivative, Caputo derivative, Riesz-Riemann-Liouville derivative and Riesz-Caputo derivative [1, 28].

Let $f(t)$ be continuous and integrable, then the left and the right Riemann-Liouville derivatives are

$$ {}_{t_1}^{\mathrm{RL}}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)}(\frac{\mathrm{d}}{\mathrm{d}t})^n \int_{t_1}^{t} (t-\xi)^{n-\alpha-1} f(\xi)\mathrm{d}\xi, \tag{2.1} $$

$$ {}_{t}^{\mathrm{RL}}D_{t_2}^{\beta} f(t) = \frac{1}{\Gamma(n-\beta)}(-\frac{\mathrm{d}}{\mathrm{d}t})^n \int_{t}^{t_2} (\xi-t)^{n-\beta-1} f(\xi)\mathrm{d}\xi, \tag{2.2} $$

the left and the right Caputo derivatives are

$$ {}_{t_1}^{\mathrm{C}}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_1}^{t} (t-\xi)^{n-\alpha-1} (\frac{\mathrm{d}}{\mathrm{d}\xi})^n f(\xi)\mathrm{d}\xi, \tag{2.3} $$

$$ {}_{t}^{\mathrm{C}}D_{t_2}^{\beta} f(t) = \frac{1}{\Gamma(n-\beta)} \int_{t}^{t_2} (\xi-t)^{n-\beta-1} (-\frac{\mathrm{d}}{\mathrm{d}\xi})^n f(\xi)\mathrm{d}\xi, \tag{2.4} $$

the Riesz-Riemann-Liouville derivative and the Riesz-Caputo derivative are

$$ {}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha} f(t) = \frac{1}{2\Gamma(n-\alpha)}(\frac{\mathrm{d}}{\mathrm{d}t})^n \int_{t_1}^{t_2} |t-\xi|^{n-\alpha-1} f(\xi)\mathrm{d}\xi, \tag{2.5} $$

$$ {}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha} f(t) = \frac{1}{2\Gamma(n-\alpha)} \int_{t_1}^{t_2} |t-\xi|^{n-\alpha-1} (\frac{\mathrm{d}}{\mathrm{d}\xi})^n f(\xi)\mathrm{d}\xi, \tag{2.6} $$

where $\Gamma(*)$ is the Gamma function, α and β are orders of fractional derivatives with $n-1 \leq \alpha, \beta < n$. When α, β are integers, the usual integer definition of a derivative is used. In addition, the Riemann-Liouville derivative of a constant is not zero while the Caputo derivative of a constant is zero, and for $\alpha = \beta = 1$, the left derivative is the negative of the right derivative.

The combined Riemann-Liouville derivative and the combined Caputo derivative are

$$ {}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}f(t) = \gamma\,{}_{t_1}^{\mathrm{RL}}D_t^{\alpha}f(t) + (-1)^n(1-\gamma)\,{}_t^{\mathrm{RL}}D_{t_2}^{\beta}f(t), \tag{2.7} $$

$$ {}^{\mathrm{C}}D_{\gamma}^{\alpha,\beta}f(t) = \gamma\,{}_{t_1}^{\mathrm{C}}D_t^{\alpha}f(t) + (-1)^n(1-\gamma)\,{}_t^{\mathrm{C}}D_{t_2}^{\beta}f(t), \tag{2.8} $$

where ${}_{t_1}D_t^{\alpha}$ and ${}_tD_{t_2}^{\beta}$ can help for dealing with dynamical systems exhibiting the arrow of time, γ determines the different quantity of information from the past and the future to keep track of the past and future of the dynamics.

From formulae (2.7) and (2.8), we have

$$ {}^{\mathrm{RL}}D_{1/2}^{\alpha,\alpha}f(t) = \frac{1}{2}[{}_{t_1}^{\mathrm{RL}}D_t^{\alpha}f(t) + (-1)^n\,{}_t^{\mathrm{RL}}D_{t_2}^{\alpha}f(t)] = {}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}f(t), \tag{2.9} $$

$$ {}^{\mathrm{C}}D_{1/2}^{\alpha,\alpha}f(t) = \frac{1}{2}[{}_{t_1}^{\mathrm{C}}D_t^{\alpha}f(t) + (-1)^n\,{}_t^{\mathrm{C}}D_{t_2}^{\alpha}f(t)] = {}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha}f(t), \tag{2.10} $$

i.e., the Riesz-Riemann-Liouville derivative and the Riesz-Caputo derivative can be deduced from the combined Riemann-Liouville derivative (2.7) and the combined Caputo derivative (2.8) by setting $\beta = \alpha$, $\gamma = \frac{1}{2}$, respectively.

What is more, when $\alpha, \beta \to 1$, we have

$$ {}_{t_1}D_t^1 = \mathrm{d}/\mathrm{d}t,\ {}_tD_{t_2}^1 = -\mathrm{d}/\mathrm{d}t,\ D_{\gamma}^{\alpha,\beta} = \gamma\,{}_{t_1}D_t^1 + (-1)^n(1-\gamma)\,{}_tD_{t_2}^1 = \mathrm{d}/\mathrm{d}t. \tag{2.11} $$

3. Fractional Birkhoff equations

In this section, fractional Pfaff actions and fractional Birkhoff equations with different fractional derivatives are listed, see [28].

Assume that a mechanical system is determined by $2n$ Birkhoffian variables a^{μ}, the Birkhoffian is $B = B(t, a^{\mu})$, and the Birkhoff functions are $R_{\nu} = R_{\nu}(t, a^{\mu})$, $\mu, \nu = 1, 2, \cdots, 2n$.

A unified fractional Pfaff action is defined as

$$ A = \int_{t_1}^{t_2} (R_{\nu}D_{\gamma}^{\alpha,\beta}a^{\nu} - B)\mathrm{d}t. \tag{3.1} $$

Case A: Based on the combined Riemann-Liouville derivative, fractional Pfaff action is expressed as

$$ A_{\mathrm{RL}} = \int_{t_1}^{t_2} (R_{\nu}\,{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}a^{\nu} - B)\mathrm{d}t. \tag{3.2} $$

When $0 < \alpha, \beta < 1$, fractional Birkhoff equations in terms of combined Riemann-Liouville derivative have the form

$$ \frac{\partial R_{\nu}}{\partial a^{\mu}}\,{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}a^{\nu} - {}^{\mathrm{C}}D_{1-\gamma}^{\beta,\alpha}R_{\mu} - \frac{\partial B}{\partial a^{\mu}} = 0, \quad \mu, \nu = 1, 2, \cdots, 2n. \tag{3.3} $$

Case B: Based on the Riesz-Riemann-Liouville derivative, fractional Pfaff action is expressed as

$$A_{\mathrm{R}} = \int_{t_1}^{t_2} (R_\nu {}^{\mathrm{R}}_{t_1}D^{\alpha}_{t_2} a^\nu - B)\mathrm{d}t. \tag{3.4}$$

When $0 < \alpha < 1$, fractional Birkhoff equations in terms of Riesz-Riemann-Liouville derivative have the form

$$\frac{\partial R_\nu}{\partial a^\mu} {}^{\mathrm{R}}_{t_1}D^{\alpha}_{t_2} a^\nu - {}^{\mathrm{RC}}_{t_1}D^{\alpha}_{t_2} R_\mu - \frac{\partial B}{\partial a^\mu} = 0, \quad \mu, \nu = 1, 2, \cdots, 2n. \tag{3.5}$$

Equations (3.5) can also be obtained from equations (3.3) when $\beta = \alpha$, $\gamma = \frac{1}{2}$.

Case C: Based on the combined Caputo derivative, fractional Pfaff action is expressed as

$$A_{\mathrm{C}} = \int_{t_1}^{t_2} (R_\nu {}^{\mathrm{C}}D^{\alpha,\beta}_{\gamma} a^\nu - B)\mathrm{d}t. \tag{3.6}$$

When $0 < \alpha, \beta < 1$, fractional Birkhoff equations in terms of combined Caputo derivative have the form

$$\frac{\partial R_\nu}{\partial a^\mu} {}^{\mathrm{C}}D^{\alpha,\beta}_{\gamma} a^\nu - {}^{\mathrm{RL}}D^{\beta,\alpha}_{1-\gamma} R_\mu - \frac{\partial B}{\partial a^\mu} = 0, \quad \mu, \nu = 1, 2, \cdots, 2n. \tag{3.7}$$

Case D: Based on the Riesz-Caputo derivative, fractional Pfaff action is expressed as

$$A_{\mathrm{RC}} = \int_{t_1}^{t_2} (R_\nu {}^{\mathrm{RC}}_{t_1}D^{\alpha}_{t_2} a^\nu - B)\mathrm{d}t. \tag{3.8}$$

When $0 < \alpha < 1$, fractional Birkhoff equations in terms of Riesz-Caputo derivative have the form

$$\frac{\partial R_\nu}{\partial a^\mu} {}^{\mathrm{RC}}_{t_1}D^{\alpha}_{t_2} a^\nu - {}^{\mathrm{R}}_{t_1}D^{\alpha}_{t_2} R_\mu - \frac{\partial B}{\partial a^\mu} = 0, \quad \mu, \nu = 1, 2, \cdots, 2n. \tag{3.9}$$

Equations (3.9) can also be obtained from equations (3.7) when $\beta = \alpha$, $\gamma = \frac{1}{2}$.

When $\alpha, \beta \to 1$, all Eqs. (3.3) (3.5) (3.7) and (3.9) reduce to the classical Birkhoff equations [34]

$$\left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu - \frac{\partial R_\mu}{\partial t} - \frac{\partial B}{\partial a^\mu} = 0, \quad \mu, \nu = 1, 2, \cdots, 2n. \tag{3.10}$$

4. Noether symmetry for fractional Birkhoffian machanics

In this section, Noether symmetry based on different fractional derivatives is investigated under the condition of $0 < \alpha, \beta < 1$.

Considering the infinitesimal transformations

$$\bar{t} = t + \triangle t, \quad \bar{a}^\nu(\bar{t}) = a^\nu(t) + \triangle a^\nu, \tag{4.1}$$

whose expanding forms are

$$\bar{t} = t + \varepsilon \xi_0(t, a^\mu), \quad \bar{a}^\nu(\bar{t}) = a^\nu(t) + \varepsilon \xi_\nu(t, a^\mu), \tag{4.2}$$

where ε is an infinitesimal parameter, ξ_0 and ξ_ν are infinitesimal generators.

Under the infinitesimal transformations (4.1), the unified fractional Pfaff action becomes

$$\bar{A}=\int_{\bar{t}_1}^{\bar{t}_2}[R_\nu(\bar{t},\bar{a}^\mu)D_\gamma^{\alpha,\beta}\bar{a}^\nu-B(\bar{t},\bar{a}^\mu)]\mathrm{d}\bar{t}. \tag{4.3}$$

Denoting the main linear part which accurates to the first-order infinitesimal of $\bar{A}-A$ as $\triangle A$.

Definition 4.1. If the condition $\triangle A=0$ holds, then the infinitesimal transformations (4.1) are called Noether symmetric transformations.

The Noether symmetry can be verified from the Noether symmetric transformations.

4.1. **Noether symmetry in terms of combined Riemann-Liouville derivative.** Under the infinitesimal transformations (4.1), the fractional Pfaff action in terms of combined Riemann-Liouville derivative becomes

$$\begin{aligned}\bar{A}_{\mathrm{RL}}=\textstyle\int_{\bar{t}_1}^{\bar{t}_2}[R_\nu(\bar{t},\bar{a}^\mu)\gamma{}_{\bar{t}_1}^{\mathrm{RL}}D_{\bar{t}}^{\alpha}\bar{a}^\nu(\bar{t})\\ +R_\nu(\bar{t},\bar{a}^\mu)(-1)^n(1-\gamma){}_{\bar{t}}^{\mathrm{RL}}D_{\bar{t}_2}^{\beta}\bar{a}^\nu(\bar{t})-B(\bar{t},\bar{a}^\mu)]\mathrm{d}\bar{t}.\end{aligned} \tag{4.4}$$

In fact,

$$\begin{aligned}{}_{\bar{t}_1}^{\mathrm{RL}}D_{\bar{t}}^{\alpha}\bar{a}^\nu(\bar{t})={}_{t_1}^{\mathrm{RL}}D_t^{\alpha}a^\nu+{}_{t_1}^{\mathrm{RL}}D_t^{\alpha}\delta a^\nu+\triangle t\tfrac{\mathrm{d}}{\mathrm{d}t}{}_{t_1}^{\mathrm{RL}}D_t^{\alpha}a^\nu\\ -\tfrac{1}{\Gamma(1-\alpha)}\tfrac{\mathrm{d}}{\mathrm{d}t}[(t-t_1)^{-\alpha}a^\nu(t_1)\triangle t_1],\end{aligned} \tag{4.5}$$

$$\begin{aligned}{}_{\bar{t}}^{\mathrm{RL}}D_{\bar{t}_2}^{\beta}\bar{a}^\nu(\bar{t})={}_t^{\mathrm{RL}}D_{t_2}^{\beta}a^\nu+{}_t^{\mathrm{RL}}D_{t_2}^{\beta}\delta a^\nu\\ +\triangle t\tfrac{\mathrm{d}}{\mathrm{d}t}{}_t^{\mathrm{RL}}D_{t_2}^{\beta}a^\nu-\tfrac{1}{\Gamma(1-\beta)}\tfrac{\mathrm{d}}{\mathrm{d}t}[(t_2-t)^{-\beta}a^\nu(t_2)\triangle t_2].\end{aligned} \tag{4.6}$$

Then using formulae (4.5) and (4.7), we get

$$\begin{aligned}\triangle A_{\mathrm{RL}}&=\int_{t_1}^{t_2}\{R_\nu(t+\triangle t,a^\mu+\triangle a^\mu)\gamma[{}_{t_1}^{\mathrm{RL}}D_t^{\alpha}a^\nu+{}_{t_1}^{\mathrm{RL}}D_t^{\alpha}\delta a^\nu\\ &+\quad\triangle t\frac{\mathrm{d}}{\mathrm{d}t}{}_{t_1}^{\mathrm{RL}}D_t^{\alpha}a^\nu-\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}(t-t_1)^{-\alpha}a^\nu(t_1)\triangle t_1]+(-1)^n(1-\gamma)\\ &\times\quad R_\nu(t+\triangle t,a^\mu+\triangle a^\mu)[{}_t^{\mathrm{RL}}D_{t_2}^{\beta}a^\nu+{}_t^{\mathrm{RL}}D_{t_2}^{\beta}\delta a^\nu+\triangle t\frac{\mathrm{d}}{\mathrm{d}t}{}_t^{\mathrm{RL}}D_{t_2}^{\beta}a^\nu\\ &-\quad\frac{1}{\Gamma(1-\beta)}\frac{\mathrm{d}}{\mathrm{d}t}(t_2-t)^{-\beta}a^\nu(t_2)\triangle t_2]-B(t+\triangle t,a^\mu+\triangle a^\mu)\}\\ &\times\quad(1+\frac{\mathrm{d}}{\mathrm{d}t}\triangle t)\mathrm{d}t-A_{\mathrm{RL}}\end{aligned}$$

$$
\begin{aligned}
= & \int_{t_1}^{t_2}\{R_\nu{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}\delta a^\nu+(R_\nu\frac{\mathrm{d}}{\mathrm{d}t}{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu+\frac{\partial R_\nu}{\partial t}{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu-\frac{\partial B}{\partial t})\triangle t \\
+ & (\frac{\partial R_\nu}{\partial a^\mu}{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu-\frac{\partial B}{\partial a^\mu})\triangle a^\mu-\gamma[R_\nu\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}(t-t_1)^{-\alpha}a^\nu(t_1)\triangle t_1] \\
+ & (R_\nu{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu-B)\frac{\mathrm{d}}{\mathrm{d}t}\triangle t+(1-\gamma)[R_\nu\frac{1}{\Gamma(1-\beta)}\frac{\mathrm{d}}{\mathrm{d}t}(t_2-t)^{-\beta} \\
\times & a^\nu(t_2)\triangle t_2]\}\mathrm{d}t. \qquad (4.7)
\end{aligned}
$$

Let $\triangle A_{\mathrm{RL}}=0$, we get

$$
\begin{aligned}
&R_\nu{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}(\xi_\nu-\dot a^\nu\xi_0)+(R_\nu\tfrac{\mathrm{d}}{\mathrm{d}t}{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu+\tfrac{\partial R_\nu}{\partial t}{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu-\tfrac{\partial B}{\partial t})\xi_0 \\
&+(\tfrac{\partial R_\nu}{\partial a^\mu}{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu-\tfrac{\partial B}{\partial a^\mu})\xi_\mu-\gamma[R_\nu\tfrac{1}{\Gamma(1-\alpha)}\tfrac{\mathrm{d}}{\mathrm{d}t}(t-t_1)^{-\alpha}a^\nu(t_1) \\
&\times\xi_0(t_1,a^\mu(t_1))]+(R_\nu{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu-B)\dot\xi_0+(1-\gamma)[R_\nu\tfrac{1}{\Gamma(1-\beta)}\tfrac{\mathrm{d}}{\mathrm{d}t}(t_2-t)^{-\beta} \\
&\times a^\nu(t_2)\xi_0(t_2,a^\mu(t_2))]=0. \qquad (4.8)
\end{aligned}
$$

Therefore, we have the following

CRITERION 4.1. *For the fractional Birkhoffian system* (3.3), *if the infinitesimal generators* ξ_0 *and* ξ_ν *satisfy formula* (4.8), *then the infinitesimal transformations* (4.2) *are Noether symmetric transformations.*

Formula (4.8) is the Noether identity for the fractional Birkhoffian system in terms of combined Riemann-Liouville derivative.

4.2. **Noether symmetry in terms of Riesz-Riemann-Liouville derivative.** Under the infinitesimal transformations (4.1), the fractional Pfaff action in terms of Riesz-Riemann-Liouville derivative becomes

$$
\bar A_{\mathrm{R}}=\int_{\bar t_1}^{\bar t_2}[R_\nu(\bar t,\bar a^\mu)\,{}_{\bar t_1}^{\mathrm{R}}D_{\bar t_2}^{\alpha}\bar a^\nu(\bar t)-B(\bar t,\bar a^\mu)]\mathrm{d}\bar t. \qquad (4.9)
$$

Since

$$
\begin{aligned}
&{}_{\bar t_1}^{\mathrm{R}}D_{\bar t_2}^{\alpha}\bar a^\nu(\bar t)={}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^\nu+{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}\delta a^\nu+\triangle t\tfrac{\mathrm{d}}{\mathrm{d}t}{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^\nu \\
&+\tfrac{1}{2\Gamma(1-\alpha)}\tfrac{\mathrm{d}}{\mathrm{d}t}|t-t_2|^{-\alpha}a^\nu(t_2)\triangle t_2-\tfrac{1}{2\Gamma(1-\alpha)}\tfrac{\mathrm{d}}{\mathrm{d}t}|t-t_1|^{-\alpha}a^\nu(t_1)\triangle t_1, \qquad (4.10)
\end{aligned}
$$

it is obtained from $\triangle A_{\mathrm{R}}=0$ that

$$
\begin{aligned}
&R_\nu{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}(\xi_\nu-\dot a^\nu\xi_0)+(R_\nu\tfrac{\mathrm{d}}{\mathrm{d}t}{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^\nu+\tfrac{\partial R_\nu}{\partial t}{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^\nu-\tfrac{\partial B}{\partial t})\xi_0+(\tfrac{\partial R_\nu}{\partial a^\mu} \\
&\times{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^\nu-\tfrac{\partial B}{\partial a^\mu})\xi_\mu+\tfrac{R_\nu}{2\Gamma(1-\alpha)}\tfrac{\mathrm{d}}{\mathrm{d}t}|t-t_2|^{-\alpha}a^\nu(t_2)\xi_0(t_2,a^\mu(t_2))+(R_\nu \\
&\times{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^\nu-B)\dot\xi_0-\tfrac{R_\nu}{2\Gamma(1-\alpha)}\tfrac{\mathrm{d}}{\mathrm{d}t}|t-t_1|^{-\alpha}a^\nu(t_1)\xi_0(t_1,a^\mu(t_1))=0. \qquad (4.11)
\end{aligned}
$$

Therefore, we have the following

Criterion 4.2. *For the fractional Birkhoffian system* (3.5)*, if the infinitesimal generators* ξ_0 *and* ξ_ν *satisfy formula* (4.11)*, then the infinitesimal transformations* (4.2) *are Noether symmetric transformations.*

Formula (4.11) is the Noether identity for the fractional Birkhoffian system in terms of Riesz-Riemann-Liouville derivative.

Formula (4.11) can also be achieved from formula (4.8) by setting $\beta = \alpha$, $\gamma = \frac{1}{2}$.

4.3. **Noether symmetry in terms of combined Caputo derivative.** Under the infinitesimal transformations (4.1), the fractional Pfaff action in terms of combined Caputo derivative becomes

$$\bar{A}_{\mathrm{C}} = \int_{\bar{t}_1}^{\bar{t}_2}[R_\nu(\bar{t},\bar{a}^\mu)\gamma{}_{\bar{t}_1}^{\mathrm{C}}D_{\bar{t}}^{\alpha}\bar{a}^\nu(\bar{t}) + R_\nu(\bar{t},\bar{a}^\mu)(-1)^n(1-\gamma){}_{\bar{t}}^{\mathrm{C}}D_{\bar{t}_2}^{\beta}\bar{a}^\nu(\bar{t}) - B(\bar{t},\bar{a}^\mu)]\mathrm{d}\bar{t}. \tag{4.12}$$

Using

$${}_{\bar{t}_1}^{\mathrm{C}}D_{\bar{t}}^{\alpha}\bar{a}^\nu(\bar{t}) = {}_{t_1}^{\mathrm{C}}D_{t}^{\alpha}a^\nu + {}_{t_1}^{\mathrm{C}}D_{t}^{\alpha}\delta a^\nu + \triangle t\frac{\mathrm{d}}{\mathrm{d}t}{}_{t_1}^{\mathrm{C}}D_{t}^{\alpha}a^\nu - \frac{1}{\Gamma(1-\alpha)}(t-t_1)^{-\alpha}\dot{a}^\nu(t_1)\triangle t_1, \tag{4.13}$$

$${}_{\bar{t}}^{\mathrm{C}}D_{\bar{t}_2}^{\beta}\bar{a}^\nu(\bar{t}) = {}_{t}^{\mathrm{C}}D_{t_2}^{\beta}a^\nu + {}_{t}^{\mathrm{C}}D_{t_2}^{\beta}\delta a^\nu + \triangle t\frac{\mathrm{d}}{\mathrm{d}t}{}_{t}^{\mathrm{C}}D_{t_2}^{\beta}a^\nu - \frac{1}{\Gamma(1-\beta)}(t_2-t)^{-\beta}\dot{a}^\nu(t_2)\triangle t_2, \tag{4.14}$$

from Definition 4.1 of the Noether symmetric transformations, we get

$$R_\nu{}^{\mathrm{C}}D_\gamma^{\alpha,\beta}(\xi_\nu - \dot{a}^\nu\xi_0) + (R_\nu\frac{\mathrm{d}}{\mathrm{d}t}{}^{\mathrm{C}}D_\gamma^{\alpha,\beta}a^\nu + \frac{\partial R_\nu}{\partial t}{}^{\mathrm{C}}D_\gamma^{\alpha,\beta}a^\nu - \frac{\partial B}{\partial t})\xi_0 + (\frac{\partial R_\nu}{\partial a^\mu} \times{}^{\mathrm{C}}D_\gamma^{\alpha,\beta}a^\nu - \frac{\partial B}{\partial a^\mu})\xi_\mu - R_\nu\frac{\gamma}{\Gamma(1-\alpha)}(t-t_1)^{-\alpha}\dot{a}^\nu(t_1)\xi_0(t_1,a^\mu(t_1)) + (R_\nu \times{}^{\mathrm{C}}D_\gamma^{\alpha,\beta}a^\nu - B)\dot{\xi}_0 + R_\nu\frac{1-\gamma}{\Gamma(1-\beta)}(t_2-t)^{-\beta}\dot{a}^\nu(t_2)\xi_0(t_2,a^\mu(t_2)) = 0. \tag{4.15}$$

Therefore, we have

Criterion 4.3. *For the fractional Birkhoffian system* (3.7)*, if the infinitesimal generators* ξ_0 *and* ξ_ν *satisfy formula* (4.15)*, then the infinitesimal transformations* (4.2) *are Noether symmetric transformations.*

Formula (4.15) is the Noether identity for the fractional Birkhoffian system in terms of combined Caputo derivative.

4.4. **Noether symmetry in terms of Riesz-Caputo derivative.** Under the infinitesimal transformations (4.1), the fractional Pfaff action in terms of Riesz-Caputo derivative becomes

$$\bar{A}_{\mathrm{RC}} = \int_{\bar{t}_1}^{\bar{t}_2}[R_\nu(\bar{t},\bar{a}^\mu){}_{\bar{t}_1}^{\mathrm{RC}}D_{\bar{t}_2}^{\alpha}\bar{a}^\nu(\bar{t}) - B(\bar{t},\bar{a}^\mu)]\mathrm{d}\bar{t}. \tag{4.16}$$

Since

$$
{}_{\bar{t}_1}^{\mathrm{RC}}D_{\bar{t}_2}^{\alpha}\bar{a}^{\nu}(\bar{t}) = {}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha}a^{\nu} + {}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha}\delta a^{\nu} + \triangle t\tfrac{\mathrm{d}}{\mathrm{d}t}{}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha}a^{\nu}
+\tfrac{1}{2\Gamma(1-\alpha)}[|t-t_2|^{-\alpha}\dot{a}^{\nu}(t_2)\triangle t_2 - |t-t_1|^{-\alpha}\dot{a}^{\nu}(t_1)\triangle t_1], \tag{4.17}
$$

and $\triangle A_{\mathrm{RC}} = 0$, we get

$$
R_{\nu}{}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha}(\xi_{\nu} - \dot{a}^{\nu}\xi_0) + (R_{\nu}\frac{\mathrm{d}}{\mathrm{d}t}{}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha}a^{\nu} + \frac{\partial R_{\nu}}{\partial t}{}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha}a^{\nu} - \frac{\partial B}{\partial t})\xi_0
+(\frac{\partial R_{\nu}}{\partial a^{\mu}}{}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha}a^{\nu} - \frac{\partial B}{\partial a^{\mu}})\xi_{\mu} + \frac{R_{\nu}}{2\Gamma(1-\alpha)}|t-t_2|^{-\alpha}\dot{a}^{\nu}(t_2)\xi_0(t_2, a^{\mu}(t_2))
+(R_{\nu}{}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha}a^{\nu} - B)\dot{\xi}_0 - \frac{R_{\nu}}{2\Gamma(1-\alpha)}|t-t_1|^{-\alpha}\dot{a}^{\nu}(t_1)\xi_0(t_1, a^{\mu}(t_1)) = 0. \tag{4.18}
$$

Therefore, we have

CRITERION 4.4. *For the fractional Birkhoffian system* (3.9), *if the infinitesimal generators* ξ_0 *and* ξ_{ν} *satisfy formula* (4.18), *then the infinitesimal transformations* (4.2) *are Noether symmetric transformations.*

Formula (4.18) is the Noether identity for the fractional Birkhoffian system in terms of Riesz-Caputo derivative.

Formula (4.18) can also be obtained from formula (4.15) by setting $\beta = \alpha$, $\gamma = \frac{1}{2}$.

When $\alpha, \beta \to 1$, considering $\Gamma(0) = \infty$, formulae (4.8) (4.11) (4.15) and (4.18) all reduce to the classical Noether identity [32]

$$
R_{\nu}\dot{\xi}_{\nu} - B\dot{\xi}_0 + (\frac{\partial R_{\nu}}{\partial t}\dot{a}^{\nu} - \frac{\partial B}{\partial t})\xi_0 + (\frac{\partial R_{\nu}}{\partial a^{\mu}}\dot{a}^{\nu} - \frac{\partial B}{\partial a^{\mu}})\xi_{\mu} = 0. \tag{4.19}
$$

5. **Conserved quantity for fractional Birkhoffian machanics**

In this section, conserved quantities based on different fractional derivatives are achieved under the condition of $0 < \alpha, \beta < 1$.

THEOREM 5.1. *For the fractional Birkhoffian system* (3.3), *if the infinitesimal transformations* (4.2) *are the Noether symmetric transformations, then there exists a conserved quantity for this system*

$$
I_{\mathrm{RL}} = (R_{\nu}{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}a^{\nu} - B)\xi_0 + \textstyle\int_{t_1}^{t}[R_{\nu}{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}(\xi_{\nu} - \dot{a}^{\nu}\xi_0) + (\xi_{\nu}
- \dot{a}^{\nu}\xi_0){}^{\mathrm{C}}D_{1-\gamma}^{\beta,\alpha}R_{\nu}]\mathrm{d}\tau - \int_{t_1}^{t}[\tfrac{\gamma R_{\nu}}{\Gamma(1-\alpha)}\tfrac{\mathrm{d}}{\mathrm{d}\tau}(\tau - t_1)^{-\alpha}a^{\nu}(t_1)\xi_0(t_1, a^{\mu}(t_1))
- \tfrac{(1-\gamma)R_{\nu}}{\Gamma(1-\beta)}\tfrac{\mathrm{d}}{\mathrm{d}\tau}(t_2 - \tau)^{-\beta}a^{\nu}(t_2)\xi_0(t_2, a^{\mu}(t_2))]\mathrm{d}\tau = \mathrm{const}\,. \tag{5.1}
$$

P r o o f. From Eqs.(3.3) and formula (4.8), we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}I_{\mathrm{RL}} &= (R_\nu{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu - B)\dot{\xi}_0 + \xi_0(\frac{\partial R_\nu}{\partial t} + \frac{\partial R_\nu}{\partial a^\mu}\dot{a}^\mu){}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu \\
&+ \xi_0 R_\nu \frac{\mathrm{d}}{\mathrm{d}t}{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu - \xi_0(\frac{\partial B}{\partial t} + \frac{\partial B}{\partial a^\mu}\dot{a}^\mu) + R_\nu{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}(\xi_\nu - \dot{a}^\nu\xi_0) \\
&+ (\xi_\nu - \dot{a}^\nu\xi_0){}^{\mathrm{C}}D_{1-\gamma}^{\beta,\alpha}R_\nu - [\frac{\gamma R_\nu}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}(t-t_1)^{-\alpha}a^\nu(t_1) \\
&\times \xi_0(t_1, a^\mu(t_1)) - \frac{(1-\gamma)R_\nu}{\Gamma(1-\beta)}\frac{\mathrm{d}}{\mathrm{d}t}(t_2 - t)^{-\beta}a^\nu(t_2)\xi_0(t_2, a^\mu(t_2))] \\
&= -R_\nu{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}(\xi_\nu - \dot{a}^\nu\xi_0) - (R_\nu\frac{\mathrm{d}}{\mathrm{d}t}{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu + \frac{\partial R_\nu}{\partial t}{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu \\
&- \frac{\partial B}{\partial t})\xi_0 - (\frac{\partial R_\nu}{\partial a^\mu}{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu - \frac{\partial B}{\partial a^\mu})\xi_\mu + \xi_0(\frac{\partial R_\nu}{\partial t} + \frac{\partial R_\nu}{\partial a^\mu}\dot{a}^\mu) \\
&\times {}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu + \xi_0 R_\nu\frac{\mathrm{d}}{\mathrm{d}t}{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu - \xi_0(\frac{\partial B}{\partial t} + \frac{\partial B}{\partial a^\mu}\dot{a}^\mu) \\
&+ R_\nu{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}(\xi_\nu - \dot{a}^\nu\xi_0) + (\xi_\nu - \dot{a}^\nu\xi_0){}^{\mathrm{C}}D_{1-\gamma}^{\beta,\alpha}R_\nu \\
&= -(\frac{\partial R_\nu}{\partial a^\mu}{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu - {}^{\mathrm{C}}D_{1-\gamma}^{\beta,\alpha}R_\mu - \frac{\partial B}{\partial a^\mu})(\xi_\mu - \dot{a}^\mu\xi_0) = 0.
\end{aligned}$$

□

THEOREM 5.2. *For the fractional Birkhoffian system* (3.5), *if the infinitesimal transformations* (4.2) *are the Noether symmetric transformations, then there exists a conserved quantity for this system*

$$\begin{aligned}
I_{\mathrm{R}} = (R_\nu{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^\nu - B)\xi_0 + \textstyle\int_{t_1}^{t}[R_\nu{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}(\xi_\nu - \dot{a}^\nu\xi_0) + (\xi_\nu \\
- \dot{a}^\nu\xi_0){}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha}R_\nu]\mathrm{d}\tau + \textstyle\int_{t_1}^{t}\frac{R_\nu}{2\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}\tau}[|\tau - t_2|^{-\alpha}a^\nu(t_2)\xi_0(t_2, a^\mu(t_2)) \\
- |\tau - t_1|^{-\alpha}a^\nu(t_1)\xi_0(t_1, a^\mu(t_1))]\mathrm{d}\tau = \mathrm{const}\,. \qquad (5.2)
\end{aligned}$$

P r o o f. Using Eqs.(3.5) and formula (4.11), we can get the result. □

THEOREM 5.3. *For the fractional Birkhoffian system* (3.7), *if the infinitesimal transformations* (4.2) *are the Noether symmetric transformations, then there exists a conserved quantity for this system*

$$\begin{aligned}
I_{\mathrm{C}} = (R_\nu{}^{\mathrm{C}}D_\gamma^{\alpha,\beta}a^\nu - B)\xi_0 + \textstyle\int_{t_1}^{t}[R_\nu{}^{\mathrm{C}}D_\gamma^{\alpha,\beta}(\xi_\nu - \dot{a}^\nu\xi_0) + (\xi_\nu \\
- \dot{a}^\nu\xi_0){}^{\mathrm{RL}}D_{1-\gamma}^{\beta,\alpha}R_\nu]\mathrm{d}\tau - \textstyle\int_{t_1}^{t}[\frac{\gamma R_\nu}{\Gamma(1-\alpha)}(\tau - t_1)^{-\alpha}\dot{a}^\nu(t_1)\xi_0(t_1, a^\mu(t_1)) \\
- \frac{(1-\gamma)R_\nu}{\Gamma(1-\beta)}(t_2 - \tau)^{-\beta}\dot{a}^\nu(t_2)\xi_0(t_2, a^\mu(t_2))]\mathrm{d}\tau = \mathrm{const}\,. \qquad (5.3)
\end{aligned}$$

P r o o f. The result can be verified from Eqs.(3.7) and formula (4.15). □

THEOREM 5.4. *For the fractional Birkhoffian system* (3.9), *if the infinitesimal transformations* (4.2) *are the Noether symmetric transformations, then there exists a conserved quantity for this system*

$$I_{\mathrm{RC}} = (R_\nu {}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha} a^\nu - B)\xi_0 + \int_{t_1}^{t} [R_\nu {}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha} (\xi_\nu - \dot{a}^\nu \xi_0) + (\xi_\nu - \dot{a}^\nu \xi_0) {}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha} R_\nu]\mathrm{d}\tau + \int_{t_1}^{t} [\tfrac{R_\nu}{2\Gamma(1-\alpha)}[|\tau - t_2|^{-\alpha} \dot{a}^\nu(t_2)\xi_0(t_2, a^\mu(t_2)) - |\tau - t_1|^{-\alpha} \dot{a}^\nu(t_1)\xi_0(t_1, a^\mu(t_1))]\mathrm{d}\tau = \mathrm{const}\,. \quad (5.4)$$

P r o o f. Taking advantage of Eqs.(3.9) and formula (4.18), we can complete this proof. □

When $\alpha, \beta \to 1$, since $\Gamma(0) = \infty$, we can get the classical conserved quantity [32].

THEOREM 5.5. *For the classical Birkhoffian system* (3.10), *if the infinitesimal generators* ξ_0 *and* ξ_ν *satisfy the classical Noether identity* (4.19), *then there exists a conserved quantity for this system*

$$I = R_\nu \xi_\nu - B\xi_0 = \mathrm{const}\,. \quad (5.5)$$

6. **Applications**

In this section, we present four applications.

APPLICATION 6.1. The Lotka biochemical oscillator model is an important class of biological model, which can well describe the competition between different species during the research of ecology [34]. The fractional Lotka biochemical oscillator model in terms of combined Riemann-Liouville derivative has the form [28]

$$\begin{aligned} &\tfrac{1}{2} {}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta} a^2 + \tfrac{1}{2} {}^{\mathrm{C}}D_{1-\gamma}^{\beta,\alpha} a^2 - \alpha_2 - \beta_2 \exp a^1 = 0, \\ &\tfrac{1}{2} {}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta} a^1 + \tfrac{1}{2} {}^{\mathrm{C}}D_{1-\gamma}^{\beta,\alpha} a^2 - \alpha_2 - \beta_2 \exp a^1 = 0, \end{aligned} \quad (6.1)$$

whose Birkhoffian and Birkhoff equations are

$$B = \alpha_2 a^1 - \alpha_1 a^2 - \beta_1 \exp a^2 + \beta_2 \exp a^1,\ R_1 = -\frac{1}{2}a^2,\ R_2 = \frac{1}{2}a^1, \quad (6.2)$$

try to find out its conserved quantity.

Since

$$\frac{\mathrm{d}}{\mathrm{d}t} {}_{t_1}^{\mathrm{RL}}D_{t}^{\alpha} a^\nu = {}_{t_1}^{\mathrm{RL}}D_{t}^{\alpha} \dot{a}^\nu + \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t}(t - t_1)^{-\alpha} a^\nu(t_1), \quad (6.3)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}{}^{\mathrm{RL}}_{t}D^{\beta}_{t_2}a^{\nu} = {}^{\mathrm{RL}}_{t}D^{\beta}_{t_2}\dot{a}^{\nu} + \frac{1}{\Gamma(1-\beta)}\frac{\mathrm{d}}{\mathrm{d}t}(t_2-t)^{-\beta}a^{\nu}(t_2), \tag{6.4}$$

it is obtained that

$$\xi_0 = 1,\ \xi_1 = \xi_2 = 0 \tag{6.5}$$

is a solution to the Noether identity (4.8). Hence, from Theorem 5.1, we obtain a conserved quantity

$$\begin{aligned} I_{\mathrm{RL}} = &-\tfrac{1}{2}a^{2\,\mathrm{RL}}D^{\alpha,\beta}_{\gamma}a^1 + \tfrac{1}{2}a^{1\,\mathrm{RL}}D^{\alpha,\beta}_{\gamma}a^2 - \alpha_2 a^1 + \alpha_1 a^2 \\ &+ \beta_1 \exp a^2 - \beta_2 \exp a^1 + \textstyle\int_{t_1}^{t}(\tfrac{1}{2}a^2\tfrac{\mathrm{d}}{\mathrm{d}\tau}{}^{\mathrm{RL}}D^{\alpha,\beta}_{\gamma}a^1 - \tfrac{1}{2}a^1\tfrac{\mathrm{d}}{\mathrm{d}\tau}{}^{\mathrm{RL}}D^{\alpha,\beta}_{\gamma}a^2 \\ &+ \tfrac{1}{2}\dot{a}^{1\,\mathrm{C}}D^{\beta,\alpha}_{1-\gamma}a^2 - \tfrac{1}{2}\dot{a}^{2\,\mathrm{C}}D^{\beta,\alpha}_{1-\gamma}a^1)\mathrm{d}\tau. \end{aligned} \tag{6.6}$$

When $\alpha,\beta \to 1$, we obtain the classical conserved quantity

$$\alpha_2 a^1 - \alpha_1 a^2 - \beta_1 \exp a^2 + \beta_2 \exp a^1 = \mathrm{const}\,. \tag{6.7}$$

Application 6.2. The Whittaker equations are important aspects of the phylogeny of classical mechanics [10]. The fractional Whittaker model in terms of Riesz-Riemann-Liouville derivative has the form [28]

$$\begin{gathered} -\tfrac{1}{2}{}^{\mathrm{R}}_{t_1}D^{\alpha}_{t_2}a^1 + \tfrac{1}{2}{}^{\mathrm{R}}_{t_1}D^{\alpha}_{t_2}a^4 - \tfrac{1}{2}{}^{\mathrm{RC}}_{t_1}D^{\alpha}_{t_2}(a^1-a^4) = 0, \\ \tfrac{1}{2}{}^{\mathrm{R}}_{t_1}D^{\alpha}_{t_2}a^2 + \tfrac{1}{2}{}^{\mathrm{RC}}_{t_1}D^{\alpha}_{t_2}a^2 - a^4 = 0, \\ -\tfrac{1}{2}{}^{\mathrm{R}}_{t_1}D^{\alpha}_{t_2}a^2 + \tfrac{1}{2}{}^{\mathrm{R}}_{t_1}D^{\alpha}_{t_2}a^3 - \tfrac{1}{2}{}^{\mathrm{RC}}_{t_1}D^{\alpha}_{t_2}a^2 + \tfrac{1}{2}{}^{\mathrm{RC}}_{t_1}D^{\alpha}_{t_2}a^3 - a^1 + a^4 = 0, \\ -\tfrac{1}{2}{}^{\mathrm{R}}_{t_1}D^{\alpha}_{t_2}a^4 - \tfrac{1}{2}{}^{\mathrm{RC}}_{t_1}D^{\alpha}_{t_2}a^4 + a^3 = 0, \end{gathered} \tag{6.8}$$

whose Birkhoffian and Birkhoff equations are

$$\begin{gathered} B = \tfrac{1}{2}[2a^1a^4 - (a^3)^2 - (a^4)^2], \\ R_1 = -\tfrac{1}{2}a^2,\ R_2 = \tfrac{1}{2}(a^1-a^4),\ R_3 = \tfrac{1}{2}a^4,\ R_4 = \tfrac{1}{2}(a^2-a^3), \end{gathered} \tag{6.9}$$

try to find out its conserved quantity.

Considering

$$\frac{\mathrm{d}}{\mathrm{d}t}{}^{\mathrm{R}}_{t_1}D^{\alpha}_{t_2}a^{\nu} = {}^{\mathrm{R}}_{t_1}D^{\alpha}_{t_2}\dot{a}^{\nu} + \frac{1}{2\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}[|t-t_1|^{-\alpha}a^{\nu}(t_1) - |t-t_2|^{-\alpha}a^{\nu}(t_2)], \tag{6.10}$$

it follows from the Noether identity (4.11) that

$$\xi_0 = 1,\ \xi_1 = \xi_2 = \xi_3 = \xi_4 = 0. \tag{6.11}$$

Therefore, from Theorem 5.2, we obtain a conserved quantity

$$\begin{aligned} I_{\mathrm{R}} = &-\tfrac{1}{2}a^{2\,\mathrm{R}}_{\ \ t_1}D^{\alpha}_{t_2}a^1 + \tfrac{1}{2}(a^1-a^4){}^{\mathrm{R}}_{t_1}D^{\alpha}_{t_2}a^2 + \tfrac{1}{2}a^{4\,\mathrm{R}}_{\ \ t_1}D^{\alpha}_{t_2}a^3 + \tfrac{1}{2}(a^2-a^3) \\ &\times {}^{\mathrm{R}}_{t_1}D^{\alpha}_{t_2}a^4 - \tfrac{1}{2}[2a^1a^4 - (a^3)^2 - (a^4)^2] + \textstyle\int_{t_1}^{t}[\tfrac{1}{2}a^2\tfrac{\mathrm{d}}{\mathrm{d}\tau}{}^{\mathrm{R}}_{t_1}D^{\alpha}_{t_2}a^1 - \tfrac{1}{2}(a^1-a^4) \\ &\times \tfrac{\mathrm{d}}{\mathrm{d}\tau}{}^{\mathrm{R}}_{t_1}D^{\alpha}_{t_2}a^2 - \tfrac{1}{2}a^4\tfrac{\mathrm{d}}{\mathrm{d}\tau}{}^{\mathrm{R}}_{t_1}D^{\alpha}_{t_2}a^3 - \tfrac{1}{2}(a^2-a^3)\tfrac{\mathrm{d}}{\mathrm{d}\tau}{}^{\mathrm{R}}_{t_1}D^{\alpha}_{t_2}a^4 + \tfrac{1}{2}\dot{a}^{1\,\mathrm{RC}}_{\ \ t_1}D^{\alpha}_{t_2}a^2 \\ &- \tfrac{1}{2}\dot{a}^{2\,\mathrm{RC}}_{\ \ t_1}D^{\alpha}_{t_2}(a^1-a^4) - \tfrac{1}{2}\dot{a}^{3\,\mathrm{RC}}_{\ \ t_1}D^{\alpha}_{t_2}a^4 - \tfrac{1}{2}\dot{a}^{4\,\mathrm{RC}}_{\ \ t_1}D^{\alpha}_{t_2}(a^2-a^3)]\mathrm{d}\tau. \end{aligned} \tag{6.12}$$

When $\alpha \to 1$, we obtain the classical conserved quantity

$$2a^1a^4 - (a^3)^2 - (a^4)^2 = \text{const}. \tag{6.13}$$

APPLICATION 6.3. The Hojman-Urrutia equations occupy an important position in the study of the inverse problem of Birkhoff mechanics and Lagrange mechanics [21, 34]. The fractional Hojman-Urrutia model in terms of combined Caputo derivative has the form [28]

$$\begin{aligned} {}^{\mathrm{RL}}D^{\beta,\alpha}_{1-\gamma}a^2 + {}^{\mathrm{RL}}D^{\beta,\alpha}_{1-\gamma}a^3 = 0,\ {}^{\mathrm{C}}D^{\alpha,\beta}_{\gamma}a^1 - a^3 = 0, \\ {}^{\mathrm{C}}D^{\alpha,\beta}_{\gamma}a^1 - {}^{\mathrm{RL}}D^{\beta,\alpha}_{1-\gamma}a^4 - (a^3 + a^2) = 0,\ {}^{\mathrm{C}}D^{\alpha,\beta}_{\gamma}a^3 + a^4 = 0, \end{aligned} \tag{6.14}$$

whose Birkhoffian and Birkhoff equations are

$$B = \frac{1}{2}[(a^3)^2 + 2a^2a^3 - (a^4)^2],\ R_1 = a^2 + a^3,\ R_3 = a^4,\ R_2 = R_4 = 0. \tag{6.15}$$

Now we discuss the Noether symmetry and conserved quantity for this fractional model.

Taking calculation of the Noether identity (4.15), where

$$\frac{\mathrm{d}}{\mathrm{d}t}{}^{\mathrm{C}}_{t_1}D^{\alpha}_{t}a^{\nu} = {}^{\mathrm{C}}_{t_1}D^{\alpha}_{t}\dot{a}^{\nu} + \frac{1}{\Gamma(1-\alpha)}(t - t_1)^{-\alpha}\dot{a}^{\nu}(t_1), \tag{6.16}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}{}^{\mathrm{C}}_{t}D^{\beta}_{t_2}a^{\nu} = {}^{\mathrm{C}}_{t}D^{\beta}_{t_2}\dot{a}^{\nu} + \frac{1}{\Gamma(1-\beta)}(t_2 - t)^{-\beta}\dot{a}^{\nu}(t_2), \tag{6.17}$$

we have

$$\xi_0 = 1,\ \xi_1 = 0,\ \xi_2 = 1,\ \xi_3 = 0,\ \xi_4 = 1. \tag{6.18}$$

From Theorem 5.3, we obtain a conserved quantity

$$\begin{aligned} I_{\mathrm{C}} = {} & (a^2 + a^3)^{\mathrm{C}}D^{\alpha,\beta}_{\gamma}a^1 + a^{4\mathrm{C}}D^{\alpha,\beta}_{\gamma}a^3 - \tfrac{1}{2}(a^3)^2 - a^2a^3 + \tfrac{1}{2}(a^4)^2 \\ & - \textstyle\int_{t_1}^{t}[(a^2 + a^3)\frac{\mathrm{d}}{\mathrm{d}\tau}{}^{\mathrm{C}}D^{\alpha,\beta}_{\gamma}a^1 + a^4\frac{\mathrm{d}}{\mathrm{d}\tau}{}^{\mathrm{C}}D^{\alpha,\beta}_{\gamma}a^3 \\ & + \dot{a}^{1\mathrm{RL}}D^{\beta,\alpha}_{1-\gamma}(a^2 + a^3) + \dot{a}^{3\mathrm{RL}}D^{\beta,\alpha}_{1-\gamma}a^4]\mathrm{d}\tau. \end{aligned} \tag{6.19}$$

When $\alpha, \beta \to 1$, we get the classical conserved quantity

$$\frac{1}{2}(a^3)^2 + a^2a^3 - \frac{1}{2}(a^4)^2 = \text{const}. \tag{6.20}$$

APPLICATION 6.4. The Hénon-Heiles model plays an important role in chaos [2, 6, 22]. The fractional Hénon-Heiles model in terms of Riesz-Caputo derivative has the form [28]

$$\begin{aligned} {}^{\mathrm{RC}}_{t_1}D^{\alpha}_{t_2}a^3 + a^1 + 2a^1a^2 = 0,\ {}^{\mathrm{RC}}_{t_1}D^{\alpha}_{t_2}a^4 + a^2 - (a^2)^2 + (a^1)^2 = 0, \\ {}^{\mathrm{R}}_{t_1}D^{\alpha}_{t_2}a^1 - a^3 = 0,\ {}^{\mathrm{R}}_{t_1}D^{\alpha}_{t_2}a^2 - a^4 = 0, \end{aligned} \tag{6.21}$$

whose Birkhoffian and Birkhoff equations are

$$\begin{aligned} B = \tfrac{1}{2}[(a^1)^2 + (a^2)^2 + (a^3)^2 + (a^4)^2 + 2a^2(a^1)^2 - \tfrac{2}{3}(a^2)^3], \\ R_1 = R_2 = 0,\ R_3 = -a^1,\ R_4 = -a^2. \end{aligned} \tag{6.22}$$

Now we discuss the Noether symmetry and conserved quantity for this fractional Hénon-Heiles model.

Making use of

$$\frac{\mathrm{d}}{\mathrm{d}t}{}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha}a^{\nu} = {}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha}\dot{a}^{\nu} + \frac{1}{2\Gamma(1-\alpha)}[|t-t_1|^{-\alpha}\dot{a}^{\nu}(t_1) - |t-t_2|^{-\alpha}\dot{a}^{\nu}(t_2)], \tag{6.23}$$

Noether identity (4.18) gives

$$\xi_0 = \xi_1 = \xi_2 = 1,\ \xi_3 = \xi_4 = 0. \tag{6.24}$$

Therefore, from Theorem 5.4, we obtain a conserved quantity

$$\begin{aligned} I_{\mathrm{RC}} = & -a^1{}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha}a^3 - a^2{}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha}a^4 - \tfrac{1}{2}[(a^1)^2 + (a^2)^2 + (a^3)^2 \\ & + (a^4)^2 + 2a^2(a^1)^2 - \tfrac{2}{3}(a^2)^3] + \textstyle\int_{t_1}^{t}(a^1\frac{\mathrm{d}}{\mathrm{d}\tau}{}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha}a^3 + a^2\frac{\mathrm{d}}{\mathrm{d}\tau}{}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha}a^4 \\ & + \dot{a}^3{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^1 + \dot{a}^4{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^2)\mathrm{d}\tau. \end{aligned} \tag{6.25}$$

When $\alpha \to 1$, we obtain the classical conserved quantity

$$(a^1)^2 + (a^2)^2 + (a^3)^2 + (a^4)^2 + 2a^2(a^1)^2 - \frac{2}{3}(a^2)^3 = \mathrm{const}\,. \tag{6.26}$$

7. **Conclusion**

A new research field called fractional Birkhoffian mechanics, which can be used to study problems of science and engineering, was constructed in Ref. [28] recently. We discuss the Noether symmetry and conserved quantity for the fractional Birkhoffian mechanics in this paper. Noether identities and conserved quantities in terms of combined Riemann-Liouville derivative, Riesz-Riemann-Liouville derivative, combined Caputo derivative and Riesz-Caputo derivative are presented, respectively. Our Theorem 5.1 - Theorem 5.4 are the new results. The classical Noether identity and conserved quantity are special cases of the results achieved in this paper.

Noether symmetries for four fractional dynamical models, including the fractional Whittaker model, the fractional Hénon-Heiles model, the fractional Lotka biochemical oscillator model and the fractional Hojman-Urrutia model are discussed in this paper, and the corresponding conserved quantities are obtained.

Considering the significance of the fractional calculus and the Birkhoffian dynamics, further research, for instance, perturbation to Noether symmetry and adiabatic invariants for the fractional Birkhoffian mechanics, can be done in the future. Besides, Lie symmetry and conserved quantity, Mei symmetry and conserved quantity for fractional mechanics, such as fractional Lagrangian mechanics, fractional Hamiltonian mechanics and fractional Birkhoffian mechanics, are also hoped to be investigated.

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[1] *College of Science, Nanjing University of Science and Technology*
No. 200 Xiaolingwei Street, Xuanwu District
Nanjing 210094, Jiangsu Province, CHINA

[2] *School of Mathematics and Physics, Suzhou University of Science and Technology, No. 1 Kerui Road, Gaoxin District*
Suzhou 215009, Jiangsu Province, CHINA
e-mail: songchuanjingsun@126.com

[3] *College of Civil Engineering, Suzhou University of Science and Technology, No. 1 Kerui Road, Gaoxin District*
Suzhou 215009, Jiangsu Province, CHINA
e-mail: zhy@mail.usts.edu.cn (Corr. author) *Received: February 10, 2017*

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