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SHOPT PAPER

ANALYTIC APPROXIMATE SOLUTIONS FOR A CLASS OF VARIABLE ORDER FRACTIONAL DIFFERENTIAL EQUATIONS USING THE POLYNOMIAL LEAST SQUARES METHOD

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Abstract

In this paper a new way to compute analytic approximate polynomial solutions for a class of nonlinear variable order fractional differential equations is proposed, based on the Polynomial Least Squares Method (PLSM). In order to emphasize the accuracy and the efficiency of the method several examples are included.

MSC 2010: Primary 26A33; Secondary 34A08, 34E10, 41A10

Key Words and Phrases: nonlinear variable order fractional differential equation, approximate analytic polynomial solution, Polynomial Least Squares Method

1. Introduction

Fractional order differential equations are successfully used in the modeling of practical problems from various fields of physics and engineering such as biophysics, aerodynamics, electrical circuits etc. ([6], [7], [11]). Recently, variable order fractional differential equations proved their usefulness in fields such as anomalous diffusion ([2], [10], [13]), processing of geographical data ([4]), heterogeneous media ([12]), noise reduction and signal-processing ([14]), Lagrangian mechanics ([1]).

With the exception of a limited number of particular cases, variable order fractional differential equations can not be solved analytically using

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traditional methods. If an exact solution of the problem can not be determined and a numerical solutions in not sufficient, an analytic approximate solution should be computed.

In the present paper we obtain analytic approximate solutions for the following class of nonlinear variable order fractional differential equations:

$$D^{\alpha(x)}u(x) + a(x) \cdot u'(x) + b(x) \cdot u(x) = f(x),\ 0 \le x \le 1, \tag{1.1}$$

$$u(0) = \mu_1,\ u(1) = \mu_2, \tag{1.2}$$

where $u \in C^n[0,1]$, $\alpha : [0,1] \to \mathbb{R}_+$, $\mu_1, \mu_2 \in \mathbb{R}$, the functions $a, b, f \in C[0,1]$ are given such that the problem (1.1),(1.2) satisfy the existence and unicity conditions for a continuous solution, and $D^{\alpha(x)}$ denotes the variable order Caputo fractional derivative ([2], [10]):

$$D^{\alpha(x)}u(x) = \frac{1}{\Gamma(n-\alpha(x))}\int_0^x (x-\tau)^{n-\alpha(x)-1}\frac{d^n u(\tau)}{d\tau^n}\,d\tau,$$
$$t > 0,\ n-1 < \alpha(x) \le n,\ n \in \mathbb{N}.$$

2. The Polynomial Least Squares Method

Attached to the problem (1.1),(1.2) we consider the following operator:

$$D(u) = D^{\alpha(x)}u + a(x) \cdot u'(x) + b(x) \cdot u(x) - f(x), \tag{2.1}$$

where $u \in C^n[0,1]$. We denote by $\tilde{u}$ an approximate solution of equation (1.1). The error obtained by replacing the exact solution u with the approximation $\tilde{u}$ is given by the remainder:

$$R(x,\tilde{u}) = D(\tilde{u}(x)),\ x \in [0,1]. \tag{2.2}$$

For $\epsilon \in \mathbb{R}_+$, we will compute approximate polynomial solutions $\tilde{u}$ of the problem (1.1),(1.2) on the interval $[0,1]$. We impose for $\tilde{u}$ the following conditions:

$$|R(x,\tilde{u})| < \epsilon, \tag{2.3}$$

$$\tilde{u}(0) = \mu_1,\ \tilde{u}(1) = \mu_2. \tag{2.4}$$

Definition 2.1. We call an *ϵ-approximate polynomial solution* of the problem (1.1),(1.2) an approximate polynomial solution $\tilde{u}$ satisfying the relations (2.3),(2.4).

Definition 2.2. We call a *weak ϵ-approximate polynomial solution* of the problem (1.1,1.2) an approximate polynomial solution $\tilde{u}$ satisfying the relation:

$$\int_0^1 |R(x,\tilde{u})|\,dx \le \epsilon,$$

together with the initial conditions (2.4).

DEFINITION 2.3. Let $P_m(x) = a_0 + a_1 x + ... + a_m x^m$, $a_i \in \mathbb{R}$, $i = 0, 1, ..., m$ be a sequence of polynomials satisfying the conditions:

$$P_m(0) = \mu_1, \ P_m(1) = \mu_2.$$

We call the sequence of polynomials $P_m(x)$ *convergent to the solution of the problem (1.1),(1.2)* if $\lim\limits_{m\to\infty} D(P_m(x)) = 0$.

We remark that from the hypothesis of the problem (1.1),(1.2) it follows that there exists a sequence of polynomials $P_m(x)$ which converges to the solution of the problem.

In the following we will compute a weak ϵ-polynomial solution of the type:

$$\tilde{u}(x) = \sum_{k=0}^{m} c_k x^k, \tag{2.5}$$

where the constants $c_0, c_1, ..., c_m$ are calculated using the following steps:

- By substituting the approximate solution (2.5) in the equation (1.1) we obtain the following expression:

$$\mathfrak{R}(x, c_0, c_1, ..., c_m) = R(x, \tilde{u}) = D^{\alpha(x)}\tilde{u} + a(x) \cdot \tilde{u}'(x) + b(x) \cdot \tilde{u}(x) - f(x). \tag{2.6}$$

  If we could find $c_0^0, c_1^0, ..., c_m^0$ such that $\mathfrak{R}(x, c_0^0, c_1^0, ..., c_m^0) = 0$ for any $x \in [0, 1]$ and the equivalents of (1.2):

$$\tilde{u}(0) = \mu_1, \ \tilde{u}(1) = \mu_2, \tag{2.7}$$

  are also satisfied, then by substituting $c_0^0, c_1^0, ..., c_m^0$ in (2.5) we obtain the exact solution of (1.1,1.2).
- Next we attach to the problem (1.1,1.2) the following real functional:

$$J(c_2, c_3, ..., c_m) = \int_0^1 \mathfrak{R}^2(x, c_0, c_1, ..., c_m)\, dx, \tag{2.8}$$

  where c_0, c_1 are computed as functions of $c_2, c_3, ..., c_m$ by using the initial conditions (2.7).
- We compute the values of $c_2^0, c_3^0, ..., c_m^0$ as the values which give the minimum of the functional (2.8) and the values of c_0^0, c_1^0 again as functions of $c_2^0, c_2^0, ..., c_m^0$ by using the initial conditions.
- Using the constants $c_0^0, c_1^0, ..., c_m^0$ thus determined, we consider the polynomial:

$$T_m(x) = \sum_{k=0}^{m} c_k^0 \ x^k. \tag{2.9}$$

The following convergence theorem holds:

THEOREM 2.1. *The sequence of polynomials $T_m(x)$ from (2.9) satisfies the property:*

$$\lim_{m\to\infty}\int_0^1 R^2(x,T_m)\,dx=0.$$

Moreover, $\forall \epsilon>0$, $\exists m_0\in\mathbb{N}$ *such that* $\forall m\in\mathbb{N}$, $m>m_0$ *it follows that* $T_m(x)$ *is a weak* ϵ*-approximate polynomial solution of the problem (1.1,1.2).*

P r o o f. Based on the way the polynomials $T_m(x)$ are computed and taking into account the relations (2.6-2.9), the following inequalities are satisfied:

$$0\le\int_0^1 R^2(x,T_m(x))\,dx\le\int_0^1 R^2(x,P_m(x))\,dx,\quad \forall m\in\mathbb{N},$$

where $P_m(x)$ is the sequence of polynomials introduced in Definition 2.3. It follows that:

$$0\le\lim_{m\to\infty}\int_0^1 R^2(x,T_m(x))\,dx\le\lim_{m\to\infty}\int_0^1 R^2(x,P_m(x))\,dx=0.$$

We obtain:

$$\lim_{m\to\infty}\int_0^1 R^2(x,T_m(x))\,dx=0.$$

From this limit we obtain that $\forall\epsilon>0$, $\exists m_0\in\mathbb{N}$ such that $\forall m\in\mathbb{N}$, $m>m_0$, it follows that $T_m(x)$ is a weak ϵ -approximate polynomial solution of the problem (1.1),(1.2). □

We remark that any ϵ-approximate polynomial solution of the problem (1.1,1.2) is also a weak ϵ^2-approximate polynomial solution, but the opposite is not always true. It follows that the set of weak approximate solutions of the problem (1.1,1.2) also contains the approximate solutions of the problem.

Taking into account the above remark, in order to find ϵ-approximate polynomial solutions of the problem (1.1,1.2) by using the Polynomial Least Squares Method (PLSM) we will first determine weak approximate polynomial solutions, $\tilde{u}$. If $|R(x,\tilde{u})|<\epsilon$ then $\tilde{u}$ is also an ϵ-approximate polynomial solution of the problem.

3. Examples

This section includes several test problems with fractional derivatives of both constant and variable order. Approximate solutions for these problems were proposed in some other papers and the comparison with our results emphasizes the accuracy of PLSM.

3.1. **Example 1.** The first example is the following fractional boundary value problem with multi-point boundary conditions:

$$\begin{cases} D^{1.3}u(x) + \cos(x) \cdot u'(x) + 2 \cdot u(x) = f(x), \ 0 \leq x \leq 1, \\ u(0) = 0, \ u(1) = u(\frac{1}{8}) + 2 \cdot u(\frac{1}{2}) + \frac{31}{49} \cdot u(\frac{7}{8}), \end{cases} \tag{3.1}$$

where $f(x) = \frac{\Gamma(3)}{\Gamma(1.7)} \cdot x^{0.7} + 2 \cdot x^2 + 2 \cdot x \cdot \cos(x)$.

The exact solution of this problem is $u(x) = x^2$.

Approximate solutions for this problem using reproducing kernel methods were proposed by Geng and Cui in [5] (with absolute errors larger than 10^{-6}) and by Li and Wu in [8] (with absolute errors larger than 10^{-8}).

Using the PLSM, we computed a solution (2.5) of the type $\tilde{u}(x) = c_0 + c_1 \cdot x + c_2 \cdot x^2$.

This solution should satisfy the multi-point boundary conditions of (3.1). Imposing the corresponding conditions $\tilde{u}(0) = 0, \ \tilde{u}(1) = \tilde{u}(\frac{1}{8}) + 2 \cdot \tilde{u}(\frac{1}{2}) + \frac{31}{49} \cdot \tilde{u}(\frac{7}{8})$, we obtain $c_0 = 0, c_1 = 0$ and the approximation becomes: $\tilde{u}(x) = c_2 \cdot x^2$.

The corresponding remainder (2.6) is in this case:

$R(x, \tilde{u}) = (c_2 - 1) \cdot \left(\frac{2 \cdot x^{0.7}}{\Gamma(1.7)} + 2 \cdot x \cdot (x + \cos(x)) \right)$.

It is clear that the value $c_2 = 1$ leads to a zero remainder and thus, by replacing this value in $\tilde{u}(x)$ we obtain the exact solution of the problem: $\tilde{u}(x) = x^2$.

3.2. **Example 2.** Next, we have the following variable order fractional differential equations with boundary conditions:

$$\begin{cases} D^{\alpha(x)}u(x) + \cos(x) \cdot u'(x) + 4 \cdot u(x) + 5 \cdot u(x^2) = f(x), \ 0 \leq x \leq 1, \\ u(0) = 0, \ u(1) = 1, \end{cases} \tag{3.2}$$

where $\alpha(x) = \dfrac{5 + \sin(x)}{4}$, $f(x) = \dfrac{2 \cdot x^{2-\alpha(x)}}{\Gamma(3 - \alpha(x))} + 5 \cdot x^4 + 4 \cdot x^2 + 2 \cdot x \cdot \cos(x)$.

The exact solution of this problem is also $u(x) = x^2$.

An approximate solution of this problem using a method based on the reproducing kernel method was proposed by Li and Wu in [9], with absolute errors larger than 10^{-8}

Using the PLSM, we compute a solution (2.5) of the type $\tilde{u}(x) = c_0 + c_1 \cdot x + c_2 \cdot x^2$.

Imposing the corresponding boundary conditions $\tilde{u}(0) = 0, \ \tilde{u}(1) = 1$, we obtain $c_0 = 0, c_1 = 1 - c_2$ and the approximation becomes: $\tilde{u}(x) = (1 - c_2) \cdot x + c_2 \cdot x^2$.

The corresponding remainder (2.6) is:

$$R(x,\tilde{u})=(c_2-1)\cdot\left(\frac{2\cdot x^{\frac{3}{4}-\frac{\sin(x)}{4}}}{\Gamma\left(\frac{1}{4}(7-\sin(x))\right)}+\left(5\cdot x^4-x^2-4\cdot x\right)+(2x-1)\cdot\cos(x)\right).$$

The value $c_2 = 1$ leads to a zero remainder and by replacing this value in $\tilde{u}(x)$ we obtain the exact solution of the problem: $\tilde{u}(x) = x^2$.

3.3. **Example 3.** The third example is the following variable order fractional differential equation with an initial condition:

$$\begin{cases} D^{\alpha(x)}u(x) - 10\cdot u'(x) + u(x) = f(x), \ 0 \le x \le 1, \\ u(0) = 5, \end{cases} \tag{3.3}$$

where $\alpha(x) = \dfrac{x+2\cdot e^x}{7}$, $f(x) = 10\cdot\left(\frac{x^{2-\alpha(x)}}{\Gamma(3-\alpha(x))}+\frac{x^{1-\alpha(x)}}{\Gamma(2-\alpha(x))}\right)+5\cdot x^2-90\cdot x-95$.

The exact solution of the problem is $u(x) = 5\cdot(1+x^2)$.

Using PLSM, we compute a solution (2.5) of the type $\tilde{u}(x) = c_0 + c_1\cdot x + c_2\cdot x^2$.

Imposing the corresponding initial condition $\tilde{u}(0) = 5$, we obtain $c_0 = 5$ and the approximation becomes: $\tilde{u}(x) = 5 + c_1\cdot x + c_2\cdot x^2$.

The corresponding remainder (2.6) is:

$$R(x,\tilde{u}) = \frac{x^{\frac{1}{7}(-2)(e^x-3)}\left(2\cdot(c_1-10)\cdot e^x - 13\cdot c_1 - 14\cdot c_2 x + 70\cdot x + 130\right)}{(2\cdot e^x - 13)\,\Gamma\left(\frac{13}{7}-\frac{2\cdot e^x}{7}\right)} + c_1\cdot(x-10) + x\cdot(c_2\cdot(x-20) - 5\cdot x + 90) + 100.$$

Next we compute the functional (2.8) (too large to be included here) and minimize it obtaining the values: $c_1 = 10.000000000000004$, $c_2 = 4.999999999999997$.

The approximate analytical solution of the problem (3.3) by PLSM is: $\tilde{u}(x) = 5 + 10.000000000000004\cdot x + 4.999999999999997\cdot x^2$.

An approximate solution of this problem using a method based on Legendre wavelets was proposed by Chen et al. in [3].

Table 1 presents the values of the absolute errors corresponding to the solution proposed by Chen et al. (denoted by LWM) and to our solution (denoted by PLSM).

4. Conclusions

The Polynomial Least Squares Method (PLSM) is presented as a straightforward and efficient method to compute approximate polynomial solutions for variable order fractional differential equations.

The examples included clearly illustrate the accuracy of the method, since for all the problems we were able to compute better approximations

	LWM	PLSM
0.2	$8 \cdot 10^{-12}$	$1 \cdot 10^{-15}$
0.4	$2 \cdot 10^{-9}$	$1 \cdot 10^{-15}$
0.6	$9 \cdot 10^{-10}$	$1 \cdot 10^{-15}$
0.8	$1 \cdot 10^{-10}$	$1 \cdot 10^{-16}$
1.0	$1 \cdot 10^{-10}$	$1 \cdot 10^{-16}$

Table 1. Comparison of absolute errors of the approximate solutions for the problem (3.3)

than the ones computed in previous papers. Moreover, for several of the problems we were able to compute the exact solution of the problem.

We mention that while in this paper we solved only a class of variable order fractional differential equations, the PLSM can be easily adapted for other types of differential and integral equations, both linear and nonlinear.

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