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RESEARCH PAPER

NONLINEAR RIEMANN-LIOUVILLE FRACTIONAL DIFFERENTIAL EQUATIONS WITH NONLOCAL ERDÉLYI-KOBER FRACTIONAL INTEGRAL CONDITIONS

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Abstract

In this paper we study a new class of Riemann-Liouville fractional differential equations subject to nonlocal Erdélyi-Kober fractional integral boundary conditions. Existence and uniqueness results are obtained by using a variety of fixed point theorems, such as Banach fixed point theorem, Nonlinear Contractions, Krasnoselskii fixed point theorem, Leray-Schauder Nonlinear Alternative and Leray-Schauder degree theory. Examples illustrating the obtained results are also presented.

MSC 2010: Primary 34A08; Secondary 34B15

Key Words and Phrases: fractional differential equations; nonlocal boundary conditions; Erdélyi-Kober fractional integral; fixed point theorem

1. Introduction

The aim of this paper is to establish the existence of solutions for the following nonlinear Riemann-Liouville fractional differential equation subject to nonlocal Erdélyi-Kober fractional integral conditions

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$$\begin{cases} D^q x(t) = f(t, x(t)), \quad t \in (0, T), \\ x(0) = 0, \quad \alpha x(T) = \displaystyle\sum_{i=1}^{m} \beta_i I_{\eta_i}^{\gamma_i, \delta_i} x(\xi_i), \end{cases} \tag{1.1}$$

where $1 < q \leq 2$, D^q is the standard Riemann-Liouville fractional derivative of order q, $I_{\eta_i}^{\gamma_i,\delta_i}$ is the Erdélyi-Kober fractional integral of order $\delta_i > 0$ with $\eta_i > 0$ and $\gamma_i \in \mathbb{R}$, $i = 1, 2, \ldots, m$, $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $\alpha, \beta_i \in \mathbb{R}$, $\xi_i \in (0, T)$, $i = 1, 2, \ldots, m$ are given constants.

In recent years, considerable interest in fractional differential equations has been stimulated due to their numerous applications in many fields of science and engineering. Important phenomena in finance, electromagnetics, acoustics, viscoelasticity, electrochemistry and material science are well described by differential equations of fractional order. For examples and recent development of the topic, see [1, 2, 3, 4, 5, 7, 6, 8, 9, 10, 13, 17, 18, 19, 23] and the references cited therein. However, it has been observed that most of the work on the topic involves either Riemann-Liouville or Caputo type fractional derivative. Besides these derivatives, the so called Erdélyi-Kober fractional derivative, as a generalization of the Riemann-Liouville fractional derivative, is often used, too. An Erdélyi-Kober operator is a fractional integration operation introduced by Arthur Erdélyi and Hermann Kober in 1940. These operators have been used by many authors, in particular, to obtain solutions of the single, dual and triple integral equations possessing special functions of mathematical physics as their kernels. For the theory and applications of the Erdélyi-Kober fractional integrals see e.g. [13, 14, 15, 20, 21, 22] and references cited therein.

In the present paper we initiate the study of boundary value problems like (1.1), in which we combine Riemann-Liouville fractional differential equations subject to the Erdélyi-Kober fractional integral boundary conditions. To the best of the author's knowledge this is the first paper dealing with Riemann-Liouville fractional differential equation subject to Erdélyi-Kober type integral boundary conditions.

Several new existence and uniqueness results are obtained by using a variety of fixed point theorems. Thus, in Theorem **4.1** we present an existence and uniqueness result via Banach's fixed point theorem, while in **4.2** we give another existence and uniqueness result nonlinear contractions. In the sequel existence results are obtained in Theorem **4.3**, via Krasnoselskii's fixed point theorem, in Theorem **4.4** via Leray-Schauder's nonlinear alternative and finally in Theorem **4.5** via Leray-Schauder's degree theory. Examples illustrating the obtained results are also presented.

The rest of the paper is organized as follows: In Section **2** we recall some some notations and definitions and lemmas from fractional calculus.

Some auxiliary lemmas, useful for the sequel, are presented in Section **3**. The main existence and uniqueness results are given in Section **4**, while the paper close with Section **5**, where enlighten examples are discussed illustrating our obtained results.

2. Preliminaries

In this section, we introduce some notations and definitions of fractional calculus and present preliminary results needed in our proofs later. For more details we refer to the books [20, 14, 19, 13], etc.

Definition **2.1**. The Riemann-Liouville fractional derivative of order $q > 0$ of a continuous function $f : (0, \infty) \to \mathbb{R}$ is defined by

$$D^q f(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-q-1} f(s)ds, \quad n-1 < q < n,$$

where $n = [q] + 1, [q]$ denotes the integer part of a real number q. Here Γ is the Gamma function defined by $\Gamma(q) = \int_0^\infty e^{-s} s^{q-1} ds$.

Definition **2.2**. The Riemann-Liouville fractional integral of order $q > 0$ of a continuous function $f : (0, \infty) \to \mathbb{R}$ is defined by

$$J^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)ds,$$

provided the integral exists.

Definition **2.3**. The Erdélyi-Kober fractional integral of order $\delta > 0$ with $\eta > 0$ and $\gamma \in \mathbb{R}$ of a continuous function $f : (0, \infty) \to \mathbb{R}$ is defined by

$$I_\eta^{\gamma,\delta} f(t) = \frac{\eta t^{-\eta(\delta+\gamma)}}{\Gamma(\delta)} \int_0^t \frac{s^{\eta\gamma+\eta-1} f(s)}{(t^\eta - s^\eta)^{1-\delta}} ds$$

provided the right side is pointwise defined on $\mathbb{R}_+$.

Remark **2.1**. For $\eta = 1$ the above operator is reduced to the Kober operator

$$I_1^{\gamma,\delta} f(t) = \frac{t^{-(\delta+\gamma)}}{\Gamma(\delta)} \int_0^t \frac{s^\gamma f(s)}{(t-s)^{1-\delta}} ds, \ \gamma, \ \delta > 0,$$

that was introduced for the first time by Kober in [15]. For $\gamma = 0$, the Kober operator is reduced to the Riemann-Liouville fractional integral with a power weight:

$$I_1^{0,\delta} f(t) = \frac{t^{-\delta}}{\Gamma(\delta)} \int_0^t \frac{f(s)}{(t-s)^{1-\delta}} ds, \ \delta > 0.$$

From the definition of the Riemann-Liouville fractional derivative and integral, we can obtain the following lemmas.

LEMMA **2.1**. (See [13]) *Let $q > 0$ and $y \in C(0,T) \cap L(0,T)$. Then the fractional differential equation $D^q y(t) = 0$ has a unique solution*

$$y(t) = c_1 t^{q-1} + c_2 t^{q-2} + \cdots + c_n t^{q-n},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$ and $n-1 < q < n$.

LEMMA **2.2**. (see [13]) *Let $q > 0$. Then for $y \in C(0,T) \cap L(0,T)$ it holds*

$$J^q D^q y(t) = y(t) + c_1 t^{q-1} + c_2 t^{q-2} + \cdots + c_n t^{q-n},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$ and $n-1 < q < n$.

3. Some auxiliary lemmas

For easy reference we include the following well known formula as a lemma (see for example, [14]).

LEMMA **3.1**. *Let $\delta, \eta > 0$ and $\gamma, q \in \mathbb{R}$. Then we have*

$$I_\eta^{\gamma,\delta} t^q = \frac{t^q \Gamma(\gamma + (q/\eta) + 1)}{\Gamma(\gamma + (q/\eta) + \delta + 1)}. \tag{3.1}$$

LEMMA **3.2**. *Let $1 < q \le 2$, $\delta_i, \eta_i > 0$, $\alpha, \gamma_i, \beta_i \in \mathbb{R}$, $\xi_i \in (0,T)$, $i = 1, 2, \ldots, m$ and $h \in C([0,T], \mathbb{R})$. Then $x \in C^2([0,T], \mathbb{R})$ is a solution of the linear Riemann-Liouville fractional differential equation subject to the Erdélyi-Kober fractional integral boundary conditions*

$$\begin{cases} D^q x(t) = h(t), \quad t \in (0,T), \\ x(0) = 0, \quad \alpha x(T) = \displaystyle\sum_{i=1}^m \beta_i I_{\eta_i}^{\gamma_i,\delta_i} x(\xi_i), \end{cases} \tag{3.2}$$

if and only if

$$x(t) = J^q h(t) - \frac{t^{q-1}}{\Lambda}\left(\alpha J^q h(T) - \sum_{i=1}^m \beta_i I_{\eta_i}^{\gamma_i,\delta_i} J^q h(\xi_i)\right), \tag{3.3}$$

where

$$\Lambda := \alpha T^{q-1} - \sum_{i=1}^m \frac{\beta_i \xi_i^{q-1} \Gamma(\gamma_i + (q-1)/\eta_i + 1)}{\Gamma(\gamma_i + (q-1)/\eta_i + \delta_i + 1)} \neq 0. \tag{3.4}$$

P r o o f. Using Lemmas **2.1**-**2.2**, the equation (3.2) can be expressed as an equivalent integral equation

$$x(t) = J^q h(t) - c_1 t^{q-1} - c_2 t^{q-2}, \tag{3.5}$$

for $c_1, c_2 \in \mathbb{R}$. The first condition of (3.2) implies that $c_2 = 0$. Taking the Erdélyi-Kober fractional integral of order $\delta_i > 0$ with $\eta_i > 0$ and $\gamma_i \in \mathbb{R}$ for (3.5) and using Lemma **3.1**, we have

$$I_{\eta_i}^{\gamma_i,\delta_i} x(t) = I_{\eta_i}^{\gamma_i,\delta_i} J^q h(t) - c_1 \frac{t^{q-1}\Gamma(\gamma_i + (q-1)/\eta_i + 1)}{\Gamma(\gamma_i + (q-1)/\eta_i + \delta_i + 1)}.$$

The second condition of (3.2) yields

$$\begin{aligned} \alpha J^q h(T) - c_1 \alpha T^{q-1} \quad = \quad & \sum_{i=1}^{m} \beta_i I_{\eta_i}^{\gamma_i,\delta_i} J^q h(\xi_i) \\ & - c_1 \sum_{i=1}^{m} \frac{\beta_i \xi_i^{q-1}\Gamma(\gamma_i + (q-1)/\eta_i + 1)}{\Gamma(\gamma_i + (q-1)/\eta_i + \delta_i + 1)}, \end{aligned}$$

which implies

$$c_1 = \frac{1}{\Lambda}\left(\alpha J^q h(T) - \sum_{i=1}^{m} \beta_i I_{\eta_i}^{\gamma_i,\delta_i} J^q h(\xi_i)\right).$$

Substituting the values of c_1 and c_2 in (3.5), we obtain the desired solution (3.3). The converse follows by direct computation. □

4. Main Results

Throughout this paper, for convenience, we use the following expressions, based on Definitions **2.2** and **2.3**:

$$J^q f(s,x(s))(z) = \frac{1}{\Gamma(q)}\int_0^z (z-s)^{q-1} f(s,x(s))ds, \quad z \in \{t, T\},$$

for $t \in [0,T]$ and

$$\begin{aligned} & I_{\eta_i}^{\gamma_i,\delta_i} J^q f(s,x(s))(\xi_i) \\ & \quad = \frac{\eta_i \xi_i^{-\eta_i(\delta_i+\gamma_i)}}{\Gamma(q)\Gamma(\delta_i)} \int_0^{\xi_i}\int_0^r \frac{r^{\eta_i\gamma_i+\eta_i-1}(r-s)^{q-1}}{(\xi_i^{\eta_i} - r^{\eta_i})^{1-\delta_i}} f(s,x(s))ds\,dr, \end{aligned}$$

where $\xi_i \in (0,T)$ for $i = 1, 2, \ldots, m$.

Let $\mathcal{C} = C([0,T],\mathbb{R})$ denote the Banach space of all continuous functions from $[0,T]$ to $\mathbb{R}$ endowed with the norm defined by $\|x\| = \sup_{t\in[0,T]} |x(t)|$. Using Lemma **3.2**, we can define an operator $\mathcal{G} : \mathcal{C} \to \mathcal{C}$ by

$$\begin{aligned} & \mathcal{G}x(t) = J^q f(s,x(s))(t) \\ & - \frac{t^{q-1}}{\Lambda}\left(\alpha J^q f(s,x(s))(T) - \sum_{i=1}^{m} \beta_i I_{\eta_i}^{\gamma_i,\delta_i} J^q f(s,x(s))(\xi_i)\right). \end{aligned} \tag{4.1}$$

It should be noticed that problem (1.1) has solution if and only if the operator $\mathcal{G}$ has fixed points.

In the following, for the sake of convenience, we set a constant

$$\Psi := \frac{T^q}{\Gamma(q+1)} + \frac{|\alpha|T^{2q-1}}{|\Lambda|\Gamma(q+1)} + \frac{T^{q-1}}{|\Lambda|\Gamma(q+1)}\sum_{i=1}^{m}\frac{|\beta_i|\xi_i^q\Gamma(\gamma_i+(q/\eta_i)+1)}{\Gamma(\gamma_i+(q/\eta_i)+\delta_i+1)}. \tag{4.2}$$

In the following subsections we prove existence, as well as existence and uniqueness results, for the boundary value problem (1.1) by using a variety of fixed point theorems.

4.1. Existence and uniqueness result via Banach's fixed point theorem

THEOREM **4.1**. *Assume that:*

(H_1) *there exists a positive constant* L *such that* $|f(t,x)-f(t,y)| \le L|x-y|$, *for each* $t\in[0,T]$ *and* $x,y\in\mathbb{R}$.

If

$$L\Psi < 1, \tag{4.3}$$

where Ψ *is defined by (4.2), then the boundary value problem* (1.1) *has a unique solution on* $[0,T]$.

P r o o f. We transform the problem (1.1) into a fixed point problem, $x=\mathcal{G}x$, where the operator $\mathcal{G}$ is defined as in (4.1). Observe that the fixed points of the operator $\mathcal{G}$ are solutions of problem (1.1). Applying the Banach contraction mapping principle, we shall show that $\mathcal{G}$ has a unique fixed point.

We let $\sup_{t\in[0,T]}|f(t,0)| = M < \infty$, and choose

$$r \ge \frac{M\Psi}{1-L\Psi}. \tag{4.4}$$

To show that $\mathcal{G}B_r \subset B_r$, where $B_r = \{x\in\mathcal{C} : \|x\|\le r\}$, we have for any $x\in B_r$ that

$$\begin{aligned}
|(\mathcal{G}x)(t)| \le{}& \sup_{t\in[0,T]}\Bigg\{J^q|f(s,x(s))|(t) + \frac{|\alpha|t^{q-1}}{|\Lambda|}J^q|f(s,x(s))|(T)\\
&+\frac{t^{q-1}}{|\Lambda|}\sum_{i=1}^{m}|\beta_i|I_{\eta_i}^{\gamma_i,\delta_i}J^q|f(s,x(s))|(\xi_i)\Bigg\}\\
\le{}& J^q(|f(s,x(s))-f(s,0)|+|f(s,0)|)(T)\\
&+\frac{|\alpha|T^{q-1}}{|\Lambda|}J^q(|f(s,x(s))-f(s,0)|+|f(s,0)|)(T)\\
&+\frac{T^{q-1}}{|\Lambda|}\sum_{i=1}^{m}|\beta_i|I_{\eta_i}^{\gamma_i,\delta_i}J^q(|f(s,x(s))-f(s,0)|+|f(s,0)|)(\xi_i)
\end{aligned}$$

$$
\begin{aligned}
&\le (L\|x\|+M)J^q(1)(T)+(L\|x\|+M)\frac{|\alpha|T^{q-1}}{|\Lambda|}J^q(1)(T)\\
&\quad +(L\|x\|+M)\frac{T^{q-1}}{|\Lambda|}\sum_{i=1}^{m}|\beta_i|I_{\eta_i}^{\gamma_i,\delta_i}J^q(1)(\xi_i)\\
&\le (Lr+M)\Psi\le r,
\end{aligned}
$$

which implies that $\mathcal{G}B_r \subset B_r$.

Next, we let $x, y \in \mathcal{C}$. Then for $t \in [0, T]$, we have

$$
\begin{aligned}
&|\mathcal{G}x(t)-\mathcal{G}y(t)|\\
&\le \sup_{t\in[0,T]}\Bigg\{J^q|f(s,x(s))-f(s,y(s))|(t)\\
&\quad +\frac{|\alpha|t^{q-1}}{|\Lambda|}J^q|f(s,x(s))-f(s,y(s))|(T)\\
&\quad +\frac{t^{q-1}}{|\Lambda|}\sum_{i=1}^{m}|\beta_i|I_{\eta_i}^{\gamma_i,\delta_i}J^q|f(s,x(s))-f(s,y(s))|(\xi_i)\Bigg\}\\
&\le L\|x-y\|J^q(1)(T)+L\|x-y\|\frac{|\alpha|T^{q-1}}{|\Lambda|}J^q(1)(T)\\
&\quad +L\|x-y\|\frac{T^{q-1}}{|\Lambda|}\sum_{i=1}^{m}|\beta_i|I_{\eta_i}^{\gamma_i,\delta_i}J^q(1)(\xi_i)\\
&= L\Psi\|x-y\|,
\end{aligned}
$$

which leads to $\|\mathcal{G}x - \mathcal{G}y\| \le L\Psi\|x - y\|$. As $L\Psi < 1$, $\mathcal{G}$ is a contraction. Therefore, we deduce, by the Banach's contraction mapping principle, that $\mathcal{G}$ has a fixed point which is the unique solution of the problem (1.1). The proof is completed. □

4.2. Existence and uniqueness result via nonlinear contractions

Definition **4.1**. Let E be a Banach space and let $\mathcal{F} : E \to E$ be a mapping. $\mathcal{F}$ is said to be a nonlinear contraction if there exists a continuous nondecreasing function $\Theta : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\Theta(0) = 0$ and $\Theta(\varepsilon) < \varepsilon$ for all $\varepsilon > 0$ with the property:

$$\|\mathcal{F}x - \mathcal{F}y\| \le \Theta(\|x - y\|), \qquad \forall x, y \in E.$$

LEMMA **4.1**. (Boyd and Wong, [11]) *Let E be a Banach space and let $\mathcal{F} : E \to E$ be a nonlinear contraction. Then $\mathcal{F}$ has a unique fixed point in E.*

THEOREM **4.2**. *Let $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the assumption:*

(H_2) $|f(t,x)-f(t,y)| \le z(t)\dfrac{|x-y|}{A^*+|x-y|}$, *for $t \in [0,T]$, $x, y \ge 0$, where $z : [0,T] \to \mathbb{R}^+$ is continuous and A^* the constant defined by*

$$A^* := \left(J^q z(T) + \frac{|\alpha|T^{q-1}}{|\Lambda|} J^q z(T) + \frac{T^{q-1}}{|\Lambda|} \sum_{i=1}^{m} |\beta_i| I_{\eta_i}^{\gamma_i,\delta_i} J^q z(\xi_i) \right).$$

Then the problem (1.1) *has a unique solution on* $[0, T]$.

P r o o f. We define the operator $\mathcal{G} : \mathcal{C} \to \mathcal{C}$ as in (4.1) and the continuous nondecreasing function $\Theta : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\Theta(\varepsilon) = \frac{A^*\varepsilon}{A^*+\varepsilon}, \qquad \forall \varepsilon \ge 0.$$

Note that the function Θ satisfies $\Theta(0) = 0$ and $\Theta(\varepsilon) < \varepsilon$ for all $\varepsilon > 0$.

For any $x, y \in \mathcal{C}$ and for each $t \in [0, T]$, we have

$$\begin{aligned}
& |\mathcal{G}x(t) - \mathcal{G}y(t)| \\
\le\ & J^q|f(s,x(s)) - f(s,y(s))|(T) + \frac{|\alpha|T^{q-1}}{|\Lambda|} J^q|f(s,x(s)) - f(s,y(s))|(T) \\
& + \frac{T^{q-1}}{|\Lambda|} \sum_{i=1}^{m} |\beta_i| I_{\eta_i}^{\gamma_i,\delta_i} J^q|f(s,x(s)) - f(s,y(s))|(\xi_i) \\
\le\ & J^q \left(z(s)\frac{|x-y|}{A^*+|x-y|} \right)(T) + \frac{|\alpha|T^{q-1}}{|\Lambda|} J^q \left(z(s)\frac{|x-y|}{A^*+|x-y|} \right)(T) \\
& + \frac{T^{q-1}}{|\Lambda|} \sum_{i=1}^{n} |\beta_i| I_{\eta_i}^{\gamma_i,\delta_i} J^q \left(z(s)\frac{|x-y|}{A^*+|x-y|} \right)(\xi_i) \\
\le\ & \frac{\Theta(\|x-y\|)}{A^*} \left(J^q z(T) + \frac{|\alpha|T^{q-1}}{|\Lambda|} J^q z(T) + \frac{T^{q-1}}{|\Lambda|} \sum_{i=1}^{m} |\beta_i| I_{\eta_i}^{\gamma_i,\delta_i} J^q z(\xi_i) \right) \\
=\ & \Theta(\|x-y\|).
\end{aligned}$$

This implies that $\|\mathcal{G}x - \mathcal{G}y\| \le \Theta(\|x-y\|)$. Therefore $\mathcal{G}$ is a nonlinear contraction. Hence, by Lemma **4.1** the operator $\mathcal{G}$ has a unique fixed point which is the unique solution of the problem (1.1). This completes the proof. $\square$

4.3. Existence result via Krasnoselskii's fixed point theorem

LEMMA **4.2**. (Krasnoselskii's fixed point theorem, [16]) *Let M be a closed, bounded, convex and nonempty subset of a Banach space X. Let A, B be the operators such that (a) $Ax + Bx \in M$ whenever $x, y \in M$; (b) A is compact and continuous; (c) B is a contraction mapping. Then there exists $z \in M$ such that $z = Az + Bz$.*

THEOREM **4.3**. *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying (H_1). In addition we assume that:*

(H_3) $|f(t,x)| \leq \varphi(t)$, $\forall (t,x) \in [0,T] \times \mathbb{R}$, *and* $\varphi \in C([0,T], \mathbb{R}^+)$.

Then the problem (1.1) *has at least one solution on* $[0,T]$ *provided*

$$L\left(\frac{T^{q-1}}{|\Lambda|\Gamma(q+1)}\sum_{i=1}^{m}\frac{|\beta_i|\xi_i^q\Gamma(\gamma_i+(q/\eta_i)+1)}{\Gamma(\gamma_i+(q/\eta_i)+\delta_i+1)}\right) < 1. \tag{4.5}$$

P r o o f. We define the operators $\mathcal{G}_1$ and $\mathcal{G}_2$ by

$$\begin{aligned}
\mathcal{G}_1x(t) &= J^q f(s,x(s))(t) - \frac{\alpha\, t^{q-1}}{\Lambda} J^q f(s,x(s))(T), \quad t \in [0,T], \\
\mathcal{G}_2x(t) &= \frac{t^{q-1}}{\Lambda}\sum_{i=1}^{m}\beta_i (I_{\eta_i}^{\gamma_i,\delta_i} J^q f(s,x(s)))(\xi_i), \quad t \in [0,T].
\end{aligned}$$

Setting $\sup_{t\in[0,T]}\varphi(t) = \|\varphi\|$ and choosing

$$\rho \geq \|\varphi\|\Psi,$$

where Ψ is defined by (4.2), we consider $B_\rho = \{x \in \mathcal{C} : \|x\| \leq \rho\}$. For any $x, y \in B_\rho$, we have

$$\begin{aligned}
&|\mathcal{G}_1x(t) + \mathcal{G}_2y(t)| \\
\leq\ & \sup_{t\in[0,T]}\left\{ J^q|f(s,x(s))|(t) + \frac{|\alpha|t^{q-1}}{|\Lambda|}J^q|f(s,x(s))|(T) \right. \\
& \left. + \frac{t^{q-1}}{|\Lambda|}\sum_{i=1}^{m}|\beta_i| I_{\eta_i}^{\gamma_i,\delta_i} J^q|f(s,x(s))|(\xi_i)\right\} \\
\leq\ & \|\varphi\|\left(\frac{T^q}{\Gamma(q+1)} + \frac{|\alpha|T^{2q-1}}{|\Lambda|\Gamma(q+1)} + \frac{T^{q-1}}{|\Lambda|\Gamma(q+1)}\sum_{i=1}^{m}\frac{|\beta_i|\xi_i^q\Gamma(\gamma_i+(q/\eta_i)+1)}{\Gamma(\gamma_i+(q/\eta_i)+\delta_i+1)}\right) \\
=\ & \|\varphi\|\Psi \leq \rho.
\end{aligned}$$

This shows that $\mathcal{G}_1x + \mathcal{G}_2y \in B_\rho$. It is easy to see using (4.5) that $\mathcal{G}_2$ is a contraction mapping.

Continuity of f implies that the operator $\mathcal{G}_1$ is continuous. Also, $\mathcal{G}_1$ is uniformly bounded on B_ρ as

$$\|\mathcal{G}_1 x\| \leq \left(\frac{T^q}{\Gamma(q+1)} + \frac{|\alpha| T^{2q-1}}{|\Lambda|\Gamma(q+1)} \right) \|\varphi\|.$$

Now we prove the compactness of the operator $\mathcal{G}_1$.

We define $\sup_{(t,x)\in[0,T]\times B_\rho} |f(t,x)| = \bar{f} < \infty$, and consequently, for $t_1, t_2 \in [0,T]$, $t_1 < t_2$, we have

$$\begin{aligned} |\mathcal{G}_1 x(t_2) - \mathcal{G}_1 x(t_1)| &= \Bigg| J^q f(s,x(s))(t_2) - \frac{\alpha\, t_2^{q-1}}{\Lambda} J^q f(s,x(s))(T) \\ &\quad - J^q f(s,x(s))(t_1) + \frac{\alpha\, t_1^{q-1}}{\Lambda} J^q f(s,x(s))(T) \Bigg| \\ &\leq \frac{\bar{f}}{\Gamma(q+1)} |t_2^q - t_1^q| + \frac{\bar{f}|\alpha| T^q}{|\Lambda|\Gamma(q+1)} \left| t_2^{q-1} - t_1^{q-1} \right|, \end{aligned}$$

which is independent of x and tend to zero as $t_2 - t_1 \to 0$. Thus, $\mathcal{G}_1$ is equicontinuous. So $\mathcal{G}_1$ is relatively compact on B_ρ. Hence, by the Arzelá-Ascoli theorem, $\mathcal{G}_1$ is compact on B_ρ. Thus all the assumptions of Lemma **4.2** are satisfied. So the conclusion of Lemma **4.2** implies that the problem (1.1) has at least one solution on $[0,T]$ □

4.4. Existence result via Leray-Schauder's Nonlinear Alternative

LEMMA **4.3**. (Nonlinear alternative for single valued maps, [12]) *Let E be a Banach space, C a closed, convex subset of E, U an open subset of C and $0 \in U$. Suppose that $\mathcal{A} : \bar{U} \to C$ is a continuous, compact (that is, $\mathcal{A}(\bar{U})$ is a relatively compact subset of C) map. Then either*

(i) $\mathcal{A}$ has a fixed point in $\bar{U}$, or

(ii) there is a $x \in \partial U$ (the boundary of U in C) and $\lambda \in (0,1)$ with $x = \lambda \mathcal{A}(x)$.

THEOREM **4.4**. *Assume that:*

(H_4) there exists a continuous nondecreasing function $\Phi : [0,\infty) \to (0,\infty)$ and a function $p \in C([0,T], \mathbb{R}^+)$ such that

$$|f(t,x)| \leq p(t)\Phi(\|x\|) \text{ for each } (t,x) \in [0,T]\times\mathbb{R};$$

(H_5) there exists a constant $N > 0$ such that

$$\frac{N}{\Phi(N)\|p\|\Psi} > 1,$$

where Ψ is defined by (4.2).

Then the problem (1.1) has at least one solution on $[0,T]$.

P r o o f. Let the operator $\mathcal{G}$ be defined by (4.1). We first show that $\mathcal{G}$ *maps bounded sets (balls) into bounded sets in* $C([0,T],\mathbb{R})$. For a positive constant r, let $B_r = \{x \in C([0,T],\mathbb{R}) : \|x\| \leq r\}$ be a bounded ball in $C([0,T],\mathbb{R})$. Then for $t \in [0,T]$ we have

$$\begin{aligned}
|\mathcal{G}x(t)| &\leq J^q|f(s,x(s))|(T) + \frac{|\alpha|T^{q-1}}{|\Lambda|}J^q|f(s,x(s))|(T) \\
&\quad + \frac{T^{q-1}}{|\Lambda|}\sum_{i=1}^{m}|\beta_i|I_{\eta_i}^{\gamma_i,\delta_i}J^q|f(s,x(s))|(\xi_i) \\
&\leq \Phi(\|x\|)J^q p(s)(T) + \Phi(\|x\|)\frac{|\alpha|T^{q-1}}{|\Lambda|}J^q p(s)(T) \\
&\quad + \Phi(\|x\|)\frac{T^{q-1}}{|\Lambda|}\sum_{i=1}^{m}|\beta_i|I_{\eta_i}^{\gamma_i,\delta_i}J^q p(s)(\xi_i) \\
&\leq \Phi(\|x\|)\|p\|\Bigg(\frac{T^q}{\Gamma(q+1)} + \frac{|\alpha|T^{2q-1}}{|\Lambda|\Gamma(q+1)} \\
&\quad + \frac{T^{q-1}}{|\Lambda|\Gamma(q+1)}\sum_{i=1}^{m}\frac{|\beta_i|\xi_i^q\Gamma(\gamma_i+(q/\eta_i)+1)}{\Gamma(\gamma_i+(q/\eta_i)+\delta_i+1)}\Bigg)
\end{aligned}$$

and consequently,

$$\|\mathcal{G}x\| \leq \Phi(r)||p||\Psi.$$

Next we will show that *the operator* $\mathcal{G}$ *maps bounded sets into equicontinuous sets of* $C([0,T],\mathbb{R})$. Let $\tau_1, \tau_2 \in [0,T]$ with $\tau_1 < \tau_2$ and $x \in B_r$. Then we have

$$\begin{aligned}
&|\mathcal{G}x(\tau_2) - \mathcal{G}x(\tau_1)| \\
\leq\ & |J^q f(s,x(s))(\tau_2) - J^q f(s,x(s))(\tau_1)| \\
&+ \frac{|\alpha||\tau_2^{q-1} - \tau_1^{q-1}|}{|\Lambda|}J^q|f(s,x(s))|(T) \\
&+ \frac{|\tau_2^{q-1} - \tau_1^{q-1}|}{|\Lambda|}\sum_{i=1}^{m}|\beta_i|I_{\eta_i}^{\gamma_i,\delta_i}J^q|f(s,x(s))|(\xi_i) \\
\leq\ & \frac{\Phi(r)}{\Gamma(q)}\left|\int_0^{\tau_1}[(\tau_2-s)^{q-1} - (\tau_1-s)^{q-1}]p(s)ds + \int_{\tau_1}^{\tau_2}(\tau_2-s)^{q-1}p(s)ds\right| \\
&+ \frac{|\tau_2^{q-1} - \tau_1^{q-1}|\Phi(r)}{|\Lambda|}\left(|\alpha|J^q p(s)(T) + \sum_{i=1}^{m}|\beta_i|I_{\eta_i}^{\gamma_i,\delta_i}J^q p(s)(\xi_i)\right).
\end{aligned}$$

As $\tau_2 - \tau_1 \to 0$, the right-hand side of the above inequality tends to zero independently of $x \in B_r$. Therefore by the Arzelá-Ascoli theorem the operator $\mathcal{G} : \mathcal{C} \to \mathcal{C}$ is completely continuous.

Let x be a solution. Then, for $t \in [0,T]$, and following the similar computations as in the first step, we have

$$|x(t)| \leq \Phi(\|x\|)||p||\Psi,$$

which leads to

$$\frac{\|x\|}{\Phi(\|x\|)||p||\Psi} \leq 1.$$

In view of (H_5), there exists N such that $\|x\| \neq N$. Let us set

$$U = \{x \in C([0,T],\mathbb{R}) \,:\, \|x\| < N\}.$$

We see that the operator $\mathcal{G} : \overline{U} \to C([0,T],\mathbb{R})$ is continuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x = \theta\mathcal{G}x$ for some $\theta \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type, we deduce that $\mathcal{G}$ has a fixed point $x \in \overline{U}$ which is a solution of the problem (1.1). This completes the proof. □

4.5. Existence result via Leray-Schauder's Degree Theory

THEOREM **4.5**. *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that:*

(H_6) *there exist constants $0 \leq \nu < \Psi^{-1}$, and $M > 0$ such that*

$$|f(t,x)| \leq \nu|x| + M \text{ for all } (t,x) \in [0,T] \times \mathbb{R},$$

where Ψ is defined by (4.2).

Then the problem (1.1) *has at least one solution on* $[0,T]$.

P r o o f. We define an operator $\mathcal{G} : \mathcal{C} \to \mathcal{C}$ as in (4.1). In view of the fixed point problem

$$x = \mathcal{G}x, \tag{4.6}$$

we are going to prove the existence of at least one solution $x \in C[0,T]$ satisfying (4.6). Set a ball $B_R \subset C[0,T]$, as

$$B_R = \{x \in \mathcal{C} : \max_{t\in[0,T]} |x(t)| < R\},$$

where a constant radius $R > 0$. Hence, we will show that the operator $\mathcal{G} : \overline{B}_R \to C[0,T]$ satisfies a condition

$$x \neq \theta\mathcal{G}x, \quad \forall x \in \partial B_R, \quad \forall \theta \in [0,1]. \tag{4.7}$$

We set

$$H(\theta,x) = \theta\mathcal{G}x, \quad x \in \mathcal{C}, \quad \theta \in [0,1].$$

As shown in Theorem **4.4** we have that the operator $\mathcal{G}$ is continuous, uniformly bounded and equicontinuous. Then, by the Arzelá-Ascoli theorem, a continuous map h_θ defined by $h_\theta(x) = x - H(\theta,x) = x - \theta\mathcal{G}x$ is completely continuous. If (4.7) holds, then the following Leray-Schauder degrees are well defined and by the homotopy invariance of topological degree, it follows that

$$\begin{aligned}\deg(h_\theta, B_R, 0) &= \deg(I - \theta\mathcal{G}, B_R, 0) = \deg(h_1, B_R, 0)\\ &= \deg(h_0, B_R, 0) = \deg(I, B_R, 0) = 1 \neq 0, \quad 0 \in B_R,\end{aligned}$$

where I denotes the unit operator. By the nonzero property of Leray-Schauder degree, we have $h_1(x) = x - \mathcal{G}x = 0$ for at least one $x \in B_R$. Let us assume that $x = \theta\mathcal{G}x$ for some $\theta \in [0,1]$ and for all $t \in [0,T]$ so that

$$\begin{aligned}|x(t)| &= |\theta\mathcal{G}x(t)|\\ &\leq J^q|f(s,x(s))|(T) + \frac{|\alpha|T^{q-1}}{|\Lambda|}J^q|f(s,x(s))|(T)\\ &\quad + \frac{T^{q-1}}{|\Lambda|}\sum_{i=1}^{m}|\beta_i|I_{\eta_i}^{\gamma_i,\delta_i}J^q|f(s,x(s))|(\xi_i)\\ &\leq (\nu|x| + M)J^q(1)(T) + (\nu|x| + M)\frac{|\alpha|T^{q-1}}{|\Lambda|}J^q(1)(T)\\ &\quad + (\nu|x| + M)\frac{T^{q-1}}{|\Lambda|}\sum_{i=1}^{m}|\beta_i|I_{\eta_i}^{\gamma_i,\delta_i}J^q(1)(\xi_i)\\ &= (\nu|x| + M)\Psi,\end{aligned}$$

which taking the norm $\sup_{t\in[0,T]}|x(t)| = \|x\|$ and solving for $\|x\|$, yields

$$\|x\| \leq \frac{M\Psi}{1 - \nu\Psi}.$$

If $R = \dfrac{M\Psi}{1 - \nu\Psi} + 1$, inequality (4.7) holds. This completes the proof. □

5. Examples

In this section, we present some examples to illustrate our results.

Example **5.1**. Consider the following nonlinear Riemann-Liouville fractional differential equation with nonlocal Erdélyi-Kober fractional integral conditions

$$\begin{cases} D^{\frac{3}{2}}x(t) = \dfrac{\sin^2(\pi t)}{2(e^t+9)^2}\left(\dfrac{1}{|x(t)|+1}+1\right)|x(t)| + \dfrac{\sqrt{3}}{4},\ t \in [0,5], \\ x(0) = 0, \\ \dfrac{2}{3}x(5) = \dfrac{e}{2} I_{\frac{\sqrt{3}}{5}}^{\frac{3}{7},\frac{5}{3}} x\left(\dfrac{4}{3}\right) + \dfrac{\pi}{3} I_{\frac{\sqrt{2}}{5}}^{\frac{\sqrt{3}}{8},\frac{2}{9}} x\left(\dfrac{3}{2}\right) + \dfrac{\sqrt{\pi}}{6} I_{\frac{e}{3}}^{\frac{e^2}{4},\frac{\sqrt{e}}{2}} x\left(\dfrac{2}{7}\right). \end{cases} \tag{5.1}$$

Here $q = 3/2$, $m = 3$, $T = 5$, $\alpha = 2/3$, $\beta_1 = e/2$, $\beta_2 = \pi/3$, $\beta_3 = \sqrt{\pi}/6$, $\eta_1 = \sqrt{3}/5$, $\eta_2 = \sqrt{2}/5$, $\eta_3 = e/3$, $\gamma_1 = 3/7$, $\gamma_2 = \sqrt{3}/8$, $\gamma_3 = e^2/4$, $\delta_1 = 5/3$, $\delta_2 = 2/9$, $\delta_3 = \sqrt{e}/2$, $\xi_1 = 4/3$, $\xi_2 = 3/2$, $\xi_3 = 2/7$, and $f(t,x) = (\sin^2(\pi t)/2(e^t+9)^2)((|x|/(|x|+1)) + 1)|x| + (\sqrt{3}/4)$. Since $|f(t,x) - f(t,y)| \leq (1/100)|x-y|$, then, (H_1) is satisfied with $L = 1/100$. By using the Maple program, we can find that $\Psi \approx 97.24231429$. Thus $L\Psi \approx 0.9724231429 < 1$. Hence, by Theorem **4.1**, the boundary value problem (5.1) has a unique solution on $[0,5]$.

Example **5.2**. Consider the following nonlinear Riemann-Liouville fractional differential equation with nonlocal Erdélyi-Kober fractional integral conditions

$$\begin{aligned} &D^{\frac{8}{5}}x(t) = \frac{t^2}{1+e^t}\left(\frac{|x(t)|}{|x(t)|+1}\right) + 2t + \frac{5}{8}, \qquad t \in [0,1], \\ &x(0) = 0, \quad \frac{3}{8}x(1) = -\frac{e}{\sqrt{2}} I_{\frac{2}{5}}^{\frac{3}{4},\frac{1}{8}} x\left(\frac{1}{2}\right) + \frac{\pi}{2} I_{\frac{1}{2}}^{\frac{4}{3},\frac{3}{2}} x\left(\frac{2}{3}\right) \\ &\quad + \frac{\pi}{4} I_{\frac{3}{2}}^{\frac{1}{3},\frac{5}{4}} x\left(\frac{3}{4}\right) + \frac{e^3}{6} I_{\frac{4}{3}}^{\frac{5}{7},\frac{7}{3}} x\left(\frac{4}{5}\right). \end{aligned} \tag{5.2}$$

Here $q = 8/5$, $m = 4$, $T = 1$, $\alpha = 3/8$, $\beta_1 = -e/\sqrt{2}$, $\beta_2 = \pi/2$, $\beta_3 = \pi/4$, $\beta_4 = e^3/6$, $\eta_1 = 2/5$, $\eta_2 = 1/2$, $\eta_3 = 3/2$, $\eta_4 = 4/3$, $\gamma_1 = 3/4$, $\gamma_2 = 4/3$, $\gamma_3 = 1/3$, $\gamma_4 = 5/7$, $\delta_1 = 1/8$, $\delta_2 = 3/2$, $\delta_3 = 5/4$, $\delta_4 = 7/3$, $\xi_1 = 1/2$, $\xi_2 = 2/3$, $\xi_3 = 3/4$, $\xi_4 = 4/5$, and $f(t,x) = (t^2/(1+e^t))(|x|/(|x|+1)) + 2t + (5/8)$. We choose $z(t) = t^2/2$ and that $A^* \approx 0.3282054040$. Clearly, we have

$$|f(t,x) - f(t,y)| \leq \frac{t^2}{2}\left(\frac{|x-y|}{0.3282054040 + |x-y|}\right).$$

Hence, by Theorem **4.2**, the boundary value problem 5.2 has a unique solution on $[0,1]$.

Example **5.3**. Consider the following nonlinear Riemann-Liouville fractional differential equation with nonlocal Erdélyi-Kober fractional integral conditions

$$\begin{cases} D^{\frac{5}{4}}x(t) = \dfrac{e^{-t^2}\sin^2 3t}{5(t+4)}\left(\dfrac{|x(t)|}{|x(t)|+1}\right) + \dfrac{2}{3t+1}, \qquad t \in [0,8], \\ x(0) = 0, \\ \dfrac{4}{9}x(8) = \dfrac{\pi}{6} I_{\frac{7}{9}}^{\frac{5}{4},\frac{2}{3}} x\left(\dfrac{3}{2}\right) + \dfrac{\ln 2}{\sqrt{3}} I_{\frac{2}{3}}^{\frac{4}{9},\frac{3}{4}} x(6) + \dfrac{\ln 3}{\sqrt{2}} I_{\frac{5}{2}}^{\frac{7}{2},\frac{3}{5}} x\left(\dfrac{7}{2}\right). \end{cases} \tag{5.3}$$

Here $q = 5/4$, $m = 3$, $T = 8$, $\alpha = 4/9$, $\beta_1 = \pi/6$, $\beta_2 = \ln 2/\sqrt{3}$, $\beta_3 = \ln 3/\sqrt{2}$, $\eta_1 = 7/9$, $\eta_2 = 2/3$, $\eta_3 = 5/2$, $\gamma_1 = 5/4$, $\gamma_2 = 4/9$, $\gamma_3 = 7/2$, $\delta_1 = 2/3$, $\delta_2 = 3/4$, $\delta_3 = 3/5$, $\xi_1 = 3/2$, $\xi_2 = 6$, $\xi_3 = 7/2$, and $f(t,x) = ((e^{-t^2}\sin^2 3t)/5(t+4))(|x|/(|x|+1)) + (2/(3t+1))$. Since $|f(t,x) - f(t,y)| \le (1/20)|x-y|$, (H_1) is satisfied with $L = 1/20$. By using Maple program, we show that

$$L\left(\frac{T^{q-1}}{|\Lambda|\Gamma(q+1)}\sum_{i=1}^{m}\frac{|\beta_i|\xi_i^q\Gamma(\gamma_i + (q/\eta_i) + 1)}{\Gamma(\gamma_i + (q/\eta_i) + \delta_i + 1)}\right) \approx 0.5830838955 < 1.$$

Clearly, we have

$$|f(t,x)| = \left|\frac{2e^{-t^2}\sin^2 3t}{t+12}\left(\frac{|x|}{|x|+1}\right) + \frac{2}{3t+1}\right| \le \frac{2}{t+12} + \frac{2}{3t+1},$$

with $\varphi(t) = (2/(t+12)) + (2/(3t+1))$. Therefore, (H_3) is satisfied. Hence, by Theorem **4.3**, the boundary value problem (5.3) has at least one solution on $[0,8]$.

Example **5.4**. Consider the following nonlinear Riemann-Liouville fractional differential equation with nonlocal Erdélyi-Kober fractional integral conditions

$$\begin{aligned} &D^{\frac{6}{5}}x(t) = \left(\frac{t^2+1}{5(1+4\pi^2)}\right)\left(\frac{x^2(t)}{|x(t)|+1} + \frac{\sqrt{|x(t)|}}{2(1+\sqrt{|x(t)|})} + \frac{1}{2}\right), \ t \in [0,2\pi] \\ &x(0) = 0, \quad \frac{5}{3}x(2\pi) = \frac{e^3}{\pi} I_{\frac{2}{5}}^{\frac{3}{7},\frac{4}{3}} x\left(\frac{\pi}{4}\right) - \frac{\sqrt{2}}{3} I_{\frac{3}{8}}^{\frac{5}{2},\frac{7}{3}} x\left(\frac{\pi}{2}\right) \\ &-\frac{3}{\sqrt{5}} I_{\frac{\sqrt{3}}{4}}^{\frac{\sqrt{5}}{8},\frac{2}{3}} x\left(\frac{11\pi}{10}\right) - \frac{2}{\sqrt{7}} I_{\frac{1}{2}}^{\frac{3}{\pi},\frac{2}{9}} x\left(\frac{3\pi}{2}\right). \end{aligned} \tag{5.4}$$

Here $q = 6/5$, $m = 4$, $T = 2\pi$, $\alpha = 5/3$, $\beta_1 = e^3/\pi$, $\beta_2 = -\sqrt{2}/3$, $\beta_3 = -3/\sqrt{5}$, $\beta_4 = -2/\sqrt{7}$, $\eta_1 = 2/5$, $\eta_2 = 3/8$, $\eta_3 = \sqrt{3}/4$, $\eta_4 = 1/2$, $\gamma_1 = 3/7$, $\gamma_2 = 5/2$, $\gamma_3 = \sqrt{5}/8$, $\gamma_4 = 3/\pi$, $\delta_1 = 4/3$, $\delta_2 = 7/3$, $\delta_3 = 2/3$, $\delta_4 = 2/9$, $\xi_1 = \pi/4$, $\xi_2 = \pi/2$, $\xi_3 = 11\pi/10$, $\xi_4 = 3\pi/2$, and $f(t,x) = ((t^2+1)/5(1+4\pi^2))((x^2/(|x|+1)) + (\sqrt{|x|}/2(1+\sqrt{|x|})) + (1/2))$. It is easy to verify that $\Psi \approx 4.196586197$. Clearly, we have

$$|f(t,x)|=\left|\left(\frac{t^2+1}{5(1+4\pi^2)}\right)\left(\frac{x^2}{|x|+1}+\frac{\sqrt{|x|}}{2(1+\sqrt{|x|})}+\frac{1}{2}\right)\right|\le\frac{(t^2+1)(|x|+1)}{5(1+4\pi^2)}.$$

Choosing $p(t) = (t^2+1)/(5(1+4\pi^2))$ and $\Phi(|x|) = |x|+1$, we can show that

$$\frac{N}{\Phi(N)\|p\|\Psi} > 1,$$

which implies $N > 5.223442990$. Hence, by Theorem **4.4**, the boundary value problem (5.4) has at least one solution on $[0, 2\pi]$.

Example **5.5**. Consider the following nonlinear Riemann-Liouville fractional differential equation with nonlocal Erdélyi-Kober fractional integral conditions

$$\begin{aligned}
&D^{\frac{7}{6}}x(t) = \frac{1}{2\pi}\sin(\pi|x(t)|)\left(\frac{|x|}{|x|+1}\right)+1, \quad t\in[0,e],\\
&x(0) = 0,\\
&\frac{6}{13}x(e) = \frac{3}{2}I_{\frac{3}{2}}^{\frac{3}{4},\frac{3}{5}}x\left(\frac{\sqrt{3}}{2}\right)+\frac{5}{3}I_{\frac{3}{\sqrt{5}}}^{\frac{5}{\sqrt{3}},\frac{8}{5}}x\left(\frac{11}{5}\right)+\frac{7}{4}I_{\frac{4}{7}}^{\frac{3}{8},\frac{6}{\sqrt{7}}}x\left(\frac{e}{2}\right).
\end{aligned} \tag{5.5}$$

Here $q = 7/6$, $m = 3$, $T = e$, $\alpha = 6/13$, $\beta_1 = 3/2$, $\beta_2 = 5/3$, $\beta_3 = 7/4$, $\eta_1 = 3/2$, $\eta_2 = 3/\sqrt{5}$, $\eta_3 = 4/7$, $\gamma_1 = 3/4$, $\gamma_2 = 5/\sqrt{3}$, $\gamma_3 = 3/8$, $\delta_1 = 3/5$, $\delta_2 = 8/5$, $\delta_3 = 6/\sqrt{7}$, $\xi_1 = \sqrt{3}/2$, $\xi_2 = 11/5$, $\xi_3 = e/2$, and $f(t,x) = (1/2\pi)\sin(\pi|x|)(|x|/(|x|+1))+1$. We can show that $\Psi \approx 1.273554230$. Since

$$|f(t,x)| = \left|\frac{1}{2\pi}\sin(\pi|x|)\left(\frac{|x|}{|x|+1}\right)+1\right| \le \frac{1}{2}|x|+1,$$

then, (H_6) is satisfied with $\nu = 1/2$ and $M = 1$ such that $\nu = 1/2 < 1/\Psi \approx 0.7852040977$. Hence, by Theorem **4.5**, the boundary value problem (5.5) has at least one solution on $[0, e]$.

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