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RESEARCH PAPER

INTEGRAL EQUATIONS OF FRACTIONAL ORDER IN LEBESGUE SPACES

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Dedicated to Professor Stefan G. Samko
on the occasion of his 75th anniversary

Abstract

We discuss solvability of a nonlinear Riemann-Liouville integral equation in Lebesgue spaces. We treat the Volterra equations of the first and the second types by applying boundedness criteria for the Riemann-Liouville integral operator. The existence of a solution to integral equations will follow from the Leray-Schauder Nonlinear Alternative.

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1. Introduction

The present paper is a study of Volterra convolution-type integral equations motivated by [3, 4]. Integral equations arising in this paper are associated with initial value problems for Riemann-Liouville equations of fractional order. Solvability of a boundary and initial value problems of fractional order in Lebesgue spaces are are discussed in [1] and [8, 9], respectively. In this paper we obtain new existence criteria of L^p-integrable solutions. In this section we introduce the auxiliaries on fractional differentiation and integration and other techniques involved in this work. For the theory and applications of fractional derivatives and integrals, we refer the reader to the monographs [11, 12, 15, 16, 17] and also mention [3, 4, 5, 9, 14] devoted to integral equations of fractional order as a part of the field of integral equations represented in the bibliography by well-known treatments

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[2, 6, 10]. The existence results are based on applications of a variant of the Leray-Schauder Nonlinear Alternative [18]:

THEOREM **1.1**. *Let $\mathcal{B}$ be a Banach space, $\mathcal{C} \subset B$ be a convex set and U be open in $\mathcal{C}$ with $0 \in U$. Let $T : \overline{U} \to \mathcal{C}$ be a continuous, compact mapping. Then either*

(i) the mapping T has a fixed point, or,
(ii) there exists $v \in \partial U$ and $\lambda \in (0,1)$ with $v = \lambda T v$.

The following theorem describing the properties of the Nemytzkii map can found in [13].

THEOREM **1.2**. *Let $f : [0,1] \times \mathbf{R}^n \to \mathbf{R}$ satisfy the Carathéodory conditions and assume that $f(\cdot, v) \in L^{p_1}[0,1]$ for every $v \in L^p[0,1]$, $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{p_1} = 1$. Then the Nemytzkii operator $F : L^p[0,1] \to L^{p_1}[0,1]$ defined by $v \mapsto f(\cdot, v)$ is continuous and bounded.*

The compactness criterion of Riesz shall be used in proving the compactness of mappings.

THEOREM **1.3**. *Let $K \subset L^p[0,1]$, $1 \leq p < \infty$. Then K is relatively compact if and only if the following hold:*

(i) K is bounded in $L^p[0,1]$;
(ii) $\int_0^1 |v(t+h) - v(t)|^p \, dt \to 0$ as $h \to 0$ uniformly in K.

The Riemann-Liouville fractional integral of order $0 < \alpha < 1$ of a function $v \in L^p(0,1)$, $1 \leq p < \infty$, is the integral

$$\mathcal{I}_{0+}^{\alpha} v(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s) \, ds. \tag{1.1}$$

The Riemann-Liouville fractional derivative of order $0 < \alpha < 1$ is defined by

$$\mathcal{D}_{0+}^{\alpha} v(t) = \frac{d}{dt} \mathcal{I}_{0+}^{1-\alpha} v(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} v(s) \, ds. \tag{1.2}$$

The relationships between (1.1) and (1.2) are stated in the next theorem (see [11, 17]).

THEOREM **1.4**. *The following hold:*

(a) The equality $\mathcal{D}_{0+}^{\alpha} \mathcal{I}_{0+}^{\alpha} f = f$ holds for every $f \in L^1(0,1)$;
(b) For $v \in L^1(0,1)$, $0 < \alpha < 1$, if $\mathcal{I}_{0+}^{1-\alpha} v \in AC[0,1]$, then

$$\mathcal{I}_{0+}^{\alpha} \mathcal{D}_{0+}^{\alpha} v(t) = v(t) - \frac{\left(\mathcal{I}_{0+}^{1-\alpha} v\right)(0)}{\Gamma(\alpha)} t^{\alpha-1}.$$

Our attention turns now to sufficient conditions for solvability of the Volterra integral equation

$$v(t) = \phi(t) + \mathcal{I}_{0+}^{\alpha} f(t, v(t)), \quad \phi(t) = \frac{\eta}{\Gamma(\alpha)} t^{\alpha-1}. \tag{1.3}$$

A solution will be sought in Lebesgue spaces $L^p(0,1)$ with $p > 1$. The integral equation above is related to the Riemann-Liouville differential equation

$$\mathcal{D}_{0+}^{\alpha} v(t) = f(t, v(t)), \quad a.e. \quad t \in (0,1), \tag{1.4}$$

of fractional order $0 < \alpha < 1$ satisfying the nonhomogeneous initial condition

$$\mathcal{I}_{0+}^{1-\alpha} v(0) = \eta. \tag{1.5}$$

The next result is well known (see, e.g., [7]).

Lemma **1.1**. *Let the mapping T be defined by*

$$Tw(t) = \int_0^1 K(t,s) w(s)\, ds, \quad t \in (0,1),$$

for $w \in L^{p_1}(0,1)$. Then $T : L^{p_1}(0,1) \to L^p(0,1)$, $1 < p_1 < \infty$, $\frac{1}{p} + \frac{1}{p_1} = 1$, is a bounded map provided

$$\|K(\cdot,\cdot)\|_{p,(0,1)\times(0,1)}^p = \int_0^1 \int_0^1 |K(t,s)|^p \, ds\, dt < \infty.$$

Moreover,

$$\|Tw\|_p \le \|K(\cdot,\cdot)\|_{p,(0,1)\times(0,1)} \|w\|_{p_1}.$$

In the integral operators occurring in this paper, we encounter a kernel $K : (0,1) \times (0,1) \to \mathbf{R}$ in the form

$$K(t,s) = \begin{cases} \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}, & 0 < s < t < 1, \\ 0, & 0 < t \le s < 1, \end{cases} \tag{1.6}$$

where $0 < \alpha < 1$. For $1 < p < \frac{1}{1-\alpha}$, we introduce

$$\begin{aligned} C_1(\alpha; p) &= \|K(\cdot,\cdot)\|_{p,(0,1)\times(0,1)} \\ &= \frac{1}{\Gamma(\alpha)} (1 - (1-\alpha)p)^{-1/p} (2 - (1-\alpha)p)^{-1/p}. \end{aligned} \tag{1.7}$$

If $(\alpha - 1)p + 1 > 0$ and $\frac{1}{p} + \frac{1}{p_1} = 1$, then, by Lemma **1.1**, $\mathcal{I}_{0+}^{\alpha} : L^{p_1}(0,1) \to L^p(0,1)$, is a bounded mapping with

$$\|\mathcal{I}_{0+}^{\alpha} w\|_p \le C_1(\alpha; p) \|w\|_{p_1} \tag{1.8}$$

in view of (1.6) and (1.7).

It is also a well-known fact [17] that $\mathcal{I}_{0+}^{\alpha} : L^p(0,1) \to L^p(0,1)$, $p \geq 1$, is a bounded mapping with

$$\|\mathcal{I}_{0+}^{\alpha} w\|_p \leq \frac{1}{\Gamma(\alpha+1)} \|w\|_p. \tag{1.9}$$

Of course, $L^{p_1}(0,1) \subset L^p(0,1)$ for $p \leq p_1$ with $\|w\|_p \leq \|w\|_{p_1}$.

Finally, we state a boundedness criterion (Theorem 3.5, [17]) for the Riemann-Liouville fractional integral, which will be used for the Volterra equation of the first kind.

Lemma **1.2**. *If $0 < \alpha < 1$ and $1 < p_1 < \frac{1}{\alpha}$, then $\mathcal{I}_{0+}^{\alpha} : L^{p_1}(0,1) \to L^p(0,1)$ is a bounded mapping for $1 < p < \frac{p_1}{1-\alpha p_1}$.*

In particular, following [17], one can show that

$$\|\mathcal{I}_{0+}^{\alpha} v\|_p \leq C_0(\alpha; p_1, p) \|v\|_{p_1}, \tag{1.10}$$

where

$$\frac{1}{C_0(\alpha; p_1, p)} = \Gamma(\alpha) \left[\frac{1}{2} \left(\frac{1}{p} - \frac{1}{p_1} + \alpha \right) \right]^{1+\frac{1}{p}-\frac{1}{p_1}} \left(\frac{p_1}{p_1 - 1} \right)^{\frac{p_1-1}{p_1}} p^{1/p}.$$

If, in addition, $\frac{1}{p} + \frac{1}{p_1} = 1$, then $1 < (1-\alpha)p < 2$ and with a little effort we can show that

$$C_2(\alpha; p) = C(\alpha; p, p_1) = \frac{1}{\Gamma(\alpha)} \left(\frac{2}{2 - p(1-\alpha)} \right)^{\frac{2}{p}}. \tag{1.11}$$

2. The existence theorems

We now impose conditions which stand throughout the paper:

(H_1) $f : [0,1] \times \mathbf{R} \to \mathbf{R}$ satisfies the Carathéodory conditions and $f(\cdot, v) \in L^{p_1}[0,1]$ for every $v \in L^p(0,1)$, $1 < p_1 < \infty$ and $\frac{1}{p_1} + \frac{1}{p} = 1$, and

(H_2) there exist a nonnegative function $a \in L^{p_1}(0,1)$ and a constant $b > 0$ such that $|f(t,z)| \leq a(t) + b|z|^{p-1}$ a. e. in $(0,1)$ and for all $z \in \mathbf{R}$.

It is convenient to introduce a technical lemma.

Lemma **2.1**. *Assume that $1 < p < \frac{1}{1-\alpha}$, $\frac{1}{p_1} + \frac{1}{p} = 1$, and let $F : L^p(0,1) \to L^{p_1}(0,1)$ satisfy (H_1) and (H_2). Then $\mathcal{I}_{0+}^{\alpha} F : L^p(0,1) \to L^p(0,1)$ with*

$$\|\mathcal{I}_{0+}^{\alpha} F v\|_p \leq D(\|a\|_{p_1} + b\|v\|_p^{p-1}), \tag{2.1}$$

where

$$D = \begin{cases} \min\left\{ \frac{1}{\Gamma(\alpha+1)}, C_1(\alpha; p) \right\}, & p \leq 2, \\ C_1(\alpha; p), & p > 2. \end{cases}$$

P r o o f. It is clear that

$$\|Fv\|_{p_1} \le \|a\|_{p_1} + b\|v\|_p^{p-1}. \tag{2.2}$$

By (1.8), we have the inequality $\|\mathcal{I}_{0+}^{\alpha} Fv\|_p \le C_1(\alpha;p)\|Fv\|_{p_1}$ for all $p > 1$. First note that the function $z : (1-1/p, 1) \to \mathbb{R}$ defined by

$$z(\alpha) = \frac{1/\Gamma(\alpha+1)}{C_1(\alpha;p)} = \frac{1}{\alpha}((\alpha-1)p+1)^{1/p}((\alpha-1)p+2)^{1/p}$$

is continuous in $(1-1/p, 1)$. Moreover,

$$\lim_{\alpha\to(1-1/p)^+} z(\alpha) = 0 \quad \text{and} \quad \lim_{\alpha\to 1^-} z(\alpha) = \frac{2^{1/p}}{\alpha} > 1.$$

If, in addition, $p \le 2 \le p_1$, we apply (1.9) to obtain $\|\mathcal{I}_{0+}^{\alpha} Fv\|_p \le \frac{1}{\Gamma(\alpha+1)}\|Fv\|_p \le \frac{1}{\Gamma(\alpha+1)}\|Fv\|_{p_1}$. Then the inequality (2.1) follows from (2.2). □

REMARK. We present numerical illustrations that the constants $1/\Gamma(\alpha+1)$ and $C_1(\alpha;p)$ indeed "compete". For example, for $\alpha = 1/4$, $C_1(\alpha;1.09) < \frac{1}{\Gamma(\alpha+1)} < C_1(\alpha;1.08)$. Also, for $p = 1.9$ and $\alpha = 0.6$, $D = \frac{1}{\Gamma(1.6)} < C_1(0.6;p)$ while, for $p = 1.9$ and $\alpha = 0.7$, $D = C_1(0.7;p) < \frac{1}{\Gamma(1.7)}$.

The main results are divided into two subsections by the type of the Volterra equation.

2.1. The Volterra equations of the second kind ($\eta \ne 0$)

THEOREM **2.1**. *Assume that* (H_1) *and* (H_2) *hold and* $1 < p < \frac{1}{1-\alpha}$ *and* $p \le 2$. *Then the integral equation (1.3) has a solution* $v_0 \in L^p(0,1)$ *with* $\|v_0\|_p < R$ *provided*

$$\|\phi\|_p + \min\left\{\frac{1}{\Gamma(\alpha+1)}, C_1(\alpha;p)\right\}\left(\|a\|_{p_1} + bR^{p-1}\right) \le R. \tag{2.3}$$

P r o o f. Let $v \in L^p(0,1)$ and define

$$Tv(t) = \phi(t) + \mathcal{I}_{0+}^{\alpha} f(t, v(t)), \quad t \in (0,1). \tag{2.4}$$

Let $K \subset L^p(0,1)$ be bounded. Define $F : L^p(0,1) \to L^{p_1}(0,1)$ by $Fv(t) = f(t, v(t))$, $t \in (0,1)$. Then $F(K)$ is a bounded subset of $L^{p_1}(0,1)$, that is, say, $\|Fv\|_{p_1} \le B$, $v \in K$. Note that $\mathcal{I}_{0+}^{\alpha} F : L^p(0,1) \to L^p(0,1)$ is a continuous mapping by Theorem **1.2** and Lemma **2.1**. An inequality similar to (2.1) shows that $T(K) \subset L^p(0,1)$ is bounded. So, it remains to check that the condition (ii) of Theorem **1.3** holds, which would imply that (2.4) is compact.

Constant factors depending only on the parameters p, α, and the constant B and varying from step to step will be replaced in the estimates below by a generic constant $c > 0$ whose exact value is unimportant to us. We rewrite (2.4) as

$$Tv(t) = \phi(t) + c\int_0^t (t-s)^{\alpha-1} Fv(s)\,ds.$$

Let $h > 0$ (the case $h < 0$ is treated similarly), so that

$$\begin{aligned}
&\int_0^1 |Tv(t+h) - Tv(t)|^p \,dt \\
&= \int_0^1 \Big|\phi(t+h) - \phi(t) \\
&+ c\int_0^{t+h} (t+h-s)^{\alpha-1} f(s, v(s))\,ds - c\int_0^t (t-s)^{\alpha-1} Fv(s)\,ds\Big|^p \,dt \\
&= \int_0^1 \Big|\phi(t+h) - \phi(t) + c\int_t^{t+h} (t+h-s)^{\alpha-1} Fv(s)\,ds \\
&+ c\int_0^t \left((t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}\right) Fv(s)\,ds\Big|^p \,dt \\
&\le c4^{p-1}\Bigg[\int_0^1 |\phi(t+h) - \phi(t)|^p \,dt \\
&+ \int_0^1 \left(\int_t^{t+h} (t+h-s)^{\alpha-1} |Fv(s)|\,ds\right)^p \,dt \\
&+ \int_0^1 \left(\int_0^t \left|(t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}\right| |Fv(s)|\,ds\right)^p \,dt\Bigg],
\end{aligned}$$

where we applied the inequality $(a+b+c)^p \le 4^{p-1}(a^p + b^p + c^p)$, $a, b, c \ge 0$.

By Hölder's inequality,

$$\begin{aligned}
&\int_0^1 |Tv(t+h) - Tv(t)|^p \,dt \\
&\le c\int_0^1 \left|(t+h)^{\alpha-1} - t^{\alpha-1}\right|^p \,dt + c\int_0^1 \int_t^{t+h} (t+h-s)^{(\alpha-1)p}\,ds\,dt\,\|Fv\|_{p_1}^p \\
&+ c\int_0^1 \int_0^t \left|(t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}\right|^p \,ds\,dt\,\|Fv\|_{p_1}^p \\
&= c\int_0^1 \left(t^{\alpha-1} - (t+h)^{\alpha-1}\right)^p \,dt + cB^p \int_0^1 \int_t^{t+h} (t+h-s)^{(\alpha-1)p}\,ds\,dt \\
&+ cB^p \int_0^1 \int_0^t \left((t-s)^{\alpha-1} - (t-s+h)^{\alpha-1}\right)^p \,ds\,dt
\end{aligned}$$

$$\leq c\int_0^1 \left(t^{(\alpha-1)p} - (t+h)^{(\alpha-1)p}\right) dt + c\int_0^1 \int_t^{t+h} (t+h-s)^{(\alpha-1)p}\, ds\, dt$$
$$+ c\int_0^1 \int_0^t \left((t-s)^{(\alpha-1)p} - (t+h-s)^{(\alpha-1)p}\right) ds\, dt, \tag{2.5}$$

where the inequality (2.5) follows from the inequality $(1-x)^p \leq 1 - x^p$, $x \in [0,1]$, $p > 1$. It is easy to see that

$$\int_0^1 \left(t^{(\alpha-1)p} - (t+h)^{(\alpha-1)p}\right) dt$$
$$= c\left(1 + h^{(\alpha-1)p+1} - (1+h)^{(\alpha-1)p+1}\right) \leq ch^{(\alpha-1)p+1},$$

$$\int_0^1 \int_0^t \left((t-s)^{(\alpha-1)p} - (t+h-s)^{(\alpha-1)p}\right) ds\, dt$$
$$= \int_0^1 c\left(t^{(\alpha-1)p+1} + h^{(\alpha-1)p+1} - (t+h)^{(\alpha-1)p+1}\right) dt \leq ch^{(\alpha-1)p+1},$$

and

$$\int_0^1 \int_t^{t+h} (t+h-s)^{(\alpha-1)p}\, ds\, dt = ch^{(\alpha-1)p+1}.$$

Combining the above,

$$\int_0^1 |Tv(t+h) - Tv(t)|^p\ dt \leq ch^{(\alpha-1)p+1}$$

for all $v \in K$ (here, again, c is constant whose exact value is insignificant). The above inequality shows that the second condition of Theorem **1.3** is verified. Hence, the set $T(K)$ is relatively compact and thus $T : L^p(0,1) \to L^p(0,1)$ is a compact mapping.

In the remainder of the proof we note that $p \leq p_1$ since $p \leq 2$ and there exists (a unique) $R_* > 0$ such that the condition (2.3) holds if $R \leq R_*$. Let $R \leq R_*$ and $U \subset \mathcal{C} = L^p(0,1)$ be defined by

$$U = \{v \in L^p(0,1) : \|v\|_p < R\}.$$

Then, assuming $v \in \partial U$ is a solution of $v = \lambda Tv$ for $\lambda \in (0,1)$, we have, by (2.1),

$$R = \|v\|_p < \|Tv\|_p \leq \|\phi\|_p + \left\|\mathcal{I}_{0+}^{\alpha} Fv\right\|_p$$
$$\leq \quad \|\phi\|_p + \min\left\{\frac{1}{\Gamma(\alpha+1)}, C_1(\alpha;p)\right\}\left(\|a\|_{p_1} + b\|v\|_p^{p-1}\right)$$
$$\leq \quad \|\phi\|_p + \min\left\{\frac{1}{\Gamma(\alpha+1)}, C_1(\alpha;p)\right\}\left(\|a\|_{p_1} + bR^{p-1}\right),$$

which contradicts (2.3). Then $u \notin \partial U$ and, by Theorem **1.1**, T has a fixed point $v_0 \in L^p(0,1)$ with $\|v_0\|_p < R$. □

Remark. If $\alpha > 1/2$, it suffices to assume in the theorem above that $p \le 2$ (see, [1]).

If $p > 2$, we can follow the previous proof to obtain the next result.

Theorem **2.2**. *Assume that (H_1) and (H_2) hold and $2 < p < \frac{1}{1-\alpha}$. Then the integral equation (1.3) has a solution $v_0 \in L^p(0,1)$ with $\|v_0\|_p < R$ provided*

$$\|\phi\|_p + C_1(\alpha;p)\left(\|a\|_{p_1} + bR^{p-1}\right) \le R. \tag{2.6}$$

2.2. The Volterra equations of the first kind ($\eta = 0$)

The following results deal with the case of $\eta = 0$. In this case, since the function $\phi \equiv 0$, the equation (1.3) becomes

$$v(t) = \mathcal{I}^{\alpha}_{0+} f(t, v(t)). \tag{2.7}$$

Certainly, Theorems **2.1** and **2.2** apply as well to the present case with an obvious modification of (2.3) and (2.6) and the same condition on p (which is initially imposed due to $\phi \in L^p(0,1)$, $p(1-\alpha) < 1$) to yield Corollaries **2.1** and **2.2**, which are stated here just for the record.

Corollary **2.1**. *Assume that (H_1) and (H_2) hold and $1 < p < \frac{1}{1-\alpha}$ and $p \le 2$. Then the integral equation (2.7) has a solution $v_0 \in L^p(0,1)$ with $\|v_0\|_p < R$ provided*

$$\min\left\{\frac{1}{\Gamma(\alpha+1)}, C_1(\alpha;p)\right\}\left(\|a\|_{p_1} + bR^{p-1}\right) \le R.$$

Corollary **2.2**. *Assume that (H_1) and (H_2) hold and $2 < p < \frac{1}{1-\alpha}$. If $\eta = 0$, then the integral equation (2.7) has a solution $v_0 \in L^p(0,1)$ with $\|v_0\|_p < R$ provided*

$$C_1(\alpha;p)\left(\|a\|_{p_1} + bR^{p-1}\right) \le R.$$

The constraint of $\phi \in L_p(0,1)$ for $p < \frac{1}{\alpha-1}$ is now removed, and thus it is possible to complement the result of Corollaries **2.1** and **2.2** by considering the case $\frac{1}{\alpha-1} < p < \frac{2}{\alpha-1}$. This will be accomplished with the aid of Lemma **1.2**. Now we give the corresponding existence result whose proof is only sketched with just enough attention to detail to emphasize that it relies on Lemma **1.2**.

THEOREM **2.3**. *Assume that* (H_1) *and* (H_2) *hold and* $\frac{1}{1-\alpha} < p < \frac{2}{1-\alpha}$. *Then the integral equation* (2.7) *has a solution* $v_0 \in L^p(0,1)$ *with* $\|v_0\|_p < R$ *provided*

$$C_2(\alpha;,p)\left(\|a\|_{p_1} + bR^{p-1}\right) \le R.$$

P r o o f. Let $v \in L^p(0,1)$ and define

$$Tv(t) = \mathcal{I}^{\alpha}_{0+} f(t, v(t)), \quad t \in (0,1).$$

Since $\frac{1}{p} + \frac{1}{p_1} = 1$ and $\frac{1}{1-\alpha} < p < \frac{2}{1-\alpha}$, then $1 < p_1 < \frac{1}{\alpha}$ and $1 < p < \frac{p_1}{1-\alpha p_1}$. Hence $\mathcal{I}^{\alpha}_{0+} : L^{p_1}(0,1) \to L^p(0,1)$ is a bounded mapping by Lemma **1.2**. Furthermore, from (1.10) and (1.11),

$$\|\mathcal{I}^{\alpha}_{0+} v\|_p \le C_0(\alpha, p_1, p)\|v\|_{p_1} = C_2(\alpha; p)\|v\|_{p_1}.$$

The rest of the proof differs from that of Theorem **2.1** in only minor details.

□

The last result demonstrates the equivalence of the initial value problem (1.4), (1.5) with $\eta = 0$ and the Volterra equation (2.7).

THEOREM **2.4**. *The function* $v \in L^p(0,1)$, $p(1-\alpha) > 1$, *is a solution of the Volterra integral equation* (2.7) *if and only if* v *is a solution of the initial value problem* (1.4) *and* (1.5) *with* $\eta = 0$.

P r o o f. Let $v \in L^p(0,1)$, $p(1-\alpha) > 1$, be a solution of the initial value problem (1.4) and (1.5) with $\eta = 0$ and $\frac{1}{p} + \frac{1}{p_1} = 1$. So, $1 < p_1 < \frac{1}{\alpha}$. Let $0 < t_1 < t_2 < 1$. Then,

$$\begin{aligned}
&\Gamma(1-\alpha)\left|\mathcal{I}^{1-\alpha}_{0+} v(t_2) - \mathcal{I}^{\alpha}_{0+} v(t_1)\right| \\
&= \left|\int_0^{t_2} (t_2 - s)^{-\alpha} v(s)\, ds - \int_0^{t_1} (t_1 - s)^{-\alpha} v(s)\, ds\right| \\
&\le \int_0^{t_1} \left((t_1 - s)^{-\alpha} - (t_2 - s)^{-\alpha}\right)|v(s)|\, ds + \int_{t_1}^{t_2} (t_2 - s)^{-\alpha}|v(s)|\, ds \\
&\le \left(\int_0^{t_1} \left((t_1 - s)^{-\alpha} - (t_2 - s)^{-\alpha}\right)^{p_1} ds\right)^{1/p_1} \|v\|_p \\
&+ \left(\int_{t_1}^{t_2} (t_2 - s)^{-\alpha p_1} ds\right)^{1/p_1} \|v\|_p
\end{aligned}$$

$$\leq \left(\int_0^{t_1} \left((t_1 - s)^{-\alpha p_1} - (t_2 - s)^{-\alpha p_1} \right) ds \right)^{1/p_1} \|v\|_p$$
$$+ \left(\int_{t_1}^{t_2} (t_2 - s)^{-\alpha p_1} ds \right)^{1/p_1} \|v\|_p$$
$$= c \left[\left(t_1^{-\alpha p_1 + 1} + (t_2 - t_1)^{-\alpha p_1 + 1} - t_2^{-\alpha p_1 + 1} \right)^{1/p_1} + (t_2 - t_1)^{-\alpha + 1/p_1} \right] \|v\|_p$$
$$\leq 2c(t_2 - t_1)^{-\alpha + 1/p_1} \|v\|_p,$$

where c is a constant, which depends only on α and p_1. This shows that $\mathcal{I}_{0+}^{1-\alpha} v \in AC[0,1]$. Thus, by Theorem **1.4** (b), and since $\mathcal{I}_{0+}^{1-\alpha} v(0) = \eta = 0$,

$$v(t) = \mathcal{I}_{0+}^{\alpha} \mathcal{D}_{0+}^{\alpha} v(t) = \mathcal{I}_{0+}^{\alpha} f(\cdot, v(\cdot))(t),$$

that is, v is a solution of the integral equation (2.7).

The converse is clear by Theorem **1.4** (a). □

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