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RESEARCH PAPER

TIME-FRACTIONAL HEAT CONDUCTION IN A TWO-LAYER COMPOSITE SLAB

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Abstract

The heat conduction equation is considered in a composite body consisting of two regions: $0 < x < L$ and $-L < x < 0$. Heat conduction in one region is described by the equation with the Caputo fractional derivative of order α, whereas in another region by the equation with the Caputo fractional derivative of order β. The integral transforms technique is used. The approximate solution valid for small values of time is analyzed in detail.

MSC 2010: Primary 26A33; Secondary 35K05, 45K05

Key Words and Phrases: fractional calculus, Mittag-Leffler type functions, fractional partial differential equations, Laplace transform

1. Introduction

The classical Fourier law states the proportionality of the heat flux $\mathbf{q}$ to the temperature gradient

$$\mathbf{q} = -k \operatorname{grad} T, \tag{1.1}$$

where k is the thermal conductivity. In combination with the law of conservation of energy, the Fourier law leads to the standard parabolic heat conduction equation.

The classical theory of heat conduction is quite acceptable for different physical situations. However, many experimental and theoretical studies testify that in media with complex internal structure the conventional parabolic heat conduction equation is no longer sufficiently accurate. For an

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extensive bibliography on this subject see [7], [8], [19], [30], [35] and references threin. The time-nonlocal dependence between the heat flux vector and the temperature gradient with the "long-tail" power kernel [21], [22], [23], [31] can be interpreted in terms of fractional calculus

$$\mathbf{q} = -k\, D_{RL}^{1-\alpha}\, \mathrm{grad}\, T, \qquad 0 < \alpha \leq 1, \tag{1.2}$$

$$\mathbf{q} = -k\, I^{\alpha-1}\, \mathrm{grad}\, T, \qquad 1 < \alpha \leq 2. \tag{1.3}$$

Here $I^{\alpha} f(t)$ and $D_{RL}^{\alpha} f(t)$ are the Riemann-Liouville fractional integral and derivative of order α, respectively [6], [9], [20]:

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, \mathrm{d}\tau, \qquad \alpha > 0, \tag{1.4}$$

$$D_{RL}^{\alpha} f(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n} \left[\frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f(\tau)\, \mathrm{d}\tau \right], \quad n-1 < \alpha < n. \tag{1.5}$$

In combination with the law of conservation of energy, the constitutive equations (1.2) and (1.3) result in the time-fractional heat conduction equation

$$\frac{\partial^{\alpha} T}{\partial t^{\alpha}} = a\, \Delta\, T \tag{1.6}$$

with the Caputo fractional derivative

$$\frac{\mathrm{d}^{\alpha} f(t)}{\mathrm{d}t^{\alpha}} = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{\mathrm{d}^n f(\tau)}{\mathrm{d}\tau^n}\, \mathrm{d}\tau, \quad n-1 < \alpha < n, \tag{1.7}$$

where $\Gamma(\alpha)$ is the gamma function.

The details of obtaining Eq. (1.6) based on (1.2) and (1.3) can be found in [24], [31]. Equation (1.6) is a mathematical model of many important physical phenomena (see [2], [16], [17], [30], [36], among others).

Some authors [13], [20] do not use a separate notation for the Riemann-Liouville fractional integral assuming that $I^{\alpha} f(t) = D_{RL}^{-\alpha} f(t),\ \alpha > 0$. Such a notation allows us to rewrite Eqs. (1.2) and (1.3) as one dependence

$$\mathbf{q} = -k\, D_{RL}^{1-\alpha}\, \mathrm{grad}\, T, \qquad 0 < \alpha \leq 2. \tag{1.8}$$

Different kinds of boundary conditions for the time-fractional heat conduction equation (1.6) were analyzed in [25], [26], [31]. The problem of fractional heat conduction in two joint half-lines was studied in [26], [28]. The central-symmetric problem for a composite medium containing a spherical inclusion was investigated in [27]. A one-dimensional problem for a composite body consisting of a layer and a half-space was considered in [29] (see also [31], where all these problems were studied in detail). In the present paper, we investigate time-fractional heat conduction in a two-layer slab occupying a region $0 < x < L$ and a region $-L < x < 0$. Heat conduction in one region is described by equation (1.6) with the Caputo fractional derivative of order α, whereas heat conduction in another region is described by equation (1.6) with the time-derivative of order β. Similar problem was considered in [10], where numerical inversion of the Laplace transform was used. We obtain the analytical solution valid for small values of time. Different problems of classical heat conduction in composite media were considered by many authors (see, for example, books [12], [18]).

2. Mathematical preliminaries

To solve the problem considered in this paper the integral transform technique will be used. Recall the Laplace transform rules for fractional integrals and derivatives [6], [9], [20]:

$$\mathcal{L}\left\{I^{\alpha} f(t)\right\} = \frac{1}{s^{\alpha}}\, f^{*}(s), \tag{2.1}$$

$$\mathcal{L}\left\{D_{RL}^{\alpha} f(t)\right\} = s^{\alpha} f^{*}(s) - \sum_{k=0}^{n-1} D^{k} I^{n-\alpha} f(0^{+}) s^{n-1-k}, \quad n-1 < \alpha < n, \tag{2.2}$$

$$\mathcal{L}\left\{\frac{\mathrm{d} f(t)}{\mathrm{d} t^{\alpha}}\right\} = s^{\alpha} f^{*}(s) - \sum_{k=0}^{n-1} f^{(k)}(0^{+}) s^{\alpha-1-k}, \quad n-1 < \alpha < n, \tag{2.3}$$

where the asterisk denotes the Laplace transform, s is the transform variable.

The finite cos-Fourier transform is the convenient reformulation of the cos-Fourier series in the domain $0 \leq x \leq L$, [33]:

$$\mathcal{F}\{f(x)\} = \widetilde{f}(\xi_m) = \int_0^L f(x) \cos(x\xi_m)\, \mathrm{d}x, \tag{2.4}$$

$$\mathcal{F}^{-1}\{\widetilde{f}(\xi_m)\} = f(x) = \frac{2}{L} \sum_{m=0}^{\infty}{}' \widetilde{f}(\xi_m) \cos(x\xi_m), \tag{2.5}$$

where

$$\xi_m = \frac{m\pi}{L}. \tag{2.6}$$

The prime near the sum denotes that the term corresponding to $m = 0$ should be multiplied by $1/2$. The finite cos-Fourier transform is used in the case of Neumann boundary condition as

$$\mathcal{F}\left\{\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}\right\} = -\xi_m^2\, \widetilde{f}(\xi_m) - \left.\frac{\mathrm{d} f(x)}{\mathrm{d}x}\right|_{x=0} + (-1)^m \left.\frac{\mathrm{d} f(x)}{\mathrm{d}x}\right|_{x=L}. \tag{2.7}$$

Similarly, in the domain $-L \leq x \leq 0$:

$$\mathcal{F}\{f(x)\} = \widetilde{f}(\xi_m) = \int_{-L}^{0} f(x) \cos(x\xi_m)\, \mathrm{d}x, \tag{2.8}$$

$$\mathcal{F}^{-1}\{\widetilde{f}(\xi_m)\} = f(x) = \frac{2}{L}\sum_{m=0}^{\infty}{}' \widetilde{f}(\xi_m)\cos(x\xi_m), \tag{2.9}$$

$$\mathcal{F}\left\{\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}\right\} = -\xi_m^2\,\widetilde{f}(\xi_m) + \left.\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right|_{x=0} - (-1)^m \left.\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right|_{x=-L}. \tag{2.10}$$

3. Statement of the problem

Consider the time-fractional heat conduction equations in a two-layer composed slab:

$$\frac{\partial^\alpha T_1}{\partial t^\alpha} = a_1\,\frac{\partial^2 T_1}{\partial x^2}, \qquad 0 < x < L, \quad 0 < \alpha \le 2, \tag{3.1}$$

$$\frac{\partial^\beta T_2}{\partial t^\beta} = a_2\,\frac{\partial^2 T_2}{\partial x^2}, \qquad -L < x < 0, \quad 0 < \beta \le 2, \tag{3.2}$$

under zero initial conditions

$$t = 0: \qquad T_1 = 0, \qquad 0 < \alpha \le 2, \quad 0 < x < L, \tag{3.3}$$

$$t = 0: \qquad \frac{\partial T_1}{\partial t} = 0, \qquad 1 < \alpha \le 2, \quad 0 < x < L, \tag{3.4}$$

$$t = 0: \qquad T_2 = 0, \qquad 0 < \beta \le 2, \quad -L < x < 0, \tag{3.5}$$

$$t = 0: \qquad \frac{\partial T_2}{\partial t} = 0, \qquad 1 < \beta \le 2, \quad -L < x < 0, \tag{3.6}$$

and the boundary conditions

$$x = L: \qquad k_1 D_{RL}^{1-\alpha}\frac{\partial T_1}{\partial x} = q(t), \qquad 0 < \alpha \le 2, \tag{3.7}$$

$$x = -L: \qquad \frac{\partial T_2}{\partial x} = 0. \tag{3.8}$$

At the contact surface, the boundary conditions of the perfect contact are fulfilled:

$$x = 0: \quad T_1 = T_2, \tag{3.9}$$

$$x = 0: \quad k_1 D_{RL}^{1-\alpha}\,\frac{\partial T_1}{\partial x} = k_2 D_{RL}^{1-\beta}\,\frac{\partial T_2}{\partial x}, \quad 0 < \alpha \le 2, \ 0 < \beta \le 2. \tag{3.10}$$

4. Solution of the problem

We introduce the unknown function

$$\varphi(t) = k_1 D_{RL}^{1-\alpha} \frac{\partial T_1}{\partial x}\bigg|_{x=0} = k_2 D_{RL}^{1-\beta} \frac{\partial T_2}{\partial x}\bigg|_{x=0} \tag{4.1}$$

and apply to the initial-boundary-value problem (3.1)-(3.8) the Laplace transform with respect to time t and the cos-Fourier transform with respect to the spatial coordinate x in the corresponding region. In such a way we get the solution in the transform domain:

$$\widetilde{T}_1^*(\xi_m, s) = \frac{a_1}{k_1}\Big[(-1)^m q^*(s) - \varphi^*(s)\Big] \frac{s^{\alpha-1}}{s^\alpha + a_1\xi_m^2}, \qquad 0 \le x \le L, \tag{4.2}$$

$$\widetilde{T}_2^*(\xi_m, s) = \frac{a_2}{k_2}\, \varphi^*(s)\, \frac{s^{\beta-1}}{s^\beta + a_2\xi_m^2}, \qquad -L \le x \le 0. \tag{4.3}$$

The inverse cos-Fourier transforms give

$$T_1^*(x, s) = \frac{2a_1}{Lk_1} \sum_{m=0}^{\infty}{}' \Big[(-1)^m q^*(s) - \varphi^*(s)\Big] \frac{s^{\alpha-1}}{s^\alpha + a_1\xi_m^2} \cos(x\xi_m), \quad 0 \le x \le L, \tag{4.4}$$

$$T_2^*(x, s) = \frac{2a_2}{Lk_2} \sum_{m=0}^{\infty}{}' \, \varphi^*(s)\, \frac{s^{\beta-1}}{s^\beta + a_2\xi_m^2} \cos(x\xi_m), \quad -L \le x \le 0. \tag{4.5}$$

Inverting the Laplace transform with the use of the convolution theorem, we have

$$T_1(x,t) = \frac{2a_1}{Lk_1} \sum_{m=0}^{\infty}{}' \cos(x\xi_m) \int_0^t \Big[(-1)^m q(t-\tau) - \varphi(t-\tau)\Big] \times E_\alpha\left(-a_1\xi_m^2\tau^\alpha\right) \mathrm{d}\tau, \qquad 0 \le x \le L, \tag{4.6}$$

$$T_2(x,t) = \frac{2a_2}{Lk_2} \sum_{m=0}^{\infty}{}' \cos(x\xi_m) \int_0^t \varphi(t-\tau) E_\beta\left(-a_2\xi_m^2\tau^\beta\right) \mathrm{d}\tau, \quad -L \le x \le 0. \tag{4.7}$$

where $E_{\alpha,\beta}(z)$ is the Mittag-Leffler function [3], [6], [9], [20],

$$E_{\alpha,\beta}(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha\, m + \beta)}, \qquad \alpha > 0,\ \beta > 0,\ z \in C, \tag{4.8}$$

and the following formula

$$\mathcal{L}^{-1}\left\{\frac{s^{\alpha-\beta}}{s^\alpha + b}\right\} = t^{\beta-1} E_{\alpha,\beta}(-bt^\alpha) \tag{4.9}$$

has been used. As usually, $E_{\alpha,1}(z) \equiv E_\alpha(z)$.

To determine the unknown function $\varphi(t)$ we use the trigonometric series presented in [32],

$$\sum_{m=0}^{\infty}{}' \frac{1}{m^2+a^2} \cos(mx) = \frac{\pi}{2a} \cosh\left[(\pi - x)a\right] \operatorname{csch}(\pi a), \quad 0 \le x \le 2\pi, \tag{4.10}$$

$$\sum_{m=0}^{\infty}{}' \frac{(-1)^m}{m^2+a^2} \cos(mx) = \frac{\pi}{2a} \cosh(xa) \operatorname{csch}(\pi a), \quad -\pi \le x \le \pi, \tag{4.11}$$

which allow us to obtain from (4.4) and (4.5):

$$\begin{aligned} T_1^*(x,s) \; &= \; \frac{\sqrt{a_1} s^{\alpha/2-1}}{k_1} \left\{ q^*(s) \cosh\left(\frac{x s^{\alpha/2}}{\sqrt{a_1}}\right) - \varphi^*(s) \cosh\left[\frac{(L-x)s^{\alpha/2}}{\sqrt{a_1}}\right] \right\} \\ &\times \operatorname{csch}\left(\frac{L s^{\alpha/2}}{\sqrt{a_1}}\right), \qquad 0 \le x \le L, \end{aligned} \tag{4.12}$$

$$T_2^*(x,s) \; = \; \frac{\sqrt{a_2}\, s^{\beta/2-1}}{k_2} \varphi^*(s) \cosh\left[\frac{(L-|x|)\, s^{\beta/2}}{\sqrt{a_2}}\right] \operatorname{csch}\left(\frac{L s^{\beta/2}}{\sqrt{a_2}}\right), \quad -L \le x \le 0. \tag{4.13}$$

From the condition of perfect thermal contact (3.9), stating the equality of temperatures at the contact surface, it follows that

$$T_1^*(0,s) = T_2^*(0,s), \tag{4.14}$$

and Eqs. (4.12)–(4.14) give

$$\varphi^*(s) = q^*(s) \operatorname{csch}\left(\frac{L s^{\alpha/2}}{\sqrt{a_1}}\right) \Delta^{-1}(s), \tag{4.15}$$

where

$$\Delta(s) = \coth\left(\frac{L s^{\alpha/2}}{\sqrt{a_1}}\right) + \gamma s^{\beta/2-\alpha/2} \coth\left(\frac{L s^{\beta/2}}{\sqrt{a_2}}\right) \tag{4.16}$$

with

$$\gamma = \frac{k_1\sqrt{a_2}}{k_2\sqrt{a_1}}.$$

5. Approximate solution

To avoid very complicated mathematical expressions and to obtain the analytical solution amenable for numerical treatment we will study the approximate solution of the considered problem valid for small values of time. In the case of classical heat conduction this method was described in [12], [18]. Based on the Tauberian theorems for the Laplace transform, for small values of time t (the large values of the transform variable s) we can neglect the exponential term in comparison with 1:

$$1 \pm \exp\left(-\frac{2Ls^{\alpha/2}}{\sqrt{a_1}}\right) \simeq 1, \qquad 1 \pm \exp\left(-\frac{2Ls^{\beta/2}}{\sqrt{a_2}}\right) \simeq 1. \tag{5.1}$$

Hence, the approximate expressions for temperatures take the form

$$T_1^*(x,s) \simeq \frac{\sqrt{a_1}s^{\alpha/2-1}}{k_1}\left\langle q^*(s)\left\{\exp\left[-\frac{(L-x)s^{\alpha/2}}{\sqrt{a_1}}\right]\right.\right.$$

$$\left.\left.+\exp\left[-\frac{(L+x)s^{\alpha/2}}{\sqrt{a_1}}\right]\right\} - \varphi^*(s)\exp\left(-\frac{xs^{\alpha/2}}{\sqrt{a_1}}\right)\right\rangle, \quad 0 \le x \le L, \tag{5.2}$$

$$T_2^*(x,s) \simeq \frac{\sqrt{a_2}s^{\beta/2-1}}{k_2}\,\varphi^*(s)\exp\left(-\frac{|x|s^{\beta/2}}{\sqrt{a_2}}\right), \quad -L \le x \le 0, \tag{5.3}$$

where

$$\varphi^*(s) \simeq 2\,q^*(s)\exp\left(-\frac{Ls^{\alpha/2}}{\sqrt{a_1}}\right)\frac{1}{1+\gamma s^{\beta/2-\alpha/2}}. \tag{5.4}$$

The solution simplifies significantly for $\alpha = \beta$. In this case

$$T_1^*(x,s) \simeq \frac{\sqrt{a_1}s^{\alpha/2-1}}{k_1}q^*(s)\left\{\exp\left[-\frac{(L-x)s^{\alpha/2}}{\sqrt{a_1}}\right]\right.$$

$$\left. + \frac{\gamma-1}{\gamma+1}\exp\left[-\frac{(L+x)s^{\alpha/2}}{\sqrt{a_1}}\right]\right\}, \quad 0 \le x \le L, \tag{5.5}$$

$$T_2^*(x,s) \simeq \frac{2\sqrt{a_2}s^{\alpha/2-1}}{(\gamma+1)k_2}\,q^*(s)\exp\left(-\frac{Ls^{\alpha/2}}{\sqrt{a_1}} - \frac{|x|s^{\alpha/2}}{\sqrt{a_2}}\right), \quad -L \le x \le 0. \tag{5.6}$$

Using the following formula for the inverse Laplace transform [14], [15]

$$\mathcal{L}^{-1}\left\{s^{\alpha-1}\exp\left(-\lambda s^{\alpha}\right)\right\} = \frac{1}{t^{\alpha}}M\left(\alpha;\lambda t^{-\alpha}\right), \quad 0<\alpha<1, \quad \lambda>0, \tag{5.7}$$

we get

$$T_1(x,t) \simeq \frac{\sqrt{a_1}}{k_1}\int_0^t q(t-\tau)\,\frac{1}{\tau^{\alpha/2}}\left[M\left(\frac{\alpha}{2};\frac{L-x}{\sqrt{a_1}\tau^{\alpha/2}}\right)\right.$$

$$\left. + \frac{\gamma-1}{\gamma+1}\,M\left(\frac{\alpha}{2};\frac{L+x}{\sqrt{a_1}\tau^{\alpha/2}}\right)\right]\mathrm{d}\tau, \quad 0 \le x \le L, \tag{5.8}$$

$$T_2(x,t) \simeq \frac{2\sqrt{a_2}}{(\gamma+1)k_2} \int_0^t q(t-\tau)\, \frac{1}{\tau^{\alpha/2}} M\left(\frac{\alpha}{2}; \frac{|x|}{\sqrt{a_2}\tau^{\alpha/2}} + \frac{L}{\sqrt{a_1}\tau^{\alpha/2}}\right) \mathrm{d}\tau, \quad -L \le x \le 0. \tag{5.9}$$

Here $M(\alpha; z)$ is the Mainardi function [14], [15], [20]

$$M(\alpha; z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{k!\,\Gamma[-\alpha k + (1-\alpha)]}, \qquad 0 < \alpha < 1,\ z \in C. \tag{5.10}$$

5.1. Dirac's delta heat flux

For Dirac's delta heat flux at the boundary

$$q(t) = Q_0\,\delta(t), \tag{5.11}$$

we have

$$T_1(x,t) \simeq \frac{Q_0\sqrt{a_1}}{k_1 t^{\alpha/2}} \left[M\left(\frac{\alpha}{2}; \frac{L-x}{\sqrt{a_1}t^{\alpha/2}}\right) + \frac{\gamma-1}{\gamma+1} M\left(\frac{\alpha}{2}; \frac{L+x}{\sqrt{a_1}t^{\alpha/2}}\right) \right], \quad 0 \le x \le L, \tag{5.12}$$

$$T_2(x,t) \simeq \frac{2Q_0\sqrt{a_2}}{(\gamma+1)k_2 t^{\alpha/2}} M\left(\frac{\alpha}{2}; \frac{|x|}{\sqrt{a_2}t^{\alpha/2}} + \frac{L}{\sqrt{a_1}t^{\alpha/2}}\right), \quad -L \le x \le 0. \tag{5.13}$$

5.2. Constant heat flux at the boundary

In this case

$$q(t) = q_0 = \mathrm{const}, \tag{5.14}$$

and to invert the Laplace transform we will use the following formula [34]

$$\mathcal{L}^{-1}\left\{s^{\beta} \exp\left(-\lambda s^{\alpha}\right)\right\} = t^{\beta-1} W\left(-\alpha, \beta; -\lambda t^{-\alpha}\right), \quad 0 < \alpha < 1, \quad \lambda > 0, \tag{5.15}$$

where $W(\alpha, \beta; z)$ is the Wright function [1], [5]

$$W(\alpha, \beta; z) = \sum_{m=0}^{\infty} \frac{z^m}{m!\,\Gamma(\alpha m + \beta)}, \qquad \alpha > -1, \quad z \in C. \tag{5.16}$$

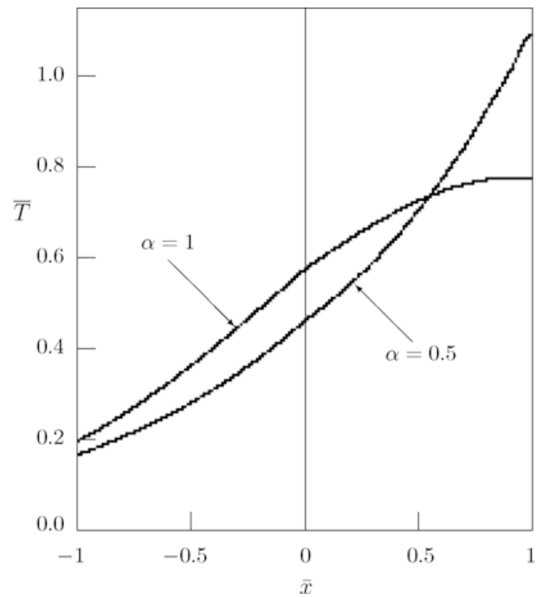

Fig. 5.1: Dependence of temperature on distance (Dirac's delta heat flux at the boundary; $\gamma = 1.5$, $\kappa = 0.75$, $\varepsilon = 0.85$).

The solution reads

$$T_1(x,t) \simeq \frac{q_0\sqrt{a_1}}{k_1 t^{\alpha/2-1}}\left[W\left(-\frac{\alpha}{2}, 2-\frac{\alpha}{2}; -\frac{L-x}{\sqrt{a_1}t^{\alpha/2}}\right) + \frac{\gamma-1}{\gamma+1}W\left(-\frac{\alpha}{2}, 2-\frac{\alpha}{2}; -\frac{L+x}{\sqrt{a_1}t^{\alpha/2}}\right)\right], \quad 0 \le x \le L, \tag{5.17}$$

$$T_2(x,t) \simeq \frac{2q_0\sqrt{a_2}}{(\gamma+1)k_2 t^{\alpha/2-1}}W\left(-\frac{\alpha}{2}, 2-\frac{\alpha}{2}; -\frac{|x|}{\sqrt{a_2}t^{\alpha/2}} - \frac{L}{\sqrt{a_1}t^{\alpha/2}}\right), \quad -L \le x \le 0. \tag{5.18}$$

The results of numerical calculations are shown in Figs. 5.1–5.3. In the calculations we use the following nondimensional quantities:

$$\bar{x} = \frac{x}{L}, \qquad \varepsilon = \sqrt{\frac{a_1}{a_2}}, \qquad \kappa = \frac{\sqrt{a_1}t^{\alpha/2}}{L},$$

for Dirac's delta boundary flux

$$\overline{T} = \frac{Lk_1}{a_1 Q_0}T,$$

for constant boundary flux

$$\overline{T} = \frac{Lk_1}{a_1 q_0 t}T.$$

To evaluate the Mainardi function and the Wright function, we have used the relations between these functions and the Mittag-Leffler function [31]:

$$M\left(\frac{\alpha}{2}; x\right) = \frac{2}{\pi}\int_0^{\infty} E_{\alpha}\left(\xi^2\right)\cos(x\xi)\,\mathrm{d}\xi, \quad 0 < \alpha < 2, \tag{5.19}$$

$$W\left(-\frac{\alpha}{2}, 2-\frac{\alpha}{2}; -x\right) = \frac{2}{\pi}\int_0^{\infty} E_{\alpha,2}\left(\xi^2\right)\cos(x\xi)\,\mathrm{d}\xi, \quad 0 < \alpha < 2. \tag{5.20}$$

For calculation of the Mittag-Leffler function we have applied the algorithm suggested in [4]. The interested reader is also referred to algorithms for evaluating the Wright function [11].

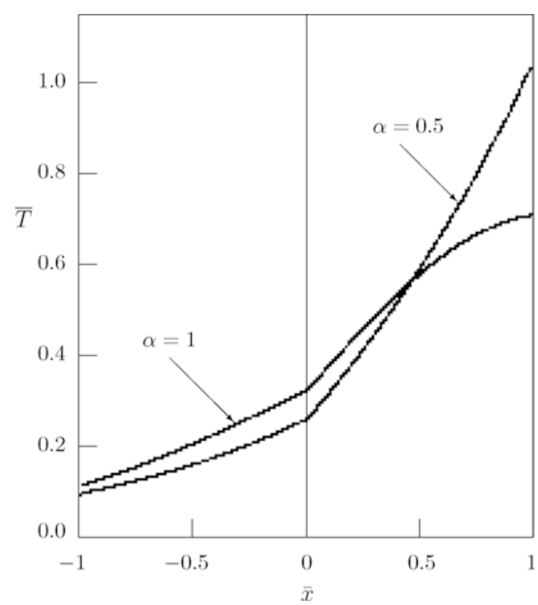

Fig. 5.2: Dependence of temperature on distance (Dirac's delta heat flux at the boundary; $\gamma = 0.5$, $\kappa = 0.75$, $\varepsilon = 0.85$).

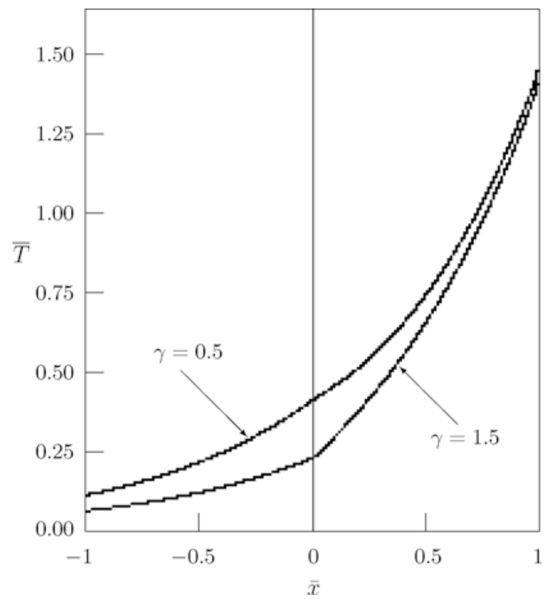

Fig. 5.3: Dependence of temperature on distance (constant heat flux at the boundary; $\alpha = 0.5$, $\kappa = 0.75$, $\varepsilon = 0.85$).

6. Concluding remarks

In this work, we have obtained the solution of the time-fractional heat conduction equations in a composite two-layer slab under conditions of perfect thermal contact. It should be emphasized that for fractional heat conduction equation the proper boundary conditions should be formulated in terms of the corresponding heat flux (1.8), not in terms of the normal derivative of temperature alone as in the case of the standard heat conduction equation and the classical Fourier law (1.1). The Laplace transform with respect to time and the finite cos-Fourier transform with respect to the spatial coordinate have been used. Based on the Tauberian theorems for the Laplace transform, the approximate solution valid for small values of time has been obtained and analyzed in detail.

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