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Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME 21, NUMBER 2 (2018)
(Print) ISSN 1311-0454
(Electronic) ISSN 1314-2224

RESEARCH PAPER

EXISTENCE OF SOLUTIONS FOR A SYSTEM OF FRACTIONAL DIFFERENTIAL EQUATIONS WITH COUPLED NONLOCAL BOUNDARY CONDITIONS

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Abstract

We study the existence of solutions for a system of nonlinear Caputo fractional differential equations with coupled boundary conditions involving Riemann-Liouville fractional integrals, by using the Schauder fixed point theorem and the nonlinear alternative of Leray-Schauder type. Two examples are given to support our main results.

MSC 2010: Primary 34A08; Secondary 34B15, 45G15

Key Words and Phrases: Caputo fractional differential equations, coupled integral boundary conditions, fixed point theorems

1. Introduction

Fractional differential equations describe many phenomena in several fields of engineering and scientific disciplines such as physics, biophysics, chemistry, biology (for example, the primary infection with HIV), economics, control theory, signal and image processing, thermoelasticity, aerodynamics, viscoelasticity, electromagnetics and rheology (see [6], [7], [8], [9], [10], [11], [12], [18], [19], [21], [22], [23], [24], [26], [27], [29], [30]). Fractional differential equations are also regarded as a better tool for the description of hereditary properties of various materials and processes than the corresponding integer order differential equations.

pp. 423–441 , DOI: 10.1515/fca-2018-0024
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Motivated by these various applications, in this paper, we investigate the following system of nonlinear fractional differential equations

$$(S)\qquad \begin{cases} {}^cD_{0+}^{\alpha}u(t) = f(t, v(t), {}^cD_{0+}^{p}v(t)), & t \in (0,1),\\ {}^cD_{0+}^{\beta}v(t) = g(t, u(t), {}^cD_{0+}^{q}u(t)), & t \in (0,1),\end{cases}$$

supplemented with the coupled nonlocal boundary conditions

$$(BC)\quad \begin{cases} u(0) = \varphi(v), \ \ u'(0) = 0, \ldots, u^{(n-2)}(0) = 0, \ \ u(1) = \lambda I_{0+}^{\gamma}v(\xi),\\ v(0) = \psi(u), \ \ v'(0) = 0, \ldots, v^{(m-2)}(0) = 0, \ \ v(1) = \mu I_{0+}^{\delta}u(\eta),\end{cases}$$

where $\alpha \in (n-1, n]$, $\beta \in (m-1, m]$, $n, m \in \mathbb{N}$, $n, m \geq 2$, $p, q \in (0,1)$, $\lambda, \mu \in \mathbb{R}$, $\gamma, \delta > 0$, $\xi, \eta \in [0,1]$, $f, g : [0,1]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $\varphi, \psi : C[0,1] \to \mathbb{R}$ are continuous functions, ${}^cD_{0+}^{\theta}$ denotes the Caputo derivative of fractional order θ (for $\theta = \alpha, \beta, p, q$), and I_{0+}^{γ}, I_{0+}^{δ} are the Riemann-Liouville fractional integrals of orders γ and δ, respectively. If $n = 2$ and $m = 2$, then the boundary conditions (BC) take the form $u(0) = \varphi(v)$, $u(1) = \lambda I_{0+}^{\gamma}v(\xi)$ and $v(0) = \psi(u)$, $v(1) = \mu I_{0+}^{\delta}u(\eta)$, respectively, that is, the derivatives of u and v at the point 0 do not appear in these conditions. Under some assumptions on the functions f and g, we will establish the existence of solutions for problem $(S)-(BC)$ by using the Schauder fixed point theorem and the nonlinear alternative of Leray-Schauder type.

The system (S) with $\alpha, \beta \in (1,2)$ subject to the uncoupled boundary conditions

$$(BC_1)\qquad \begin{cases} u(0) = \varphi(u), \ \ u(1) = aI_{0+}^{\theta_1}u(\eta),\\ v(0) = \psi(v), \ \ v(1) = bI_{0+}^{\theta_2}v(\xi),\end{cases}$$

with $\xi, \eta \in [0,1]$, $\theta_1, \theta_2 > 0$, $a, b \in \mathbb{R}$, was studied in paper Ref. [25]. We also mention the paper Ref. [4], where the authors investigated the system of fractional differential equations equipped with nonlocal coupled boundary conditions

$$(P)\qquad \begin{cases} {}^cD_{0+}^{\alpha}x(t) = f(t, x(t), y(t), {}^cD_{0+}^{\gamma}y(t)), & t \in [0,T],\\ {}^cD_{0+}^{\beta}y(t) = g(t, x(t), {}^cD_{0+}^{\delta}x(t), y(t)), & t \in [0,T],\\ x(0) = h(y), \ \ \int_0^T y(s)\,ds = \mu_1 x(\eta), \\ y(0) = \phi(x), \ \ \int_0^T x(s)\,ds = \mu_2 y(\xi),\end{cases}$$

where $\alpha, \beta \in (1,2]$, $\gamma, \delta \in (0,1)$, $\eta, \xi \in (0,T)$, $f, g : [0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $h, \phi : C([0,T],\mathbb{R}) \to \mathbb{R}$ are continuous functions, and μ_1, μ_2 are real constants. By using the Banach contraction mapping principle and the Leray-Schauder nonlinear alternative, the authors proved in Ref. [4] the existence and uniqueness, and the existence of solutions for problem (P). For other recent developments on the fractional differential equations with nonlocal boundary conditions we refer the reader to the papers [1], [2], [3],

[5], [13], [14], [15], [16], [17], [20], [28], [31], [32], [33] and the references therein.

The paper is organized as follows. In Section 2, we present some auxiliary results for a nonlocal boundary value problem of linear fractional differential equations. Section 3 contains the main theorems for the existence of solutions for problem $(S)-(BC)$, while Section 4 contains two examples which illustrate our main results.

2. **Auxiliary results**

In this section we present some definitions and auxiliary results that will be used to prove our main theorems.

Definition 2.1. ([18]) The Riemann-Liouville fractional integral of order $\alpha > 0$ is given by

$$I_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\, ds, \ \ t > 0, \ \ \alpha > 0,$$

provided that the right-hand side is pointwise defined on $(0,\infty)$, where Γ is the gamma function, defined by $\Gamma(p) = \int_0^\infty t^{p-1} e^{-t}\, dt$.

Definition 2.2. ([18]) For an $(n-1)$-times absolutely continuous function $u : [0,\infty) \to \mathbb{R}$, the Caputo derivative of fractional order α is defined as

$${}^{c}\!D^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} u^{(n)}(s)\, ds, \ t > 0, \ n-1 < \alpha < n, \ n \in N.$$

If $\alpha = m \in N$ and $u \in C^m[0,\infty)$, then ${}^{c}\!D^{\alpha} u(t) = u^{(m)}(t)$.

We consider the fractional differential system

$$\begin{cases} {}^{c}\!D_{0+}^{\alpha} u(t) = x(t), \ \ t \in (0,1), \\ {}^{c}\!D_{0+}^{\beta} v(t) = y(t), \ \ t \in (0,1), \end{cases} \tag{2.1}$$

with the coupled boundary conditions

$$\begin{cases} u(0) = u_0, \ \ u'(0) = 0, \ldots, u^{(n-2)}(0) = 0, \ \ u(1) = \lambda I_{0+}^{\gamma} v(\xi), \\ v(0) = v_0, \ \ v'(0) = 0, \ldots, v^{(m-2)}(0) = 0, \ \ v(1) = \mu I_{0+}^{\delta} u(\eta), \end{cases} \tag{2.2}$$

where $\alpha \in (n-1, n]$, $\beta \in (m-1, m]$, n, $m \in \mathbb{N}$, n, $m \geq 2$, λ, $\mu \in \mathbb{R}$, γ, $\delta > 0$, ξ, $\eta \in [0,1]$, u_0, $v_0 \in \mathbb{R}$, and x, y are real functions.

In the sequel we set $\Delta = 1 - \dfrac{\lambda\mu\xi^{\gamma+m-1}\eta^{\delta+n-1}(m-1)!(n-1)!}{\Gamma(\gamma+m)\Gamma(\delta+n)}$.

LEMMA 2.1. *If $x,\ y \in C[0,1]$, $u_0,\ v_0 \in \mathbb{R}$ and $\Delta \neq 0$, then the unique solution $(u,v) \in C^n[0,1] \times C^m[0,1]$ of the boundary value problem (2.1)-(2.2) is given by*

$$\begin{aligned}
u(t) &= u_0 + I_{0+}^{\alpha}x(t) + \frac{t^{n-1}}{\Delta}\Bigg[-u_0 - I_{0+}^{\alpha}x(1) + \frac{\lambda v_0 \xi^{\gamma}}{\Gamma(\gamma+1)} \\
&+\lambda I_{0+}^{\beta+\gamma}y(\xi) - \frac{\lambda v_0 \xi^{\gamma+m-1}(m-1)!}{\Gamma(\gamma+m)} - \frac{\lambda \xi^{\gamma+m-1}(m-1)!}{\Gamma(\gamma+m)} I_{0+}^{\beta}y(1) \\
&+ \frac{\lambda\mu u_0 \xi^{\gamma+m-1}\eta^{\delta}(m-1)!}{\Gamma(\delta+1)\Gamma(\gamma+m)} + \frac{\lambda\mu\xi^{\gamma+m-1}(m-1)!}{\Gamma(\gamma+m)} I_{0+}^{\alpha+\delta}x(\eta)\Bigg], \\
v(t) &= v_0 + I_{0+}^{\beta}y(t) + \frac{t^{m-1}}{\Delta}\Bigg[-v_0 - I_{0+}^{\beta}y(1) + \frac{\mu u_0 \eta^{\delta}}{\Gamma(\delta+1)} \\
&+\mu I_{0+}^{\alpha+\delta}x(\eta) - \frac{\mu u_0 \eta^{\delta+n-1}(n-1)!}{\Gamma(\delta+n)} - \frac{\mu\eta^{\delta+n-1}(n-1)!}{\Gamma(\delta+n)} I_{0+}^{\alpha}x(1) \\
&+ \frac{\lambda\mu v_0 \xi^{\gamma}\eta^{\delta+n-1}(n-1)!}{\Gamma(\gamma+1)\Gamma(\delta+n)} + \frac{\lambda\mu\eta^{\delta+n-1}(n-1)!}{\Gamma(\delta+n)} I_{0+}^{\beta+\gamma}y(\xi)\Bigg],
\end{aligned} \tag{2.3}$$

for all $t \in [0,1]$.

P r o o f. The solution $u \in C^n[0,1]$ of equation ${}^cD_{0+}^{\alpha}u(t) = x(t)$ is

$$u(t) = \widetilde{c}_1 + \widetilde{c}_2 t + \cdots + \widetilde{c}_n t^{n-1} + I_{0+}^{\alpha}x(t),\ \ t \in [0,1],$$

where $\widetilde{c}_1, \widetilde{c}_2, \ldots, \widetilde{c}_n \in \mathbb{R}$. By using the conditions $u(0) = u_0$, $u'(0) = \cdots = u^{(n-2)}(0) = 0$, we obtain $\widetilde{c}_1 = u_0$, $\widetilde{c}_2 = \cdots = \widetilde{c}_{n-1} = 0$, and so we deduce $u(t) = u_0 + \widetilde{c}_n t^{n-1} + I_{0+}^{\alpha}x(t),\ t \in [0,1]$.

In a similar manner the solution $v \in C^m[0,1]$ of equation ${}^cD_{0+}^{\beta}v(t) = y(t)$ satisfying the conditions $v(0) = v_0$, $v'(0) = \cdots = v^{(m-2)}(0) = 0$ is $v(t) = v_0 + \widetilde{d}_m t^{m-1} + I_{0+}^{\beta}y(t),\ t \in [0,1]$, with $\widetilde{d}_m \in \mathbb{R}$.

Imposing the conditions $u(1) = \lambda I_{0+}^{\gamma}v(\xi)$ and $v(1) = \mu I_{0+}^{\delta}u(\eta)$ on the above functions u and v, we obtain the following system in the unknown constants $\widetilde{c}_n$ and $\widetilde{d}_m$

$$\begin{cases}
\widetilde{c}_n - \dfrac{\lambda \widetilde{d}_m \xi^{\gamma+m-1}(m-1)!}{\Gamma(\gamma+m)} = -u_0 - I_{0+}^{\alpha}x(1) + \dfrac{\lambda v_0 \xi^{\gamma}}{\Gamma(\gamma+1)} + \lambda I_{0+}^{\beta+\gamma}y(\xi), \\
-\dfrac{\mu \widetilde{c}_n \eta^{\delta+n-1}(n-1)!}{\Gamma(\delta+n)} + \widetilde{d}_m = -v_0 - I_{0+}^{\beta}y(1) + \dfrac{\mu u_0 \eta^{\delta}}{\Gamma(\delta+1)} + \mu I_{0+}^{\alpha+\delta}x(\eta).
\end{cases}$$

In view of the given assumption $\Delta \neq 0$, the unique solution $(\widetilde{c}_n, \widetilde{d}_m)$ of the above system is

$$\begin{aligned}
\widetilde{c}_n &= \frac{1}{\Delta}\Bigg[-u_0 - I_{0+}^{\alpha}x(1) + \frac{\lambda v_0 \xi^{\gamma}}{\Gamma(\gamma+1)} + \lambda I_{0+}^{\beta+\gamma}y(\xi) - \frac{\lambda v_0 \xi^{\gamma+m-1}(m-1)!}{\Gamma(\gamma+m)} \\
&-\frac{\lambda \xi^{\gamma+m-1}(m-1)!}{\Gamma(\gamma+m)} I_{0+}^{\beta}y(1) + \frac{\lambda\mu u_0 \xi^{\gamma+m-1}\eta^{\delta}(m-1)!}{\Gamma(\delta+1)\Gamma(\gamma+m)}
\end{aligned}$$

$$+\frac{\lambda\mu\xi^{\gamma+m-1}(m-1)!}{\Gamma(\gamma+m)}I_{0+}^{\alpha+\delta}x(\eta)\Bigg],$$

$$\widetilde{d}_m=\frac{1}{\Delta}\Bigg[-v_0-I_{0+}^{\beta}y(1)+\frac{\mu u_0\eta^{\delta}}{\Gamma(\delta+1)}+\mu I_{0+}^{\alpha+\delta}x(\eta)-\frac{\mu u_0\eta^{\delta+n-1}(n-1)!}{\Gamma(\delta+n)}$$
$$-\frac{\mu\eta^{\delta+n-1}(n-1)!}{\Gamma(\delta+n)}I_{0+}^{\alpha}x(1)+\frac{\lambda\mu v_0\xi^{\gamma}\eta^{\delta+n-1}(n-1)!}{\Gamma(\gamma+1)\Gamma(\delta+n)}$$
$$+\frac{\lambda\mu\eta^{\delta+n-1}(n-1)!}{\Gamma(\delta+n)}I_{0+}^{\beta+\gamma}y(\xi)\Bigg].$$

Substituting the values of $\widetilde{c}_n$ and $\widetilde{d}_m$ in the expressions for u and v, we obtain the solution (u,v) of problem (2.1)-(2.2) given by formula (2.3). $\square$

By direct computation, we can obtain the following result.

Lemma 2.2. *If x, $y\in C[0,1]$, then the following inequalities are satisfied*:

a) $|I_{0+}^{\alpha}x(t)|\le\dfrac{\|x\|}{\Gamma(\alpha+1)}$;

b) $|I_{0+}^{\alpha}x(1)|\le\dfrac{\|x\|}{\Gamma(\alpha+1)}$;

c) $|I_{0+}^{\alpha+\delta}x(\eta)|\le\dfrac{\eta^{\alpha+\delta}\|x\|}{\Gamma(\alpha+\delta+1)}$; *d)* $|I_{0+}^{\beta}y(t)|\le\dfrac{\|y\|}{\Gamma(\beta+1)}$;

e) $|I_{0+}^{\beta}y(1)|\le\dfrac{\|y\|}{\Gamma(\beta+1)}$; *f)* $|I_{0+}^{\beta+\gamma}y(\xi)|\le\dfrac{\xi^{\beta+\gamma}\|y\|}{\Gamma(\beta+\gamma+1)}$,

where $\|x\|=\sup_{t\in[0,1]}|x(t)|$ *and* $\|y\|=\sup_{t\in[0,1]}|y(t)|$.

Now we state the Schauder fixed point theorem and the nonlinear alternative of Leray-Schauder type, which play a pivotal role in the forthcoming analysis.

Theorem 2.1. *Let X be a Banach space and $Y\subset X$ a nonempty, bounded, convex and closed subset. If the operator $A:Y\to Y$ is completely continuous, then A has at least one fixed point.*

Theorem 2.2. *Let X be a Banach space and U be an open and bounded subset of X, $0\in U$ and $A:\bar{U}\to X$ be a completely continuous operator. Then either A has a fixed point in $\bar{U}$, or there exist $u\in\partial U$ and $\nu\in(0,1)$ such that $u=\nu Au$.*

3. Main results

We consider the spaces $X = \{u \in C[0,1],\ {}^cD_{0+}^q u \in C[0,1]\}$ and $Y = \{v \in C[0,1], {}^cD_{0+}^p v \in C[0,1]\}$ equipped respectively with the norms $\|u\|_X = \|u\| + \|{}^cD_{0+}^q u\|$ and $\|v\|_Y = \|v\| + \|{}^cD_{0+}^p v\|$, where $\|\cdot\|$ is the supremum norm, that is $\|w\| = \sup_{t\in[0,1]} |w(t)|$ for $w \in C[0,1]$. The spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are Banach spaces, and the product space $X \times Y$ endowed with the norm $\|(u,v)\|_{X\times Y} = \|u\|_X + \|v\|_Y$ is also a Banach space.

By using Lemma 2.1, we introduce the operator $T : X \times Y \to X \times Y$ defined by $T(u,v) = (T_1(u,v), T_2(u,v))$ for $(u,v) \in X \times Y$, where the operators $T_1 : X \times Y \to X$ and $T_2 : X \times Y \to Y$ are given by

$$
\begin{aligned}
T_1(u,v)(t) &= \varphi(v)\left[1 - \frac{t^{n-1}}{\Delta} + \frac{t^{n-1}\lambda\mu\xi^{\gamma+m-1}\eta^{\delta}(m-1)!}{\Delta\Gamma(\delta+1)\Gamma(\gamma+m)}\right] \\
&\quad + \psi(u)\frac{t^{n-1}\lambda\xi^{\gamma}}{\Delta}\left[\frac{1}{\Gamma(\gamma+1)} - \frac{\xi^{m-1}(m-1)!}{\Gamma(\gamma+m)}\right] \\
&\quad + I_{0+}^{\alpha} f(t, v(t), {}^cD_{0+}^p v(t)) + \frac{t^{n-1}}{\Delta}\Bigg[- I_{0+}^{\alpha} f(t, v(t), {}^c D_{0+}^p v(t))|_{t=1} \\
&\quad + \frac{\lambda\mu\xi^{\gamma+m-1}(m-1)!}{\Gamma(\gamma+m)} I_{0+}^{\alpha+\delta} f(t, v(t), {}^cD_{0+}^p v(t))|_{t=\eta} \\
&\quad + \lambda I_{0+}^{\beta+\gamma} g(t, u(t), {}^cD_{0+}^q u(t))|_{t=\xi} \\
&\quad - \frac{\lambda\xi^{\gamma+m-1}(m-1)!}{\Gamma(\gamma+m)} I_{0+}^{\beta} g(t, u(t), {}^cD_{0+}^q u(t))|_{t=1}\Bigg], \\
T_2(u,v)(t) &= \varphi(v)\frac{t^{m-1}\mu\eta^{\delta}}{\Delta}\left[\frac{1}{\Gamma(\delta+1)} - \frac{\eta^{n-1}(n-1)!}{\Gamma(\delta+n)}\right] \\
&\quad + \psi(u)\left[1 - \frac{t^{m-1}}{\Delta} + \frac{t^{m-1}\lambda\mu\xi^{\gamma}\eta^{\delta+n-1}(n-1)!}{\Delta\Gamma(\gamma+1)\Gamma(\delta+n)}\right] \\
&\quad + I_{0+}^{\beta} g(t, u(t), {}^cD_{0+}^q u(t)) + \frac{t^{m-1}}{\Delta}\Bigg[\mu I_{0+}^{\alpha+\delta} f(t, v(t), {}^cD_{0+}^p v(t))|_{t=\eta} \\
&\quad - \frac{\mu\eta^{\delta+n-1}(n-1)!}{\Gamma(\delta+n)} I_{0+}^{\alpha} f(t, v(t), {}^cD_{0+}^p v(t))|_{t=1} - I_{0+}^{\beta} g(t, u(t), {}^cD_{0+}^q u(t))|_{t=1} \\
&\quad + \frac{\lambda\mu\eta^{\delta+n-1}(n-1)!}{\Gamma(\delta+n)} I_{0+}^{\beta+\gamma} g(t, u(t), {}^cD_{0+}^q u(t))|_{t=\xi}\Bigg],
\end{aligned}
$$

for all $(u,v) \in X \times Y$ and $t \in [0,1]$.

The pair (u,v) is a solution of problem $(S)-(BC)$ if and only if (u,v) is a fixed point of operator T.

Now we enlist the assumptions that we need in this section:

(*H*1) $\alpha,\ \beta \in \mathbb{R},\ \alpha \in (n-1,n],\ \beta \in (m-1,m],\ n,m \in \mathbb{N},\ n,m \geq 2$; $p,\ q \in (0,1)$; $\gamma,\ \delta > 0$; $\lambda,\ \mu \in \mathbb{R}$; $\xi,\ \eta \in [0,1]$;
$$\Delta = 1 - \frac{\lambda\mu\xi^{\gamma+m-1}\eta^{\delta+n-1}(m-1)!(n-1)!}{\Gamma(\gamma+m)\Gamma(\delta+n)} \neq 0.$$

(*H*2) $f,\ g : [0,1]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are continuous functions and there exist the constants $a_i,\ b_i \geq 0,\ i=0,1,2$, and $l_j,\ m_j \in (0,1),\ j=1,2$ such that
$$|f(t,u,v)| \leq a_0 + a_1|u|^{l_1} + a_2|v|^{l_2},\ \ \forall\, t\in[0,1],\ u,\ v\in\mathbb{R},$$
$$|g(t,u,v)| \leq b_0 + b_1|u|^{m_1} + b_2|v|^{m_2},\ \ \forall\, t\in[0,1],\ u,\ v\in\mathbb{R}.$$

(*H*3) $\varphi,\ \psi : C[0,1] \to \mathbb{R}$ are continuous functions, $\varphi(0)=\psi(0)=0$ and there exist constants $L_1,\ L_2 > 0$, $\theta_1,\ \theta_2 \in (0,1)$ such that
$$|\varphi(x)| \leq L_1\|x\|^{\theta_1},\ \ |\psi(x)| \leq L_2\|x\|^{\theta_2},\ \ \forall\, x\in C[0,1].$$

(*H*4) $f,\ g : [0,1]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are continuous functions, and there exist $c_i,\ d_i \geq 0,\ i=0,1,2$ and nondecreasing functions $h_j,\ k_j \in C([0,\infty),[0,\infty)),\ j=1,2$ such that
$$|f(t,u,v)| \leq c_0 + c_1h_1(|u|) + c_2h_2(|v|),\ \ \forall\, t\in[0,1],\ u,\ v\in\mathbb{R},$$
$$|g(t,u,v)| \leq d_0 + d_1k_1(|u|) + d_2k_2(|v|),\ \ \forall\, t\in[0,1],\ u,\ v\in\mathbb{R}.$$

(*H*5) There exists $L_0 > 0$ such that
$$(M_1+\widetilde{M}_1+N_1+\widetilde{N}_1)L_0^{\theta_1} + (M_2+\widetilde{M}_2+N_2+\widetilde{N}_2)L_0^{\theta_2}$$
$$+A_1(M_3+\widetilde{M}_3+N_3+\widetilde{N}_3) + A_2(M_4+\widetilde{M}_4+N_4+\widetilde{N}_4) < L_0,$$
where $A_1 = c_0+c_1h_1(L_0)+c_2h_2(L_0)$, $A_2 = d_0+d_1k_1(L_0)+d_2k_2(L_0)$, $\theta_1,\ \theta_2$ are given in (*H*3), $c_i,\ d_i,\ i=0,1,2$ and $h_j,\ k_j,\ j=1,2$ are given in (*H*4), and the constants $M_i,\ \widetilde{M}_i,\ N_i,\ \widetilde{N}_i,\ i=1,2,3,4$ are given below.

For computational convenience, we introduce the following notations:
$$M_1 = L_1\left(1+\frac{1}{|\Delta|}+\frac{\lambda\mu\xi^{\gamma+m-1}\eta^{\delta}(m-1)!}{|\Delta|\Gamma(\delta+1)\Gamma(\gamma+m)}\right),$$
$$M_2 = \frac{L_2\lambda\xi^{\gamma}}{|\Delta|}\left(\frac{1}{\Gamma(\gamma+1)}+\frac{\xi^{m-1}(m-1)!}{\Gamma(\gamma+m)}\right),$$
$$M_3 = \frac{1}{\Gamma(\alpha+1)}+\frac{1}{|\Delta|\Gamma(\alpha+1)}+\frac{\lambda\mu\xi^{\gamma+m-1}\eta^{\alpha+\delta}(m-1)!}{|\Delta|\Gamma(\gamma+m)\Gamma(\alpha+\delta+1)},$$
$$M_4 = \frac{\lambda\xi^{\beta+\gamma}}{|\Delta|\Gamma(\beta+\gamma+1)}+\frac{\lambda\xi^{\gamma+m-1}(m-1)!}{|\Delta|\Gamma(\gamma+m)\Gamma(\beta+1)},$$

$$\widetilde{M}_1 = \frac{L_1(n-1)}{\Gamma(2-q)|\Delta|}\left(1+\frac{\lambda\mu\xi^{\gamma+m-1}\eta^{\delta}(m-1)!}{\Gamma(\delta+1)\Gamma(\gamma+m)}\right),$$
$$\widetilde{M}_2 = \frac{L_2(n-1)\lambda\xi^{\gamma}}{\Gamma(2-q)|\Delta|}\left(\frac{1}{\Gamma(\gamma+1)}+\frac{\xi^{m-1}(m-1)!}{\Gamma(\gamma+m)}\right),$$
$$\widetilde{M}_3 = \frac{1}{\Gamma(2-q)}\left(\frac{1}{\Gamma(\alpha)}+\frac{n-1}{|\Delta|\Gamma(\alpha+1)}+\frac{(n-1)\lambda\mu\xi^{\gamma+m-1}\eta^{\alpha+\delta}(m-1)!}{|\Delta|\Gamma(\gamma+m)\Gamma(\alpha+\delta+1)}\right),$$
$$\widetilde{M}_4 = \frac{n-1}{\Gamma(2-q)|\Delta|}\left(\frac{\lambda\xi^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)}+\frac{\lambda\xi^{\gamma+m-1}(m-1)!}{\Gamma(\gamma+m)\Gamma(\beta+1)}\right),$$
$$N_1 = \frac{L_1\mu\eta^{\delta}}{|\Delta|}\left(\frac{1}{\Gamma(\delta+1)}+\frac{\eta^{n-1}(n-1)!}{\Gamma(\delta+n)}\right),$$
$$N_2 = L_2\left(1+\frac{1}{|\Delta|}+\frac{\lambda\mu\xi^{\gamma}\eta^{\delta+n-1}(n-1)!}{|\Delta|\Gamma(\gamma+1)\Gamma(\delta+n)}\right),$$
$$N_3 = \frac{\mu\eta^{\alpha+\delta}}{|\Delta|\Gamma(\alpha+\delta+1)}+\frac{\mu\eta^{\delta+n-1}(n-1)!}{|\Delta|\Gamma(\delta+n)\Gamma(\alpha+1)},$$
$$N_4 = \frac{1}{\Gamma(\beta+1)}+\frac{1}{|\Delta|\Gamma(\beta+1)}+\frac{\lambda\mu\xi^{\beta+\gamma}\eta^{\delta+n-1}(n-1)!}{|\Delta|\Gamma(\delta+n)\Gamma(\beta+\gamma+1)},$$
$$\widetilde{N}_1 = \frac{L_1(m-1)\mu\eta^{\delta}}{\Gamma(2-p)|\Delta|}\left(\frac{1}{\Gamma(\delta+1)}+\frac{\eta^{n-1}(n-1)!}{\Gamma(\delta+n)}\right),$$
$$\widetilde{N}_2 = \frac{L_2(m-1)}{\Gamma(2-p)|\Delta|}\left(1+\frac{\lambda\mu\xi^{\gamma}\eta^{\delta+n-1}(n-1)!}{\Gamma(\gamma+1)\Gamma(\delta+n)}\right),$$
$$\widetilde{N}_3 = \frac{m-1}{\Gamma(2-p)|\Delta|}\left(\frac{\mu\eta^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)}+\frac{\mu\eta^{\delta+n-1}(n-1)!}{\Gamma(\delta+n)\Gamma(\alpha+1)}\right),$$
$$\widetilde{N}_4 = \frac{1}{\Gamma(2-p)}\left(\frac{1}{\Gamma(\beta)}+\frac{m-1}{|\Delta|\Gamma(\beta+1)}+\frac{(m-1)\lambda\mu\eta^{\delta+n-1}\xi^{\beta+\gamma}(n-1)!}{|\Delta|\Gamma(\delta+n)\Gamma(\beta+\gamma+1)}\right).$$

THEOREM 3.1. *Assume that* $(H1)-(H3)$ *hold. Then problem* $(S)-(BC)$ *has at least one solution on* $[0,1]$.

P r o o f. Let $\bar{B}_R = \{(u,v)\in X\times Y,\ \|(u,v)\|_{X\times Y}\le R\}$, where

$$\begin{aligned}R \ge \max\Big\{&[8(M_1+\widetilde{M}_1)]^{\frac{1}{1-\theta_1}},\ [8(M_2+\widetilde{M}_2)]^{\frac{1}{1-\theta_2}},\ 24a_0(M_3+\widetilde{M}_3),\\ &[24a_1(M_3+\widetilde{M}_3)]^{\frac{1}{1-l_1}},\ [24a_2(M_3+\widetilde{M}_3)]^{\frac{1}{1-l_2}},\ 24b_0(M_4+\widetilde{M}_4),\\ &[24b_1(M_4+\widetilde{M}_4)]^{\frac{1}{1-m_1}},\ [24b_2(M_4+\widetilde{M}_4)]^{\frac{1}{1-m_2}},\ [8(N_1+\widetilde{N}_1)]^{\frac{1}{1-\theta_1}},\\ &[8(N_2+\widetilde{N}_2)]^{\frac{1}{1-\theta_2}},\ 24a_0(N_3+\widetilde{N}_3),\ [24a_1(N_3+\widetilde{N}_3)]^{\frac{1}{1-l_1}},\\ &[24a_2(N_3+\widetilde{N}_3)]^{\frac{1}{1-l_2}},\ 24b_0(N_4+\widetilde{N}_4),\ [24b_1(N_4+\widetilde{N}_4)]^{\frac{1}{1-m_1}},\\ &[24b_2(N_4+\widetilde{N}_4)]^{\frac{1}{1-m_2}}\Big\}.\end{aligned}$$

Let us first show that $T:\bar{B}_R\to\bar{B}_R$. For $(u,v)\in\bar{B}_R$, it follows by Lemma 2.2 that

$$
\begin{aligned}
&|T_1(u,v)(t)| \le |\varphi(v)| \left(1+\frac{1}{|\Delta|}+\frac{\lambda\mu\xi^{\gamma+m-1}\eta^{\delta}(m-1)!}{|\Delta|\Gamma(\delta+1)\Gamma(\gamma+m)}\right)\\
&+\frac{|\psi(u)|\lambda\xi^{\gamma}}{|\Delta|}\left(\frac{1}{\Gamma(\gamma+1)}+\frac{\xi^{m-1}(m-1)!}{\Gamma(\gamma+m)}\right)\\
&+|I_{0+}^{\alpha}f(t,v(t),{}^{c}D_{0+}^{p}v(t))|+\frac{1}{|\Delta|}\Big[|I_{0+}^{\alpha}f(t,v(t),{}^{c}D_{0+}^{p}v(t))|_{t=1}|\\
&+\frac{\lambda\mu\xi^{\gamma+m-1}(m-1)!}{\Gamma(\gamma+m)}|I_{0+}^{\alpha+\delta}f(t,v(t),{}^{c}D_{0+}^{p}v(t))|_{t=\eta}|\\
&+\lambda|I_{0+}^{\beta+\gamma}g(t,u(t),{}^{c}D_{0+}^{q}u(t))|_{t=\xi}|\\
&+\frac{\lambda\xi^{\gamma+m-1}(m-1)!}{\Gamma(\gamma+m)}|I_{0+}^{\beta}g(t,u(t),{}^{c}D_{0+}^{q}u(t))|_{t=1}|\Big]\\
&\le L_1R^{\theta_1}\left(1+\frac{1}{|\Delta|}+\frac{\lambda\mu\xi^{\gamma+m-1}\eta^{\delta}(m-1)!}{|\Delta|\Gamma(\delta+1)\Gamma(\gamma+m)}\right)\\
&+\frac{L_2R^{\theta_2}\lambda\xi^{\gamma}}{|\Delta|}\left(\frac{1}{\Gamma(\gamma+1)}+\frac{\xi^{m-1}(m-1)!}{\Gamma(\gamma+m)}\right)+(a_0+a_1R^{l_1}+a_2R^{l_2})\\
&\times\left(\frac{1}{\Gamma(\alpha+1)}+\frac{1}{|\Delta|\Gamma(\alpha+1)}+\frac{\lambda\mu\xi^{\gamma+m-1}\eta^{\alpha+\delta}(m-1)!}{|\Delta|\Gamma(\gamma+m)\Gamma(\alpha+\delta+1)}\right)\\
&+(b_0+b_1R^{m_1}+b_2R^{m_2})\left(\frac{\lambda\xi^{\beta+\gamma}}{|\Delta|\Gamma(\beta+\gamma+1)}+\frac{\lambda\xi^{\gamma+m-1}(m-1)!}{|\Delta|\Gamma(\gamma+m)\Gamma(\beta+1)}\right)\\
&=M_1R^{\theta_1}+M_2R^{\theta_2}+M_3(a_0+a_1R^{l_1}+a_2R^{l_2})\\
&+M_4(b_0+b_1R^{m_1}+b_2R^{m_2}),\ \forall\, t\in[0,1].
\end{aligned}
$$

On the other hand, by Definition 2.2, we have

$$
{}^{c}D_{0+}^{q}T_1(u,v)(t)=\frac{1}{\Gamma(1-q)}\int_0^t\frac{(T_1(x,y))'(s)}{(t-s)^q}\,ds,
$$

where

$$
\begin{aligned}
&(T_1(u,v))'(t)=\varphi(v)\left(-\frac{(n-1)t^{n-2}}{\Delta}+\frac{(n-1)t^{n-2}\lambda\mu\xi^{\gamma+m-1}\eta^{\delta}(m-1)!}{\Delta\Gamma(\delta+1)\Gamma(\gamma+m)}\right)\\
&+\psi(u)\frac{(n-1)t^{n-2}\lambda\xi^{\gamma}}{\Delta}\left(\frac{1}{\Gamma(\gamma+1)}-\frac{\xi^{m-1}(m-1)!}{\Gamma(\gamma+m)}\right)\\
&+I_{0+}^{\alpha-1}f(t,v(t),{}^{c}D_{0+}^{p}v(t))+\frac{(n-1)t^{n-2}}{\Delta}\Big[-I_{0+}^{\alpha}f(t,v(t),{}^{c}D_{0+}^{p}v(t))|_{t=1}\\
&+\frac{\lambda\mu\xi^{\gamma+m-1}(m-1)!}{\Gamma(\gamma+m)}I_{0+}^{\alpha+\delta}f(t,v(t),{}^{c}D_{0+}^{p}v(t))|_{t=\eta}\\
&+\lambda I_{0+}^{\beta+\gamma}g(t,u(t),{}^{c}D_{0+}^{q}u(t))|_{t=\xi}\\
&-\frac{\lambda\xi^{\gamma+m-1}(m-1)!}{\Gamma(\gamma+m)}I_{0+}^{\beta}g(t,u(t),{}^{c}D_{0+}^{q}u(t))|_{t=1}\Big],\ \forall\, t\in(0,1).
\end{aligned}
$$

Then, by Lemma 2.2, we obtain

$$\begin{aligned}
&|(T_1(u,v))'(t)| \le |\varphi(v)| \left(\frac{n-1}{|\Delta|} + \frac{(n-1)\lambda\mu\xi^{\gamma+m-1}\eta^{\delta}(m-1)!}{|\Delta|\Gamma(\delta+1)\Gamma(\gamma+m)} \right) \\
&+ \frac{|\psi(u)|(n-1)\lambda\xi^{\gamma}}{|\Delta|} \left(\frac{1}{\Gamma(\gamma+1)} + \frac{\xi^{m-1}(m-1)!}{\Gamma(\gamma+m)} \right) \\
&+ |I_{0+}^{\alpha-1} f(t, v(t), {}^cD_{0+}^{p} v(t))| + \frac{n-1}{|\Delta|} \Bigg[|I_{0+}^{\alpha} f(t, v(t), {}^cD_{0+}^{p} v(t))|_{t=1}| \\
&+ \frac{\lambda\mu\xi^{\gamma+m-1}(m-1)!}{\Gamma(\gamma+m)} |I_{0+}^{\alpha+\delta} f(t, v(t), {}^cD_{0+}^{p} v(t))|_{t=\eta}| \\
&+ \lambda |I_{0+}^{\beta+\gamma} g(t, u(t), {}^cD_{0+}^{q} u(t))|_{t=\xi}| \\
&+ \frac{\lambda\xi^{\gamma+m-1}(m-1)!}{\Gamma(\gamma+m)} |I_{0+}^{\beta} g(t, u(t), {}^cD_{0+}^{q} u(t))|_{t=1}| \Bigg]
\end{aligned}$$

$$\begin{aligned}
&\le L_1 R^{\theta_1} \left(\frac{n-1}{|\Delta|} + \frac{(n-1)\lambda\mu\xi^{\gamma+m-1}\eta^{\delta}(m-1)!}{|\Delta|\Gamma(\delta+1)\Gamma(\gamma+m)} \right) \\
&+ \frac{L_2 R^{\theta_2}(n-1)\lambda\xi^{\gamma}}{|\Delta|} \left(\frac{1}{\Gamma(\gamma+1)} + \frac{\xi^{m-1}(m-1)!}{\Gamma(\gamma+m)} \right) + (a_0 + a_1 R^{l_1} + a_2 R^{l_2}) \\
&\times \left(\frac{1}{\Gamma(\alpha)} + \frac{n-1}{|\Delta|\Gamma(\alpha+1)} + \frac{(n-1)\lambda\mu\xi^{\gamma+m-1}\eta^{\alpha+\delta}(m-1)!}{|\Delta|\Gamma(\gamma+m)\Gamma(\alpha+\delta+1)} \right) \\
&+ (b_0 + b_1 R^{m_1} + b_2 R^{m_2}) \left(\frac{(n-1)\lambda\xi^{\beta+\gamma}}{|\Delta|\Gamma(\beta+\gamma+1)} + \frac{(n-1)\lambda\xi^{\gamma+m-1}(m-1)!}{|\Delta|\Gamma(\gamma+m)\Gamma(\beta+1)} \right),
\end{aligned}$$

for all $t \in (0,1)$. In consequence, we obtain

$$\begin{aligned}
&|{}^cD_{0+}^{q} T_1(u,v)(t)| \le \frac{1}{\Gamma(2-q)} \|(T_1(u,v))'\| \\
&\le \frac{L_1 R^{\theta_1}}{\Gamma(2-q)} \left(\frac{n-1}{|\Delta|} + \frac{(n-1)\lambda\mu\xi^{\gamma+m-1}\eta^{\delta}(m-1)!}{|\Delta|\Gamma(\delta+1)\Gamma(\gamma+m)} \right) \\
&+ \frac{L_2 R^{\theta_2}(n-1)\lambda\xi^{\gamma}}{|\Delta|\Gamma(2-q)} \left(\frac{1}{\Gamma(\gamma+1)} + \frac{\xi^{m-1}(m-1)!}{\Gamma(\gamma+m)} \right) \\
&+ \frac{1}{\Gamma(2-q)} (a_0 + a_1 R^{l_1} + a_2 R^{l_2}) \left(\frac{1}{\Gamma(\alpha)} + \frac{n-1}{|\Delta|\Gamma(\alpha+1)} \right. \\
&\left. + \frac{(n-1)\lambda\mu\xi^{\gamma+m-1}\eta^{\alpha+\delta}(m-1)!}{|\Delta|\Gamma(\gamma+m)\Gamma(\alpha+\delta+1)} \right) + \frac{1}{\Gamma(2-q)} (b_0 + b_1 R^{m_1} + b_2 R^{m_2}) \\
&\times \left(\frac{(n-1)\lambda\xi^{\beta+\gamma}}{|\Delta|\Gamma(\beta+\gamma+1)} + \frac{(n-1)\lambda\xi^{\gamma+m-1}(m-1)!}{|\Delta|\Gamma(\gamma+m)\Gamma(\beta+1)} \right) \\
&= \widetilde{M}_1 R^{\theta_1} + \widetilde{M}_2 R^{\theta_2} + \widetilde{M}_3 (a_0 + a_1 R^{l_1} + a_2 R^{l_2}) + \widetilde{M}_4 (b_0 + b_1 R^{m_1} + b_2 R^{m_2}),
\end{aligned}$$

for all $t \in [0,1]$. Thus we have

$$\begin{aligned}
&\|T_1(u,v)\|_X = \|T_1(u,v)\| + \|{}^cD_{0+}^{q} T_1(u,v)\| \\
&\le (M_1 + \widetilde{M}_1) R^{\theta_1} + (M_2 + \widetilde{M}_2) R^{\theta_2} + (M_3 + \widetilde{M}_3)(a_0 + a_1 R^{l_1} + a_2 R^{l_2}) \\
&+ (M_4 + \widetilde{M}_4)(b_0 + b_1 R^{m_1} + b_2 R^{m_2}) \le \tfrac{R}{8} + \tfrac{R}{8} + \tfrac{R}{8} + \tfrac{R}{8} = \tfrac{R}{2}.
\end{aligned}$$

In a similar manner, we can obtain

$$\begin{aligned}&\|T_2(u,v)\|_Y = \|T_2(u,v)\| + \|{}^cD^p_{0+}T_2(u,v)\|\\
&\le (N_1+\widetilde{N}_1)R^{\theta_1} + (N_2+\widetilde{N}_2)R^{\theta_2} + (N_3+\widetilde{N}_3)(a_0+a_1R^{l_1}+a_2R^{l_2})\\
&+(N_4+\widetilde{N}_4)(b_0+b_1R^{m_1}+b_2R^{m_2}) \le \tfrac{R}{8}+\tfrac{R}{8}+\tfrac{R}{8}+\tfrac{R}{8}=\tfrac{R}{2}.\end{aligned}$$

By the foregoing arguments, we deduce that

$$\|T(u,v)\|_{X\times Y} = \|T_1(u,v)\|_X + \|T_2(u,v)\|_Y \le \frac{R}{2}+\frac{R}{2} = R,$$

which implies that $T : \bar{B}_R \to \bar{B}_R$.

From the continuity of the functions f, g, φ and ψ, we can easily show that the operator T is continuous.

Next we will show that the operator $T : \bar{B}_R \to \bar{B}_R$ is equicontinuous. We denote

$$\Lambda_1 = \max_{t\in[0,1],\, |u|\le R,\, |v|\le R} |f(t,u,v)|$$

and

$$\Lambda_2 = \max_{t\in[0,1],\, |u|\le R,\, |v|\le R} |g(t,u,v)|.$$

For any $(u,v) \in \bar{B}_R$ and $t_1,\, t_2 \in [0,1]$, $t_1 < t_2$, we have

$$\begin{aligned}&|T_1(u,v)(t_2) - T_1(u,v)(t_1)|\\
&\le \left|\varphi(v)\left(1 - \frac{t_2^{n-1}}{\Delta} + \frac{t_2^{n-1}\lambda\mu\xi^{\gamma+m-1}\eta^\delta(m-1)!}{\Delta\Gamma(\delta+1)\Gamma(\gamma+m)}\right.\right.\\
&\left.\left.-\ 1 + \frac{t_1^{n-1}}{\Delta} - \frac{t_1^{n-1}\lambda\mu\xi^{\gamma+m-1}\eta^\delta(m-1)!}{\Delta\Gamma(\delta+1)\Gamma(\gamma+m)}\right)\right|\\
&+\left|\psi(u)\lambda\xi^\gamma\frac{t_2^{n-1}-t_1^{n-1}}{\Delta}\left(\frac{1}{\Gamma(\gamma+1)} - \frac{\xi^{m-1}(m-1)!}{\Gamma(\gamma+m)}\right)\right|\\
&+\left|I^\alpha_{0+}f(t,v(t),{}^cD^p_{0+}v(t))|_{t=t_2} - I^\alpha_{0+}f(t,v(t),{}^cD^p_{0+}v(t))|_{t=t_1}\right|\\
&+\frac{t_2^{n-1}-t_1^{n-1}}{|\Delta|}\Bigg[|I^\alpha_{0+}f(t,v(t),{}^cD^p_{0+}v(t))|_{t=1}|\\
&+\frac{\lambda\mu\xi^{\gamma+m-1}(m-1)!}{\Gamma(\gamma+m)}|I^{\alpha+\delta}_{0+}f(t,v(t),{}^cD^p_{0+}v(t))|_{t=\eta}|\\
&+\lambda|I^{\beta+\gamma}_{0+}g(t,u(t),{}^cD^q_{0+}u(t))|_{t=\xi}|\\
&+\frac{\lambda\xi^{\gamma+m-1}(m-1)!}{\Gamma(\gamma+m)}|I^\beta_{0+}g(t,u(t),{}^cD^q_{0+}u(t))|_{t=1}|\Bigg]\\
&\le L_1R^{\theta_1}\frac{t_2^{n-1}-t_1^{n-1}}{|\Delta|}\left(1+\frac{\lambda\mu\xi^{\gamma+m-1}\eta^\delta(m-1)!}{\Gamma(\delta+1)\Gamma(\gamma+m)}\right)\\
&+L_2R^{\theta_2}\frac{t_2^{n-1}-t_1^{n-1}}{|\Delta|}\lambda\xi^\gamma\left(\frac{1}{\Gamma(\gamma+1)}+\frac{\xi^{m-1}(m-1)!}{\Gamma(\gamma+m)}\right)\\
&+\frac{1}{\Gamma(\alpha)}\left|\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]f(s,v(s),{}^cD^p_{0+}v(s))\,ds\right|\end{aligned}$$

$$+\frac{1}{\Gamma(\alpha)}\left|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}f(s,v(s),{}^cD_{0+}^p v(s))\,ds\right|$$
$$+\frac{t_2^{n-1}-t_1^{n-1}}{|\Delta|}\left[\frac{\Lambda_1}{\Gamma(\alpha+1)}+\frac{\lambda\mu\xi^{\gamma+m-1}\eta^{\alpha+\delta}(m-1)!\Lambda_1}{\Gamma(\gamma+m)\Gamma(\alpha+\delta+1)}\right.$$
$$\left.+\frac{\lambda\xi^{\beta+\gamma}\Lambda_2}{\Gamma(\beta+\gamma+1)}+\frac{\lambda\xi^{\gamma+m-1}(m-1)!\Lambda_2}{\Gamma(\gamma+m)\Gamma(\beta+1)}\right],$$

which further implies that

$$\begin{aligned}
&|T_1(u,v)(t_2)-T_1(u,v)(t_1)|\\
&\leq L_1R^{\theta_1}\frac{t_2^{n-1}-t_1^{n-1}}{|\Delta|}\left(1+\frac{\lambda\mu\xi^{\gamma+m-1}\eta^{\delta}(m-1)!}{\Gamma(\delta+1)\Gamma(\gamma+m)}\right)\\
&+L_2R^{\theta_2}\lambda\xi^{\gamma}\frac{t_2^{n-1}-t_1^{n-1}}{|\Delta|}\left(\frac{1}{\Gamma(\gamma+1)}+\frac{\xi^{m-1}(m-1)!}{\Gamma(\gamma+m)}\right)\\
&+\frac{\Lambda_1}{\Gamma(\alpha+1)}(t_2^{\alpha}-t_1^{\alpha})+\frac{t_2^{n-1}-t_1^{n-1}}{|\Delta|}\\
&\times\left[\frac{\Lambda_1}{\Gamma(\alpha+1)}+\frac{\lambda\mu\xi^{\gamma+m-1}\eta^{\alpha+\delta}(m-1)!\Lambda_1}{\Gamma(\gamma+m)\Gamma(\alpha+\delta+1)}\right.\\
&\left.+\frac{\lambda\xi^{\beta+\gamma}\Lambda_2}{\Gamma(\beta+\gamma+1)}+\frac{\lambda\xi^{\gamma+m-1}(m-1)!\Lambda_2}{\Gamma(\gamma+m)\Gamma(\beta+1)}\right].
\end{aligned}\tag{3.1}$$

On the other hand, we obtain

$$\begin{aligned}
&|{}^cD_{0+}^q T_1(u,v)(t_2)-{}^cD_{0+}^q T_1(u,v)(t_1)|\\
&=\frac{1}{\Gamma(1-q)}\left|\int_0^{t_2}\frac{(T_1(u,v))'(s)}{(t_2-s)^q}\,ds-\int_0^{t_1}\frac{(T_1(u,v))'(s)}{(t_1-s)^q}\,ds\right|\\
&\leq\frac{1}{\Gamma(1-q)}\left|\int_0^{t_1}\left[\frac{1}{(t_2-s)^q}-\frac{1}{(t_1-s)^q}\right](T_1(u,v))'(s)\,ds\right|\\
&+\frac{1}{\Gamma(1-q)}\left|\int_{t_1}^{t_2}\frac{(T_1(u,v))'(s)}{(t_2-s)^q}\,ds\right|.
\end{aligned}$$

Because

$$\begin{aligned}
&|(T_1(u,v))'(t)|\leq L_1R^{\theta_1}\left(\frac{n-1}{|\Delta|}+\frac{(n-1)\lambda\mu\xi^{\gamma+m-1}\eta^{\delta}(m-1)!}{|\Delta|\Gamma(\delta+1)\Gamma(\gamma+m)}\right)\\
&+L_2R^{\theta_2}\lambda\xi^{\gamma}\frac{n-1}{|\Delta|}\left(\frac{1}{\Gamma(\gamma+1)}+\frac{\xi^{m-1}(m-1)!}{\Gamma(\gamma+m)}\right)\\
&+\frac{\Lambda_1}{\Gamma(\alpha)}+\frac{n-1}{|\Delta|}\left[\frac{\Lambda_1}{\Gamma(\alpha+1)}+\frac{\lambda\mu\xi^{\gamma+m-1}\eta^{\alpha+\delta}(m-1)!\Lambda_1}{\Gamma(\gamma+m)\Gamma(\alpha+\delta+1)}\right.\\
&\left.+\frac{\lambda\xi^{\beta+\gamma}\Lambda_2}{\Gamma(\beta+\gamma+1)}+\frac{\lambda\xi^{\gamma+m-1}(m-1)!\Lambda_2}{\Gamma(\gamma+m)\Gamma(\beta+1)}\right]=:D_1,\ \forall\,t\in(0,1),
\end{aligned}$$

we deduce by the above inequalities that

$$|{}^cD_{0+}^q T_1(u,v)(t_2)-{}^cD_{0+}^q T_1(u,v)(t_1)|$$

$$\begin{aligned}
&\leq \frac{D_1}{\Gamma(1-q)}\left[\int_0^{t_1}[(t_1-s)^{-q}-(t_2-s)^{-q}]\,ds+\int_{t_1}^{t_2}(t_2-s)^{-q}\,ds\right] \\
&= \frac{D_1}{\Gamma(2-q)}[2(t_2-t_1)^{1-q}-t_2^{1-q}+t_1^{1-q}].
\end{aligned} \tag{3.2}$$

In a similar manner, we find that

$$\begin{aligned}
&|T_2(u,v)(t_2)-T_2(u,v)(t_1)| \\
&\leq L_1R^{\theta_1}\mu\eta^{\delta}\frac{t_2^{m-1}-t_1^{m-1}}{|\Delta|}\left(\frac{1}{\Gamma(\delta+1)}+\frac{\eta^{n-1}(n-1)!}{\Gamma(\delta+n)}\right) \\
&+L_2R^{\theta_2}\frac{t_2^{m-1}-t_1^{m-1}}{|\Delta|}\left(1+\frac{\lambda\mu\xi^{\gamma}\eta^{\delta+n-1}(n-1)!}{\Gamma(\gamma+1)\Gamma(\delta+n)}\right)
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
&+\frac{\Lambda_2}{\Gamma(\beta)}(t_2^{\beta}-t_1^{\beta})+\frac{t_2^{m-1}-t_1^{m-1}}{|\Delta|}\left[\frac{\mu\eta^{\alpha+\delta}\Lambda_1}{\Gamma(\alpha+\delta+1)}+\frac{\mu\eta^{\delta+n-1}(n-1)!\Lambda_1}{\Gamma(\delta+n)\Gamma(\alpha+1)}\right. \\
&\left.+\frac{\Lambda_2}{\Gamma(\beta+1)}+\frac{\lambda\mu\eta^{\delta+n-1}\xi^{\beta+\gamma}(n-1)!\Lambda_2}{\Gamma(\delta+n)\Gamma(\beta+\gamma+1)}\right].
\end{aligned}$$

Using the estimate

$$\begin{aligned}
&|(T_2(u,v))'(t)|\leq L_1R^{\theta_1}\mu\eta^{\delta}\frac{m-1}{|\Delta|}\left(\frac{1}{\Gamma(\delta+1)}+\frac{\eta^{n-1}(n-1)!}{\Gamma(\delta+n)}\right) \\
&+L_2R^{\theta_2}\frac{m-1}{|\Delta|}\left(1+\frac{\lambda\mu\xi^{\gamma}\eta^{\delta+n-1}(n-1)!}{\Gamma(\gamma+1)\Gamma(\delta+n)}\right) \\
&+\frac{\Lambda_2}{\Gamma(\beta)}+\frac{m-1}{|\Delta|}\left[\frac{\mu\eta^{\alpha+\delta}\Lambda_1}{\Gamma(\alpha+\delta+1)}+\frac{\mu\eta^{\delta+n-1}(n-1)!\Lambda_1}{\Gamma(\delta+n)\Gamma(\alpha+1)}\right. \\
&\left.+\frac{\Lambda_2}{\Gamma(\beta+1)}+\frac{\lambda\mu\eta^{\delta+n-1}\xi^{\beta+\gamma}(n-1)!\Lambda_2}{\Gamma(\delta+n)\Gamma(\beta+\gamma+1)}\right]=:D_2,\ \forall\, t\in(0,1),
\end{aligned}$$

we get

$$\begin{aligned}
&|{}^cD_{0+}^{p}T_2(u,v)(t_2)-{}^cD_{0+}^{p}T_2(u,v)(t_1)| \\
&\leq \frac{D_2}{\Gamma(2-p)}[2(t_2-t_1)^{1-p}-t_2^{1-p}+t_1^{1-p}].
\end{aligned} \tag{3.4}$$

By the relations (3.1)-(3.4), we deduce that $T:\bar{B}_R\to\bar{B}_R$ is equicontinuous. Thus the Arzela-Ascoli theorem applies and that the set $T(\bar{B}_R)$ is relatively compact, and thus T is a completely continuous operator. Therefore, by Schauder fixed point theorem (Theorem 2.1), we deduce that the operator T has at least one fixed point (u,v) in $\bar{B}_R$, which is a solution of problem $(S)-(BC)$. □

THEOREM 3.2. *Assume that* $(H1)$, $(H3)$, $(H4)$ *and* $(H5)$ *hold. Then problem* $(S)-(BC)$ *has at least one solution on* $[0,1]$.

P r o o f. With L_0 given by $(H5)$, we consider a set $\bar{B}_{L_0} = \{(u,v) \in X \times Y,\ \|(u,v)\|_{X\times Y} \le L_0\}$ and show that $T(\bar{B}_{L_0}) \subset \bar{B}_{L_0}$. For $(u,v) \in \bar{B}_{L_0}$ and $t \in [0,1]$, we obtain

$$
\begin{aligned}
&|T_1(u,v)(t)| \le L_1 L_0^{\theta_1}\left(1 + \frac{1}{|\Delta|} + \frac{\lambda\mu\xi^{\gamma+m-1}\eta^{\delta}(m-1)!}{|\Delta|\Gamma(\delta+1)\Gamma(\gamma+m)}\right)\\
&+L_2 L_0^{\theta_2}\frac{\lambda\xi^{\gamma}}{|\Delta|}\left(\frac{1}{\Gamma(\gamma+1)} + \frac{\xi^{m-1}(m-1)!}{\Gamma(\gamma+m)}\right) + (c_0 + c_1 h_1(L_0) + c_2 h_2(L_0))\\
&\times\left(\frac{1}{\Gamma(\alpha+1)} + \frac{1}{|\Delta|\Gamma(\alpha+1)} + \frac{\lambda\mu\xi^{\gamma+m-1}\eta^{\alpha+\delta}(m-1)!}{|\Delta|\Gamma(\gamma+m)\Gamma(\alpha+\delta+1)}\right)\\
&+(d_0 + d_1 k_1(L_0) + d_2 k_2(L_0))\left(\frac{\lambda\xi^{\beta+\gamma}}{|\Delta|\Gamma(\beta+\gamma+1)} + \frac{\lambda\xi^{\gamma+m-1}(m-1)!}{|\Delta|\Gamma(\gamma+m)\Gamma(\beta+1)}\right)\\
&= M_1 L_0^{\theta_1} + M_2 L_0^{\theta_2} + M_3(c_0 + c_1 h_1(L_0) + c_2 h_2(L_0))\\
&+M_4(d_0 + d_1 k_1(L_0) + d_2 k_2(L_0)),
\end{aligned}
$$

and

$$
\begin{aligned}
&|{}^cD_{0+}^q T_1(u,v)(t)| \le \frac{L_1 L_0^{\theta_1}}{\Gamma(2-q)}\left(\frac{n-1}{|\Delta|} + \frac{(n-1)\lambda\mu\xi^{\gamma+m-1}\eta^{\delta}(m-1)!}{|\Delta|\Gamma(\delta+1)\Gamma(\gamma+m)}\right)\\
&+\frac{L_2 L_0^{\theta_2}(n-1)\lambda\xi^{\gamma}}{|\Delta|\Gamma(2-q)}\left(\frac{1}{\Gamma(\gamma+1)} + \frac{\xi^{m-1}(m-1)!}{\Gamma(\gamma+m)}\right)\\
&+\frac{1}{\Gamma(2-q)}(c_0 + c_1 h_1(L_0) + c_2 h_2(L_0))\left(\frac{1}{\Gamma(\alpha)} + \frac{n-1}{|\Delta|\Gamma(\alpha+1)}\right.\\
&\left.+\frac{(n-1)\lambda\mu\xi^{\gamma+m-1}\eta^{\alpha+\gamma}(m-1)!}{|\Delta|\Gamma(\gamma+m)\Gamma(\alpha+\delta+1)}\right) + \frac{1}{\Gamma(2-q)}(d_0 + d_1 k_1(L_0) + d_2 k_2(L_0))\\
&\times\left(\frac{(n-1)\lambda\xi^{\beta+\gamma}}{|\Delta|\Gamma(\beta+\gamma+1)} + \frac{(n-1)\lambda\xi^{\gamma+m-1}(m-1)!}{|\Delta|\Gamma(\gamma+m)\Gamma(\beta+1)}\right)\\
&= \widetilde{M}_1 L_0^{\theta_1} + \widetilde{M}_2 L_0^{\theta_2} + \widetilde{M}_3(c_0 + c_1 h_1(L_0) + c_2 h_2(L_0))\\
&+\widetilde{M}_4(d_0 + d_1 k_1(L_0) + d_2 k_2(L_0)).
\end{aligned}
$$

In view of the above estimates, we find that

$$
\begin{aligned}
&\|T_1(u,v)\|_X = \|T_1(u,v)\| + \|{}^cD_{0+}^q T_1(u,v)\| \le (M_1 + \widetilde{M}_1)L_0^{\theta_1}\\
&+(M_2 + \widetilde{M}_2)L_0^{\theta_2} + (M_3 + \widetilde{M}_3)(c_0 + c_1 h_1(L_0) + c_2 h_2(L_0))\\
&+(M_4 + \widetilde{M}_4)(d_0 + d_1 k_1(L_0) + d_2 k_2(L_0)).
\end{aligned}
\tag{3.5}
$$

In a similar manner we obtain

$$
\begin{aligned}
&\|T_2(u,v)\|_Y \le (N_1 + \widetilde{N}_1)L_0^{\theta_1} + (N_2 + \widetilde{N}_2)L_0^{\theta_2} + (N_3 + \widetilde{N}_3)\\
&\times(c_0 + c_1 h_1(L_0) + c_2 h_2(L_0)) + (N_4 + \widetilde{N}_4)(d_0 + d_1 k_1(L_0) + d_2 k_2(L_0)).
\end{aligned}
\tag{3.6}
$$

Using (3.5) and (3.6), we get

$$\begin{aligned}
&\|T(u,v)\|_{X\times Y} \le (M_1+\widetilde{M}_1+N_1+\widetilde{N}_1)L_0^{\theta_1} + (M_2+\widetilde{M}_2+N_2+\widetilde{N}_2)L_0^{\theta_2} \\
&+(M_3+\widetilde{M}_3+N_3+\widetilde{N}_3)(c_0+c_1h_1(L_0)+c_2h_2(L_0)) \\
&+(M_4+\widetilde{M}_4+N_4+\widetilde{N}_4)(d_0+d_1k_1(L_0)+d_2k_2(L_0)) \\
&= (M_1+\widetilde{M}_1+N_1+\widetilde{N}_1)L_0^{\theta_1} + (M_2+\widetilde{M}_2+N_2+\widetilde{N}_2)L_0^{\theta_2} \\
&+A_1(M_3+\widetilde{M}_3+N_3+\widetilde{N}_3)+A_2(M_4+\widetilde{M}_4+N_4+\widetilde{N}_4) < L_0,
\end{aligned}$$

which shows that $T(\bar{B}_{L_0}) \subset \bar{B}_{L_0}$. As argued in the proof of Theorem 3.1, it can be shown that the operator T is completely continuous.

Next we suppose that there exists $(u,v) \in \partial B_{L_0}$ such that $(u,v) = \nu T(u,v)$ for some $\nu \in (0,1)$. Then

$$\begin{aligned}
&\|(u,v)\|_{X\times Y} \le \|T(u,v)\|_{X\times Y} \le (M_1+\widetilde{M}_1+N_1+\widetilde{N}_1)L_0^{\theta_1} \\
&+(M_2+\widetilde{M}_2+N_2+\widetilde{N}_2)L_0^{\theta_2} + A_1(M_3+\widetilde{M}_3+N_3+\widetilde{N}_3) \\
&+A_2(M_4+\widetilde{M}_4+N_4+\widetilde{N}_4) < L_0,
\end{aligned}$$

which contradicts that $(u,v) \in \partial B_{L_0}$. Thus, by the nonlinear alternative of Leray-Schauder type (Theorem 2.2), we deduce that the operator T has a fixed point $(u,v) \in \bar{B}_{L_0}$, and so problem $(S)-(BC)$ has at least one solution on $[0,1]$. □

4. **Examples**

EXAMPLE 4.1. Letting $\alpha = 10/3$ $(n=3)$, $\beta = 9/2$ $(m=4)$, $p=3/4$, $q=1/2$, $\gamma = 5/3$, $\delta = 11/5$, $\xi = 1/2$, $\eta = 1/3$, $\lambda = 1$ and $\mu = 2$, we consider the following system of fractional differential equations

$$\begin{cases}
{}^cD_{0+}^{10/3}u(t) = \dfrac{1}{\sqrt{4+t^2}}\sin t + \dfrac{1}{4}(v(t))^{1/3} - \dfrac{1}{3(1+t)}\left({}^cD_{0+}^{3/4}v(t)\right)^{2/5}, \\
{}^cD_{0+}^{9/2}v(t) = \dfrac{e^{-t}}{1+t^2} - \dfrac{1}{2}(u(t))^{2/3} + \dfrac{1}{5(3+t)}\arctan\left({}^cD_{0+}^{1/2}u(t)\right)^{1/7},
\end{cases} \tag{S_0}$$

for $t \in (0,1)$, supplemented with the boundary conditions

$$\begin{cases}
u(0) = 2\left(\int_0^1 v(t)\,ds\right)^{1/5}, \ u'(0)=0, \ u(1) = I_{0+}^{5/3}v(1/2), \\
v(0) = 4\left(\int_0^1 u(t)\,dt\right)^{1/3}, \ v'(0)=v''(0)=0, \ v(1) = 2I_{0+}^{11/5}u(1/3).
\end{cases} \tag{BC_0}$$

With the given data, it is found that $\Delta \approx 0.9999958$ $(\Delta \neq 0)$ and

$$\begin{aligned}
|f(t,u,v)| &= \left|\tfrac{1}{\sqrt{4+t^2}}\sin t + \tfrac{1}{4}u^{1/3} - \tfrac{1}{3(1+t)}v^{2/5}\right| \le \tfrac{1}{2} + \tfrac{1}{4}|u|^{1/3} + \tfrac{1}{3}|v|^{2/5}, \\
|g(t,u,v)| &= \left|\tfrac{e^{-t}}{1+t^2} - \tfrac{1}{2}u^{2/3} + \tfrac{1}{5(3+t)}\arctan(v^{1/7})\right| \le 1 + \tfrac{1}{2}|u|^{2/3} + \tfrac{1}{15}|v|^{1/7},
\end{aligned}$$

for all $t \in [0,1]$, $u, v \in \mathbb{R}$ and $\varphi(0) = \psi(0) = 0$, $|\varphi(v)| \le 2\|v\|^{1/5}$, $|\psi(u)| \le 4\|u\|^{1/3}$, where

$$\varphi(v) = 2\left(\int_0^1 v(t)\,dt\right)^{1/5}, \ \psi(u) = 4\left(\int_0^1 u(t)\,dt\right)^{1/3}, \ \forall\, u, v \in C[0,1].$$

Here $l_1 = 1/3$, $l_2 = 2/5$, $m_1 = 2/3$, $m_2 = 1/7$, $\theta_1 = 1/5$, $\theta_2 = 1/3$, $L_1 = 2$ and $L_2 = 4$. Clearly the assumptions $(H1) - (H3)$ are satisfied. Thus, by Theorem 3.1, we deduce that the problem $(S_0) - (BC_0)$ has at least one solution on $[0,1]$.

Example 4.2. Let us choose $\alpha = 5/2$ ($n = 2$), $\beta = 11/3$ ($m = 3$), $p = 1/2$, $q = 1/3$, $\gamma = 9/4$, $\delta = 7/2$, $\lambda = 2$, $\mu = 3$, $\xi = 1/5$, $\eta = 1/2$, and consider the system

$$\begin{cases} {}^cD_{0+}^{5/2}u(t) = \dfrac{(1+t)^3}{200} - \dfrac{v^4(t)}{500(1+|v(t)|)} + \dfrac{t^2}{400}\left({}^cD_{0+}^{1/2}v(t)\right)^3, \\ {}^cD_{0+}^{11/3}v(t) = \dfrac{(1-t)^2}{300} + \dfrac{(1-t)u^2(t)}{400(1+u^2(t))} - \dfrac{t^3}{100}\left({}^cD_{0+}^{1/3}u(t)\right)^4, \end{cases} \qquad (\widetilde{S}_0)$$

with the boundary conditions

$$\begin{cases} u(0) = \dfrac{1}{300}\left(\max\limits_{t\in[0,1]}|v(t)|\right)^{1/2}, \ u(1) = 2I_{0+}^{9/4}v(1/5), \\ v(0) = \dfrac{1}{200}\left(\max\limits_{t\in[0,1]}|u(t)|\right)^{1/4}, \ v'(0) = 0, \ v(1) = 3I_{0+}^{7/2}u(1/2), \end{cases} \qquad (\widetilde{BC}_0)$$

where

$$f(t,u,v) = \tfrac{(1+t)^3}{200} - \tfrac{u^4}{500(1+|u|)} + \tfrac{t^2v^3}{400},$$
$$g(t,u,v) = \tfrac{(1-t)^2}{300} + \tfrac{(1-t)u^2}{400(1+u^2)} - \tfrac{t^3v^4}{100}, \ \forall\, t\in[0,1], \ u, v \in \mathbb{R},$$
$$\varphi(x) = \frac{1}{300}\left(\max_{t\in[0,1]}|x(t)|\right)^{1/2}, \ \psi(x) = \frac{1}{200}\left(\max_{t\in[0,1]}|x(t)|\right)^{1/4}, \ \forall\, x \in C[0,1].$$

Obviously

$$|f(t,u,v)| \le \tfrac{1}{25} + \tfrac{1}{500}|u|^4 + \tfrac{1}{400}|v|^3, \ |g(t,u,v)| \le \tfrac{1}{300} + \tfrac{1}{400}|u|^2 + \tfrac{1}{100}|v|^4,$$
$$|\varphi(x)| \le \tfrac{1}{300}\|x\|^{1/2}, \ |\psi(x)| \le \tfrac{1}{200}\|x\|^{1/4},$$

for all $t \in [0,1]$, $u, v \in \mathbb{R}$, $x \in C[0,1]$ and that $h_1(x) = x^4$, $h_2(x) = x^3$, $k_1(x) = x^2$, $k_2(x) = x^4$. One can notice that $c_0 = \frac{1}{25}$, $c_1 = \frac{1}{500}$, $c_2 = \frac{1}{400}$, $d_0 = \frac{1}{300}$, $d_1 = \frac{1}{400}$, $d_2 = \frac{1}{100}$, $\theta_1 = \frac{1}{2}$, $\theta_2 = \frac{1}{4}$, $L_1 = \frac{1}{300}$, $L_2 = \frac{1}{200}$. Using the given values, we obtain $\Delta \approx 0.99999969$, $M_1 \approx 0.00666668$, $M_2 \approx 0.00010554$, $M_3 \approx 0.60180232$, $M_4 \approx 8.49974 \times 10^{-6}$, $\widetilde{M}_1 \approx 0.00369245$, $\widetilde{M}_2 \approx 0.00011691$, $\widetilde{M}_3 \approx 1.16661255$, $\widetilde{M}_4 \approx 9.41543 \times 10^{-6}$, $N_1 \approx 0.00008443$, $N_2 \approx 0.01000027$, $N_3 \approx 0.00082728$, $N_4 \approx 0.13594897$, $\widetilde{N}_1 \approx 0.00019054$, $\widetilde{N}_2 \approx 0.01128439$, $\widetilde{N}_3 \approx 0.00186696$, and $\widetilde{N}_4 \approx 0.43463894$. Taking $L_0 = 3$, we find that $A_1 \approx 0.26949999$ and $A_2 \approx 0.83583333$, and the assumption $(H5)$ holds true as

$$(M_1 + \widetilde{M_1} + N_1 + \widetilde{N}_1)L_0^{\theta_1} + (M_2 + \widetilde{M_2} + N_2 + \widetilde{N}_2)L_0^{\theta_2}$$
$$+A_1(M_3 + \widetilde{M_3} + N_3 + \widetilde{N}_3) + A_2(M_4 + \widetilde{M_4} + N_4 + \widetilde{N}_4) \approx 1.001 < 3.$$

Therefore, the conclusion of Theorem 3.2 applies and consequently problem $(\widetilde{S}_0) - (\widetilde{BC}_0)$ has at least one solution on $[0, 1]$.

References

[1] R. Agarwal, S. Hristova, D. O'Regan, Non-instantaneous impulses in Caputo fractional differential equations. *Fract. Calc. Appl. Anal.* **20**, No 3 (2017), 595–622; DOI: 10.1515/fca-2017-0032; https://www.degruyter.com/view/j/fca.2017.20.issue-3/issue-files/fca.2017.20.issue-3.xml.

[2] B. Ahmad, S.K. Ntouyas, Existence results for a coupled system of Caputo type sequential fractional differential equations with nonlocal integral boundary conditions. *Appl. Math. Comput.* **266** (2015), 615–622.

[3] B. Ahmad, S.K. Ntouyas, Existence results for Caputo type sequential fractional differential inclusions with nonlocal integral boundary conditions. *J. Appl. Math. Comput.* **50** (2016), 157–174.

[4] B. Ahmad, S. Ntouyas, A. Alsaedi, On a coupled system of fractional differential equations with coupled nonlocal and integral boundary conditions. *Chaos Solitons Fractals* **83** (2016), 234–241.

[5] A. Alsaedi, S.K. Ntouyas, R.P. Agarwal, B. Ahmad, On Caputo type sequential fractional differential equations with nonlocal integral boundary conditions. *Adv. Differ. Equ.* **2015**, No 33 (2015), 1–12.

[6] A.A.M. Arafa, S.Z. Rida, M. Khalil, Fractional modeling dynamics of HIV and $CD4^+$ T-cells during primary infection. *Nonlinear Biomed. Phys.* **6**, No 1 (2012), 1–7.

[7] D. Baleanu, K. Diethelm, E. Scalas, J.J. Trujillo, *Fractional Calculus Models and Numerical Methods*. Series on Complexity, Nonlinearity and Chaos. World Scientific, Boston (2012).

[8] K. Cole, Electric conductance of biological systems. In: *Proc. Cold Spring Harbor Symp. Quant. Biol.*, Col Springer Harbor Laboratory Press, New York (1993), 107–116.

[9] S. Das, *Functional Fractional Calculus for System Identification and Controls*. Springer, New York (2008).

[10] Y. Ding, H. Ye, A fractional-order differential equation model of HIV infection of $CD4^+$ T-cells. *Math. Comp. Model.* **50** (2009), 386–392.

[11] V. Djordjevic, J. Jaric, B. Fabry, J. Fredberg, D. Stamenovic, Fractional derivatives embody essential features of cell rheological behavior.

Ann. Biomed. Eng. **31** (2003), 692–699.

[12] Z.M. Ge, C.Y. Ou, Chaos synchronization of fractional order modified Duffing systems with parameters excited by a chaotic signal. *Chaos Solitons Fractals* **35** (2008), 705–717.

[13] M. Gunendi, I. Yaslan, Positive solutions of higher-order nonlinear multi-point fractional equations with integral boundary conditions. *Fract. Calc. Appl. Anal.* **19**, No 4 (2016), 989–1009; DOI: 10.1515/fca-2016-0054; https://www.degruyter.com/view/j/fca.2016.19.issue-4/issue-files/fca.2016.19.issue-4.xml.

[14] J. Henderson, R. Luca, *Boundary Value Problems for Systems of Differential, Difference and Fractional Equations. Positive Solutions.* Elsevier, Amsterdam (2016).

[15] J. Henderson, R. Luca, Existence of positive solutions for a singular fractional boundary value problem. *Nonlinear Anal. Model. Control* **22**, No 1 (2017), 99–114.

[16] J. Henderson, R. Luca, A. Tudorache, On a system of fractional differential equations with coupled integral boundary conditions. *Fract. Calc. Appl. Anal.* **18**, No 2 (2015), 361–386; DOI: 10.1515/fca-2015-0024; https://www.degruyter.com/view/j/fca.2015.18.issue-2/issue-files/fca.2015.18.issue-2.xml.

[17] J. Jiang, L. Liu, Y. Wu, Positive solutions to singular fractional differential system with coupled boundary conditions. *Comm. Nonlinear Sc. Num. Sim.* **18**, No 11 (2013), 3061–3074.

[18] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations.* North-Holland Mathematics Studies, 204, Elsevier Science B.V., Amsterdam (2006).

[19] J. Klafter, S.C. Lim, R. Metzler (Eds.), *Fractional Dynamics in Physics.* Singapore, World Scientific (2011).

[20] R. Luca, A. Tudorache, Positive solutions to a system of semipositone fractional boundary value problems. *Adv. Difference Equ.* **2014**, No 179 (2014), 1–11.

[21] R. Metzler, J. Klafter, The random walks guide to anomalous diffusion: a fractional dynamics approach. *Phys. Rep.* **339** (2000), 1–77.

[22] M. Ostoja-Starzewski, Towards thermoelasticity of fractal media. *J. Therm. Stress.* **30** (2007), 889–896.

[23] I. Podlubny, *Fractional Differential Equations.* Academic Press, San Diego (1999).

[24] Y.Z. Povstenko, *Fractional Thermoelasticity.* Springer, N. York (2015).

[25] T. Qi, Y. Liu, Y. Cui, Existence of solutions for a class of coupled fractional differential systems with nonlocal boundary conditions. *J. Funct. Spaces* **2017**, Article ID 6703860 (2017), 1–9.

[26] J. Sabatier, O.P. Agrawal, J.A.T. Machado (Eds.), *Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering*. Springer, Dordrecht (2007).

[27] S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*. Gordon and Breach, Yverdon (1993).

[28] C.S. Sin, L. Zheng, Existence and uniqueness of global solutions of Caputo-type fractional differential equations. *Fract. Calc. Appl. Anal.* **19** No 3 (2016), 765–774; DOI: 10.1515/fca-2016-0040; https://www.degruyter.com/view/j/fca.2016.19.issue-3/issue-files/fca.2016.19.issue-3.xml.

[29] I.M. Sokolov, J. Klafter, A. Blumen, A fractional kinetics. *Phys. Today* **55** (2002), 48–54.

[30] J.A. Tenreiro Machado, V. Kiryakova, The chronicles of fractional calculus. *Fract. Calc. Appl. Anal.* **20**, No 2 (2017), 307–336, DOI: 10.1515/fca-2017-0017; https://www.degruyter.com/view/j/fca.2017.20.issue-2/issue-files/fca.2017.20.issue-2.xml.

[31] Y. Wang, L. Liu, Y. Wu, Positive solutions for a class of higher-order singular semipositone fractional differential systems with coupled integral boundary conditions and parameters. *Adv. Differ. Equ.* **2014**, No 268 (2014), 1–24.

[32] S. Xie, Y. Xie, Positive solutions of higher-order nonlinear fractional differential systems with nonlocal boundary conditions. *J. Appl. Anal. Comput.* **6**, No 4 (2016), 1211–1227.

[33] L. Zhang, B. Ahmad, G. Wang, Existence and approximation of positive solutions for nonlinear fractional integro-differential boundary value problems on an unbounded domain. *Appl. Comput. Math.* **15** (2016), 149–158.

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Received: September 7, 2017

Please cite to this paper as published in:

Fract. Calc. Appl. Anal., Vol. **21**, No 2 (2018), pp. 423–441,
DOI: 10.1515/fca-2018-0024