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RESEARCH PAPER

OPERATOR METHOD FOR CONSTRUCTION OF SOLUTIONS OF LINEAR FRACTIONAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

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Abstract

One of the effective methods to find explicit solutions of differential equations is the method based on the operator representation of solutions. The essence of this method is to construct a series, whose members are the relevant iteration operators acting to some classes of sufficiently smooth functions. This method is widely used in the works of B. Bondarenko for construction of solutions of differential equations of integer order. In this paper, the operator method is applied to construct solutions of linear differential equations with constant coefficients and with Caputo fractional derivatives. Then the fundamental solutions are used to obtain the unique solution of the Cauchy problem, where the initial conditions are given in terms of the unknown function and its derivatives of integer order. Comparison is made with the use of Mikusinski operational calculus for solving similar problems.

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1. Introduction

The fractional differential equations have achieved in recent years a considerable interest both in mathematics and in applications. They have been used in modeling of many physical and chemical processes and in engineering (see, for example, [1] -[3], [5], [9]-[11], [23], [27],[29]. In its turn,

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mathematical aspects of fractional differential equations and methods of their solution have been discussed by many authors, including [1],[7],[8],[9], [15]-[17], [24],[26] and [28].

Let m be a positive integer and $m-1<\alpha\le m$. The Caputo fractional derivative of order α is defined, for f a C^m function on its interval of definition, as (see [24], [17])

$$D^\alpha f(t) := I^{m-\alpha}\frac{d^m}{dt^m}f(t), \quad t>0,$$

where

$$I^\beta f(t) := \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}f(\tau)d\tau, \ t>0, \quad \beta>0,$$

is the Riemann-Liouville fractional integral of order β.

If $\beta\to 0$, it is easy to verify that $I^\beta f(t)\to f(t)$ almost everywhere [28]. Therefore we may define $I^0 f(t) := f(t)$, which leads for $\alpha=m$ to the equality

$$D^m f(t) = \frac{d^m}{dt^m}f(t).$$

Consider a homogenous linear fractional differential equation

$$D^\alpha y(t) - a_1 D^{\alpha-1}y(t) - ... - a_{m-1}D^{\alpha-(m-1)}y(t) - a_m y(t) = 0, \ \ t>0 \quad (1.1)$$

with constant real coefficients a_j, $j=1,...,m$ and $m>1$.

We look for solutions $y\in C^m([0,\infty))$.

If $m=1$, i.e. $0<\alpha\le 1$, then we have the equation $D^\alpha y(t)-a_1y(t)=0$ and $y\in C^1([0,\infty))$.

When $\alpha=m$ equation (1.1) coincides with integer order linear differential equation and in this case the construction of the fundamental solutions and of the solution of the Cauchy problem with the initial data

$$y^{(n)}(0) := \frac{d^n}{dt^n}y(0) = b_n, \ \ n=0,1,...,m-1, \quad (1.2)$$

is well known. This fundamental theory, based on the characteristic equation

$$\lambda^m - a_1\lambda^{m-1} - \ldots - a_{m-1}\lambda - a_m = 0,$$

can be found in any textbook on differential equations. The main goal of the present paper is to construct the solution of the Cauchy problem (1.1), (1.2) and to obtain the fundamental solutions of equation (1.1). For this purpose we modify and use the technique based on the method of operator algorithms introduced by B.A. Bondarenko in [4] and then developed by V.V. Karachik [14] for ordinary differential equations.

There are different methods of solving Cauchy problems for differential equations of fractional order. A detailed survey of these methods can be found in [17]. In the paper of M.M. Dzerbashyan and A.B. Nersesyan [8] it was investigated the Cauchy problem for the special class of differential equations of fractional order. To solve this problem the authors reduced it to an equivalent integral equation. One of the common methods of solving differential equations of fractional order is the method of integral transformations. A detailed description of this method can be found in the paper [15] and books by [17], [24], etc. An effective method for constructing explicit solutions and solving the Cauchy problem for differential equations of fractional order is based on the Mikusinski operational calculus. In the papers of Yu. Luchko et al. [12],[19]-[22] this method has been applied for solving linear differential equations of fractional order with constant coefficients and with derivatives of type Riemann-Liouville, Caputo and Hilfer. After that, this method has been applied for a general equation with the operator of R. Hilfer [18]. In the paper A. Pskhu [25] it has been formulated and solved the initial problem for linear ordinary differential equations of fractional order with Riemann-Liouville derivatives. He reduced the problem to an integral equation and constructed the explicit solution in terms of the Wright function. We also note that in papers [6] and [13] the Cauchy problem for differential equations of fractional order has been studied by the Adomian decomposition method.

The main idea of the method used in the present paper is based on the properties of the normed system of functions and consists on the following:

Let us introduce the notations

$$L_1 = D^{\alpha},\ L_2 = a_1 D^{\alpha-1} + ... + a_{m-1} D^{\alpha-(m-1)} + a_m,$$

and $\mathbb{R}_+ = (0, +\infty)$. Then equation (1.1) can be written as $L_1 y(t) = L_2 y(t)$, $t \in \mathbb{R}_+$.

A system of functions $\{f_k(t)\}_{k=0}^{\infty}$ is called to be f-normed with respect to operator L_1 in the domain $\mathbb{R}_+$, if the equations $L_1 f_0(t) = f(t)$, and $L_1 f_k(t) = f_{k-1}(t)$ hold everywhere in $\mathbb{R}_+$ (see [14]). In the case $f(t) \equiv 0$, the system $\{f_k(t)\}$ is called 0-normed with respect to L_1.

Now let the system $\{f_k(t)\}_{k=0}^{\infty}$ be 0-normed with respect to L_1 in the domain $\mathbb{R}_+$ satisfying the following two conditions everywhere in $\mathbb{R}_+$:

(i) $L_1 L_2 f_k(t) = L_2 L_1 f_k(t), k = 1, 2, ...$

(ii) the series

$$y(t) = \sum_{k=0}^{\infty} L_2^k f_k(t) \tag{1.3}$$

converges and allows term-wise application of L_1.

Then it is easy to verify that the function defined in (1.3) is a solution of (1.1). Indeed,

$$L_1 y(t) = \sum_{k=1}^{\infty} L_2^k L_1 f_k(t) = \sum_{k=1}^{\infty} L_2^k f_{k-1}(t) = L_2 \sum_{k=0}^{\infty} L_2^k f_k(t) = L_2 y(t).$$

If we consider, instead of a 0-normed system, a f-normed system, then we may construct solutions of non-homogeneous equations.

We note also that a similar method was used to solve the Cauchy problem for the equation $(D^{\alpha,\beta} - \lambda)^N y(t) = f(t)$ in [30], where $D^{\alpha,\beta}$ is the generalized Riemann-Liouville fractional derivative introduced by R. Hilfer (see [12]).

2. Homogenous equations

In this section we consider equation (1.1) and construct a 0-normed system of functions with respect to operator L_1. Based on this system of functions we find the fundamental system of solutions of equation (1.1).

For $s = 0, 1, ..., m-1$, we introduce the following system of functions

$$f_{s,k}(t) = \frac{t^{\alpha k+s}}{\Gamma(\alpha k + s + 1)}, \;\; k = 0, 1, 2, ... \tag{2.1}$$

First we show that for any s the system $\{f_{s,k}(t)\}_{k=0}^{\infty}$ is 0-normed with respect to L_1 and, from Section **1**, it satisfies conditions (i) and (ii).

LEMMA **2.1**. *For any $s = 0, 1, ..., m-1$ the system of functions $\{f_{s,k}(t)\}_{k=0}^{\infty}$ is 0-normed with respect to L_1 in the domain $\mathbb{R}_+$, i.e. for all $t \in \mathbb{R}_+$*

$$D^\alpha f_{s,0}(t) = 0, \quad D^\alpha f_{s,k}(t) = f_{s,k-1}(t), \;\; k \geq 1,$$

P r o o f. Obviously $D^\alpha t^s = 0$ for all $s = 0, 1, ..., m-1$. Therefore $D^\alpha f_{s,0}(t) = 0$ for these s.

Let $k \geq 1$. Then by the definition of derivatives D^α one has

$$\begin{aligned} D^\alpha t^{\alpha k+s} &= \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{d^m}{d\tau^m} \tau^{\alpha k+s} d\tau \\ &= \frac{(\alpha k+s)\cdots(\alpha k+s-(m-1))}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \tau^{\alpha k+s-m} d\tau \\ &= \frac{(\alpha k+s)\cdots(\alpha k+s-(m-1))}{\Gamma(m-\alpha)} \\ &\quad \times \frac{\Gamma(m-\alpha)\Gamma(\alpha k+s+1-m)}{\Gamma(\alpha k+s+1-\alpha)} t^{\alpha k+s-\alpha} \\ &= \frac{\Gamma(\alpha k+s+1)}{\Gamma(\alpha(k-1)+s+1)} t^{\alpha(k-1)+s}. \end{aligned}$$

Thus

$$D^\alpha t^{\alpha k+s} = \frac{\Gamma(\alpha k+s+1)}{\Gamma(\alpha(k-1)+s+1)} t^{\alpha(k-1)+s}. \tag{2.2}$$

Therefore,

$$D^\alpha f_{s,k}(t) = \frac{1}{\Gamma(\alpha k+s+1)} \frac{\Gamma(\alpha k+s+1)}{\Gamma(\alpha(k-1)+s+1)} t^{\alpha(k-1)+s} = f_{s,k-1}(t).$$

□

LEMMA **2.2**. *For any $s = 0, 1, .., m-1$, $k \geq 1$ and all $t \in \mathbb{R}_+$, it is satisfied that $L_1 L_2 f_{s,k}(t) = L_2 L_1 f_{s,k}(t)$, i.e.*

$$D^\alpha D^{\alpha-j} f_{s,k}(t) = D^{\alpha-j} D^\alpha f_{s,k}(t), \quad j = 1, ..., m-1. \tag{2.3}$$

P r o o f. Since $m-1-j < \alpha - j \leq m-j$ then, by the definition of the Caputo derivatives, we have

$$\begin{aligned}
D^{\alpha-j} t^{\alpha k+s} &= I^{m-j-(\alpha-j)} \frac{d^{m-j}}{dt^{m-j}} t^{\alpha k+s} = I^{m-\alpha} \frac{d^{m-j}}{dt^{m-j}} t^{\alpha k+s} \\
&= \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{d^{m-j}}{d\tau^{m-j}} \tau^{\alpha k+s} d\tau \\
&= \frac{(\alpha k+s)\cdots(\alpha k+s-(m-j-1))}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \tau^{\alpha k+s-(m-j)} d\tau \\
&= \frac{(\alpha k+s)\cdots(\alpha k+s-(m-j-1))}{\Gamma(m-\alpha)} \frac{\Gamma(m-\alpha)\Gamma(\alpha k+s+1-(m-j))}{\Gamma(\alpha k+s+1-(\alpha-j))} \\
&\times t^{\alpha k+s-(\alpha-j)} = \frac{\Gamma(\alpha k+s+1)}{\Gamma(\alpha k+s+1-(\alpha-j))} t^{\alpha k+s-(\alpha-j)}.
\end{aligned}$$

Thus

$$D^{\alpha-j} f_{s,k}(t) = \frac{1}{\Gamma(\alpha k+s+1-(\alpha-j))} t^{\alpha k+s-(\alpha-j)}.$$

Therefore

$$\begin{aligned}
D^\alpha D^{\alpha-j} f_{s,k}(t) &= \frac{1}{\Gamma(\alpha k+s+1-(\alpha-j))} \frac{1}{\Gamma(m-\alpha)} \\
&\times \int_0^t (t-\tau)^{m-\alpha-1} \frac{d^m}{d\tau^m} \tau^{\alpha k+s-(\alpha-j)} d\tau.
\end{aligned}$$

If the number $\alpha k + s - (\alpha - j) \leq m-1$ and it is integer, i.e. if $k = 1$ and $s \in \{0, 1, ..., m-j-1\}$, then

$$D^\alpha D^{\alpha-j} f_{s,k}(t) = 0. \tag{2.4}$$

Otherwise one has

$$\begin{aligned}
D^\alpha D^{\alpha-j} f_{s,k}(t) &= \frac{(\alpha k+s-(\alpha-j))\cdots(\alpha k+s-(\alpha-j)-(m-1))}{\Gamma(\alpha k+s+1-(\alpha-j))\Gamma(m-\alpha)} \\
&\times \int_0^t (t-\tau)^{m-\alpha-1} \tau^{\alpha k+s-(\alpha-j)-m} d\tau,
\end{aligned}$$

and the integral can be written as

$$\frac{\Gamma(m-\alpha)\Gamma(\alpha k+s+1-(\alpha-j)-m)}{\Gamma(\alpha k+s+1-(\alpha-j)-\alpha)}t^{\alpha k+s-(\alpha-j)-\alpha}.$$

Therefore, if $k\geq 1$ and $s\notin\{0,1,...,m-j-1\}$, then

$$D^{\alpha}D^{\alpha-j}f_{s,k}(t)=\frac{1}{\Gamma(\alpha k+s+1-(\alpha-j)-\alpha)}t^{\alpha k+s-(\alpha-j)-\alpha}. \tag{2.5}$$

On the other hand, from (2.2) we have

$$D^{\alpha-j}D^{\alpha}f_{s,k}(t)=\frac{1}{\Gamma(\alpha(k-1)+s+1)}D^{\alpha-j}t^{\alpha(k-1)+s}.$$

Obviously, if $k=1$ and $s\in\{0,1,...,m-j-1\}$, then

$$D^{\alpha-j}D^{\alpha}f_{s,k}(t)=0. \tag{2.6}$$

Otherwise one has

$$\begin{aligned}D^{\alpha-j}D^{\alpha}f_{s,k}(t)&=\frac{1}{\Gamma(\alpha(k-1)+s+1)\Gamma(m-\alpha)}\\&\quad\times\int_0^t(t-\tau)^{m-\alpha-1}\frac{d^{m-j}}{d\tau^{m-j}}\tau^{\alpha(k-1)+s}d\tau\\&=\frac{(\alpha k-\alpha+s)\cdots(\alpha k-\alpha+s-(m-j-1)}{\Gamma(\alpha(k-1)+s+1)\Gamma(m-\alpha)}\\&\quad\times\int_0^t(t-\tau)^{m-\alpha-1}\tau^{\alpha k-\alpha+s-(m-j)}d\tau.\end{aligned}$$

The last integral has the form

$$\frac{\Gamma(m-\alpha)\Gamma(\alpha k+s+1-\alpha-(m-j))}{\Gamma(\alpha k+s+1-(\alpha-j)-\alpha)}t^{\alpha k+s-(\alpha-j)-\alpha}.$$

Therefore, if $k\geq 1$ and $s\notin\{0,1,...,m-j-1\}$, then

$$D^{\alpha-j}D^{\alpha}f_{s,k}(t)=\frac{1}{\Gamma(\alpha k+s+1-(\alpha-j)-\alpha)}t^{\alpha k+s-(\alpha-j)-\alpha}. \tag{2.7}$$

Comparing equalities (2.4) with (2.6) and (2.5) with (2.7) we deduce the equality (2.3). $\square$

According to equation (1.3) we introduce the following m functions

$$\begin{aligned}y_s(t)&=\sum_{k=0}^{\infty}L_2^k f_{s,k}(t)\\&=\sum_{k=0}^{\infty}(a_1D^{\alpha-1}+\cdots+a_{m-1}D^{\alpha-(m-1)}+a_m)^k\frac{t^{\alpha k+s}}{\Gamma(\alpha k+s+1)},\ t\geq 0,\end{aligned} \tag{2.8}$$

where $s=0,1,...,m-1$.

Note that it is well known (see [25], page 12), that for the Gamma function the asymptotic estimation

$$\Gamma(x+1) = \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left\{1 + O\left(\frac{1}{x}\right)\right\},$$

holds as $x \to \infty$.

THEOREM **2.1**. *Series (2.8) converges uniformly on any segment* $[0, T]$. *Moreover it can be differentiated term-wise for all* $t \in \mathbb{R}_+$ *and at any natural order, and operator* L_1 *may be applied term-wise.*

P r o o f. If $m = 1$, i.e. $0 < \alpha \leq 1$, then the statement of theorem follows from the asymptotic estimation of the Gamma function.

Let us assume $m > 1$ and denote $\varepsilon = \alpha - (m-1)$ and $a = \max\{|a_1|, ..., |a_m|\}$. Then it is not hard to verify that

$$\begin{aligned} &\left|(a_1 D^{m-2+\varepsilon} + \cdots + a_{m-1} D^{\varepsilon} + a_m)^k \frac{t^{\alpha k+s}}{\Gamma(\alpha k + s + 1)}\right| \\ &\leq a^k (D^{m-2+\varepsilon} + \cdots + D^{\varepsilon} + 1)^k \frac{t^{\alpha k+s}}{\Gamma(\alpha k + s + 1)}. \end{aligned}$$

Note that $\sum_{i_1+\cdots+i_m=k} \binom{k}{i_1 ... i_m} = m^k$.

Therefore

$$\begin{aligned} & (D^{m-2+\varepsilon} + \cdots + D^{\varepsilon} + 1)^k f_{s,k}(t) \\ = & \sum_{i_1+\cdots+i_m=k} \binom{k}{i_1 ... i_m} D^{(m-2+\varepsilon)i_1} \cdots D^{\varepsilon i_{m-1}} f_{s,k}(t) \\ = & \sum_{n=0}^{(m-2)k} \sum_{(m-2)i_1+(m-3)i_2 \cdots + i_{m-2}=n} \binom{k}{i_1 ... i_m} D^n D^{\varepsilon(i_1+\cdots+i_{m-1})} f_{s,k}(t) \\ \leq & \; m^k \sum_{n=0}^{(m-2)k} D^n \sum_{j=0}^{k} D^{\varepsilon j} f_{s,k}(t), \end{aligned}$$

since the corresponding derivatives of $f_{s,k}(t)$ are positive and

$$D^{j+\varepsilon} f_{s,k}(t) = D^j D^{\varepsilon} f_{s,k}(t).$$

Let $(D-1) g_{s,k}(t) = f_{s,k}(t)$ and $(D^{\varepsilon} - 1) h_{s,k}(t) = g_{s,k}(t)$, i.e.

$$g_{s,k}(t) = \int_0^t e^{\tau} f_{s,k}(t-\tau) d\tau$$

and

$$h_{s,k}(t) = \int_0^t \tau^{\varepsilon-1} E_{\varepsilon,\varepsilon}(\tau^{\varepsilon}) g_{s,k}(t-\tau) d\tau,$$

where

$$E_{\varepsilon,\mu}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\varepsilon k+\mu)} \tag{2.9}$$

is the Mittag-Leffler function (see [28]).

Then $f_{s,k}(t)=(D-1)(D^{\varepsilon}-1)h_{s,k}(t)$, and therefore

$$\begin{aligned}
m^k\sum_{n=0}^{(m-2)k}D^n\sum_{j=0}^{k}D^{\varepsilon j}f_{s,k}(t) &= m^k(D^{(m-2)k+1}-1)(D^{\varepsilon(k+1)}-1)h_{s,k}(t)\\
&= m^k(D^{(\alpha-1)k+1+\varepsilon}-D^{(m-2)k+1}\\
&= -D^{\varepsilon(k+1)}+1)h_{s,k}(t).
\end{aligned}$$

After some routine calculation as below, we have the following estimate for $h_{s,k}(t)$:

$$\begin{aligned}
h_{s,k}(t) &\le \frac{E_{\varepsilon,\varepsilon}(t^{\varepsilon})e^t}{\Gamma(\alpha k+s+1)}\int_0^t\tau^{\varepsilon-1}\int_0^{t-\tau}(t-\tau-p)^{\alpha k+s}dpd\tau\\
&= \frac{E_{\varepsilon,\varepsilon}(\tau^{\varepsilon})e^t}{\Gamma(\alpha k+s+2)}\int_0^t\tau^{\varepsilon-1}(t-\tau)^{\alpha k+s+1}d\tau\\
&= \frac{E_{\varepsilon,\varepsilon}(\tau^{\varepsilon})e^t}{\Gamma(\alpha k+s+2)}\frac{\Gamma(\varepsilon)\Gamma(\alpha k+s+2)}{\Gamma(\alpha k+s+2+\varepsilon)}t^{\alpha k+s+1+\varepsilon}\\
&= \frac{G(t)t^{\alpha k+s+1+\varepsilon}}{\Gamma(\alpha k+s+2+\varepsilon)},
\end{aligned}$$

where $G(t):=\Gamma(\varepsilon)E_{\varepsilon,\varepsilon}(\tau^{\varepsilon})e^t$ is a bounded function in any segment $[0,T]$.

Let $N-1<\beta\le N$, and $0<N\le(m-1)k+2$ be an integer number. Let the integer k_0 be such that $k_0\,\varepsilon>1$. From here on, in this section it is assumed that $k\ge k_0$.

We apply the operator D^{β} to the function $h_{s,k}(t)$. First, since all the corresponding derivatives of $f_{s,k}(t)$, up to order N–1, are zero at the origin, we conclude that

$$\frac{d^N}{dt^N}g_{s,k}(t)=\int_0^t e^{\tau}\frac{d^N}{dt^N}f_{s,k}(t-\tau)d\tau.$$

In the same way one has that

$$\frac{d^N}{dt^N}h_{s,k}(t)=\int_0^t\tau^{\varepsilon-1}E_{\varepsilon,\varepsilon}(\tau^{\varepsilon})\frac{d^N}{dt^N}g_{s,k}(t-\tau)d\tau.$$

Therefore,

$$D^{\beta}h_{s,k}(t)=I^{N-\beta}\frac{d^N}{dt^N}h_{s,k}(t)=\frac{1}{\Gamma(N-\beta)}\int_0^t(t-x)^{N-\beta-1}\frac{d^N}{dx^N}h_{s,k}(x)dx$$

$$= \int_0^t \tau^{\varepsilon-1} E_{\varepsilon,\varepsilon}(\tau^\varepsilon)\big[\frac{1}{\Gamma(N-\beta)}\int_\tau^t (t-x)^{N-\beta-1}\frac{d^N}{dx^N} g_{s,k}(x-\tau)dx\big]d\tau$$

$$= \int_0^t \tau^{\varepsilon-1} E_{\varepsilon,\varepsilon}(\tau^\varepsilon)\big[\frac{1}{\Gamma(N-\beta)}\int_0^{t-\tau} (t-\tau-p)^{N-\beta-1}\frac{d^N}{dp^N} g_{s,k}(p)dp\big]d\tau.$$

Thus

$$D^\beta h_{s,k}(t) = \int_0^t \tau^{\varepsilon-1} E_{\varepsilon,\varepsilon}(\tau^\varepsilon)(D^\beta g_{s,k})(t-\tau)d\tau,$$

or, by using the same argument,

$$D^\beta h_{s,k}(t) = \int_0^t \tau^{\varepsilon-1} E_{\varepsilon,\varepsilon}(\tau^\varepsilon)\int_0^{t-\tau} e^p (D^\beta f_{s,k})(t-\tau-p)dpd\tau.$$

To prove this result, we estimate $D^\beta h_{s,k}(t)$. First, by direct calculation (see proof of Lemma **2.1**), one has

$$D^\beta f_{s,k}(t) = D^\beta \frac{t^{\alpha k+s}}{\Gamma(\alpha k+s+1)} = \frac{t^{\alpha k+s-\beta}}{\Gamma(\alpha k+s+1-\beta)}.$$

Therefore, in a similar manner as we estimated $h_{s,k}(t)$, we have

$$D^\beta h_{s,k}(t) \le \frac{G(t)t^{\alpha k+s+1+\varepsilon-\beta}}{\Gamma(\alpha k+s+2+\varepsilon-\beta)}.$$

Making use of this estimate and the one of $h_{s,k}(t)$, we easily obtain

$$\big|(D^{(\alpha-1)k+1+\varepsilon} - D^{(m-2)k+1} - D^{\varepsilon(k+1)}+1)h_{s,k}(t)\big| \le (D^{(\alpha-1)k+1+\varepsilon}+1)h_{s,k}(t)$$

$$\le \frac{G(t)t^{k+s}}{\Gamma(k+s+1)} + \frac{G(t)t^{\alpha k+s+1+\varepsilon}}{\Gamma(\alpha k+s+2+\varepsilon)}.$$

Therefore the asymptotic estimation of the Gamma function implies:

$$\sum_{k=k_0}^\infty \big|(a_1 D^{m-2+\varepsilon} + \cdots + a_{m-1}D^\varepsilon + a_m)^k \frac{t^{\alpha k+s}}{\Gamma(\alpha k+s+1)}\big|$$

$$\le G(t)\left\{t^s \sum_{k=k_0}^\infty \frac{(amt)^k}{\Gamma(k+s+1)} + t^{s+1+\varepsilon}\sum_{k=k_0}^\infty \frac{(am)^k t^{\alpha k}}{\Gamma(\alpha k+s+2+\varepsilon)}\right\} < \infty.$$

Thus series (2.8) converges uniformly on any segment $[0,T]$.

Moreover, if $t \in \mathbb{R}_+$, then it is not hard to verify that

$$\sum_{k=k_0+n}^\infty \big|\frac{d^n}{dt^n}\big[\big(a_1 D^{m-2+\varepsilon} + \cdots + a_{m-1}D^\varepsilon + a_m\big)^k \frac{t^{\alpha k+s}}{\Gamma(\alpha k+s+1)}\big]\big|$$

$$\le G(t)\frac{d^n}{dt^n}\left\{t^s \sum_{k=k_0+n}^\infty \frac{(amt)^k}{\Gamma(k+s+1)}\right.$$

$$+\, t^{s+1+\varepsilon} \sum_{k=k_0+n}^{\infty} \frac{(am)^k t^{\alpha k}}{\Gamma(\alpha k + s + 2 + \varepsilon)} \Bigg\} < \infty,$$

which implies the convergence of the series on the left hand side.

Hence, when $t \in \mathbb{R}_+$ one may differentiate series (2.8) term-wise at any natural order.

Similarly, for any $t \in \mathbb{R}_+$ one obtains

$$\sum_{k=k_0+m}^{\infty} \left| L_1\big(a_1 D^{m-2+\varepsilon} + \cdots + a_{m-1} D^{\varepsilon} + a_m\big)^k \frac{t^{\alpha k+s}}{\Gamma(\alpha k + s + 1)} \right|$$

$$\leq G(t) \left\{ \sum_{k=k_0+m}^{\infty} \frac{t^{s-\alpha}(amt)^k}{\Gamma(k+s-\alpha+1)} + \sum_{k=k_0+m}^{\infty} \frac{t^{s-\alpha+1+\varepsilon}(am)^k t^{\alpha k}}{\Gamma(\alpha k + s - \alpha + 2 + \varepsilon)} \right\} < \infty,$$

i.e. the series on the left hand side converges. Hence, when $t \in \mathbb{R}_+$ we can apply operator L_1 term-wise to (2.8) and the result holds. $\square$

As consequence of the previous assertions, we conclude that for each value of $s = 0, 1, ..., m-1$ functions (2.8) are solutions of equation (1.1).

THEOREM **2.2**. *The functions $y_s(t)$, $s = 0, 1, ..., m-1$ are linearly independent on any segment $[t_1, t_2] \subset \mathbb{R}_+$.*

P r o o f. The result holds by contradiction. Let us assume that functions $y_s(t)$, $s = 0, 1, ..., m-1$ are not linearly independent on some segment $[t_1, t_2] \subset \mathbb{R}_+$, i.e. there exist constants C_s, not all of them are equal to zero, such that

$$\varphi(t) := \sum_{s=0}^{m-1} C_s y_s(t) = 0, \;\; t \in [t_1, t_2].$$

According to Theorem **2.1**, the functions $y_s(t)$ are power series, converging in $\mathbb{R}_+$. Hence the function $\varphi(t)$ is a power series too, converging in $\mathbb{R}_+$. Therefore $\varphi(t) = 0, t \in [t_1, t_2]$ implies the equality $\varphi(t) = 0, \; t \in \mathbb{R}_+$ and, in particular, $\varphi(0) = 0$.

Now it is easy to verify that $0 = \sum_{s=0}^{m-1} C_s y_s(0) = C_0$. Hence,

$$\sum_{s=1}^{m-1} C_s y_s(t) = 0, \;\; t \in \mathbb{R}_+.$$

If we differentiate this equality, taking into account that $y_s^{(1)}(t) = y_{s-1}(t)$, we have that

$$\sum_{s=1}^{m-1} C_s y_{s-1}(t) = \sum_{s=0}^{m-2} C_{s+1} y_s(t) = 0.$$

Using the same arguments as above, we deduce that $C_s = 0$ for all $s = 0, 1, ..., m-1$. Thus we arrive to a contradiction and we deduce the linear independence of functions $y_s(t)$, $s = 0, 1, ..., m-1$, on $[t_1, t_2]$. $\square$

REMARK **2.1**. The function $y_s(t)$ can be written in terms of multivariate Mittag-Leffler function (see, for example, [20]):

$$E_{(b_1,b_2,\ldots,b_m),b}(z_1, z_2, \ldots, z_m) := \sum_{k=0}^{\infty} \sum_{i_1+\cdots+i_m=k} C_{k,i_1\ldots i_m} \frac{\prod_{j=1}^{m} z_j^{i_j}}{\Gamma(b + \sum_{j=1}^{m} b_j\, i_j)}, \tag{2.10}$$

where $C_{k,i_1\ldots i_m} = \frac{k!}{i_1!\cdots i_{m-1}! i_m!}$, $i_1, i_2, \ldots, i_m \geq 0$, are the multinomial coefficients.

Indeed, let $\alpha_j = \alpha - j, j = 1, 2, ..., m-1$. Then

$$L_2^k f_{s,k}(t) = (a_1 D^{\alpha_1} + a_2 D^{\alpha_2} + ... + a_{m-1} D^{\alpha_{m-1}} + a_m)^k f_{s,k}(t)$$

$$= \sum_{i_1+...+i_{m-1}+i_m=k} C_{k,i_1\ldots i_m} a_1^{i_1} ... a_{m-1}^{i_{m-1}} a_m^{i_m} \frac{D^{\alpha_1 i_1} ... D^{\alpha_{m-1} i_{m-1}} t^{\alpha(i_1+...+i_{m-1}+i_m)+s}}{\Gamma(\alpha(i_1 + ... + i_{m-1} + i_m) + s + 1)}$$

$$= \sum_{i_1+...+i_{m-1}+i_m=k} C_{k,i_1\ldots i_m} a_1^{i_1} ... a_{m-1}^{i_{m-1}} a_m^{i_m} \frac{t^{(\alpha-\alpha_1)i_1} ... t^{(\alpha-\alpha_{m-1})i_{m-1}} t^{\alpha i_m}}{\Gamma\left(s + 1 + \sum_{j=1}^{m-1} (\alpha - \alpha_j) i_j + \alpha i_m\right)}.$$

Further, since $\alpha_j = \alpha - j, j = 1, 2, ..., m-1$, then

$$y_s(t) = t^s \times$$

$$\sum_{k=0}^{\infty} \sum_{i_1+...+i_{m-1}+i_m=k} C_{k,i_1\ldots i_m} a_1^{i_1} ... a_{m-1}^{i_{m-1}} a_m^{i_m} \frac{t^{i_1} ... t^{(m-1)i_{m-1}} t^{\alpha i_m}}{\Gamma\left(s + 1 + \sum_{j=1}^{m-1} j i_j + \alpha i_m\right)}$$

$$= t^s\, E_{(1,\ldots,m-1,\alpha),s+1}(a_1 t, ..., a_{m-1} t^{m-1}, a_m t^{\alpha}).$$

DEFINITION **2.1**. The linearly independent functions $y_s(t)$, $s = 0, 1, ..., m-1$, form the fundamental system of solutions of equation (1.1).

EXAMPLE **2.1**. By means of Remark **4.1**, we have that Theorem **2.1** generalizes [17, Theorem 5.12]. Indeed, let m be a positive integer and $m - 1 < \alpha \leq m$. Suppose that $a_j = 0, j = 1, 2, ..., m-1$, and $a_m = \lambda \neq 0$. Then equation (1.1) has the form

$$D^{\alpha} y(t) - \lambda y(t) = 0,$$

and according to Theorem **2.1** the following functions

$$y_s(t) = \sum_{k=0}^{\infty} L_2^k \frac{t^{\alpha k+s}}{\Gamma(\alpha k+s+1)} = \sum_{k=0}^{\infty} \lambda^k \frac{t^{\alpha k+s}}{\Gamma(\alpha k+s+1)}$$
$$= t^s \sum_{k=0}^{\infty} \frac{(\lambda t^\alpha)^k}{\Gamma(\alpha k+s+1)} = t^s E_{\alpha,s+1}(\lambda t^\alpha), \quad s = 0, 1, ..., m-1, \qquad (2.11)$$

form the fundamental system of solutions, where $E_{\alpha,s+1}(\lambda t^\alpha)$ is the Mittag-Leffler function (2.9).

3. The fundamental matrix and the Cauchy problem

In this section we consider the Cauchy problem (1.1), (1.2) and find its solution. Note that existence and uniqueness of solutions of the Cauchy problem, even for more general equations than (1.1), were proved by many authors (see, for example, [20]).

Let $y_s(t)$, $s = 0, 1, ..., m-1$, be the fundamental system defined in (2.11).

Definition **3.1**. The following matrix

$$Y(t) = \begin{pmatrix} y_0(t) & y_1(t) & ... & y_{m-1}(t) \\ y_0^{(1)}(t) & y_1^{(1)}(t) & ... & y_{m-1}^{(1)}(t) \\ ... & ... & ... & ... \\ y_0^{(m-1)}(t) & y_1^{(m-1)}(t) & ... & y_{m-1}^{(m-1)}(t) \end{pmatrix}$$

is called the fundamental matrix of equation (1.1).

Based on this matrix one can easily find the solution of the Cauchy problem. Indeed, if $y(t) = \sum_{s=0}^{m-1} C_s y_s(t)$ is a solution of (1.1), then $y^{(n)}(t) = \sum_{s=0}^{m-1} C_s y_s^{(n)}(t)$ and therefore one has

$$\begin{pmatrix} y(t) \\ y^{(1)}(t) \\ ... \\ y^{(m-1)}(t) \end{pmatrix} = Y(t) \begin{pmatrix} C_0 \\ C_1 \\ ... \\ C_{m-1} \end{pmatrix}.$$

Thus, if the vector $\mathbf{C} = (C_0, ..., C_{m-1})^T$ satisfies equation $Y(0)\mathbf{C} = \mathbf{b}$, where $\mathbf{b} = (b_0, ..., b_{m-1})^T$, then $y(t)$ is the solution of the Cauchy problem (1.1), (1.2). In other words, if we choose $\mathbf{C} = Y^{-1}(0)\mathbf{b}$, then the solution of the Cauchy problem has the form

$$y(t) = (Y^{-1}(0)\mathbf{b}, \mathbf{y}_F(t)), \qquad (3.1)$$

where $\mathbf{y}_F(t) = (y_0(t), ..., y_{m-1}(t))$.

Obviously, in order to ensure the existence of the solution defined in (3.1) one should verify that $\det Y(0) \neq 0$.

PROPOSITION **3.1**. $\det Y(0) \neq 0$.

P r o o f. The proof will be deduced by contradiction. For this let us assume that $\det Y(0) = 0$. In this case there exists a constant vector $\mathbf{C} = (C_0, ..., C_{m-1})^T$, such that not all coordinates C_s are zero and $Y(0)\mathbf{C} = \mathbf{0}$, where $\mathbf{0}$ is zero vector. This implies that $y(t) = \sum_{s=0}^{m-1} C_s y_s(t)$ is the solution of equation (1.1) with the initial data $y^{(n)}(0) = 0$, $n = 0, 1, ..., m-1$. But the Cauchy problem has the unique solution and therefore

$$y(t) = C_0 y_0(t) + C_1 y_1(t) + \cdots + C_{m-1} y_{m-1}(t) \equiv 0.$$

Since not all C_s are zero, previous expression implies linear dependence of the system $y_s(t)$. Thus we have a contradiction, which proves the proposition. □

Next we show that the maximal number of linearly independent solutions of equation (1.1) is m.

PROPOSITION **3.2**. *Let $x(t)$, $t \geq 0$, be any solution of equation (1.1). Then $x(t)$ is a linear combination of solutions $y_s(t)$, $s = 0, ..., m-1$.*

P r o o f. Let $x(t)$ be a solution of (1.1) and $x^{(n)}(0) = x_n$, $n = 0, 1, ..., m-1$. Obviously, $y(t) = (Y^{-1}(0)\mathbf{x}_0, \mathbf{y}_F(t))$ is a solution of equation (1.1), where $\mathbf{x}_0 = (x_0, ..., x_{m-1})^T$ satisfies the same initial conditions. Since the Cauchy problem has a unique solution, then $x(t) = (Y^{-1}(0)\mathbf{x}_0, \mathbf{y}_F(t))$. □

Thus, formula (3.1) gives the expression of the solution of the Cauchy problem (1.1), (1.2).

So, we are interested to find the explicit form of the matrix $Y^{-1}(0)$.

To this end, let $p \geq 0$ be any real number. Consider the functions

$$y_p(t) = \sum_{k=0}^{\infty} (a_1 D^{\alpha-1} + \cdots + a_{m-1} D^{\alpha-(m-1)} + a_m)^k \frac{t^{\alpha k + p}}{\Gamma(\alpha k + p + 1)},\ t \geq 0. \tag{3.2}$$

Obviously, if $p = 0, 1, ..., m-1$, then y_p belongs to the set of fundamental solutions of equation (1.1).

In the sequel we denote $y^{(\beta)}(t) = D^\beta y(t)$ for any positive real number β.

THEOREM **3.1**. *Let $p \geq 0$, n be integer and $n < \alpha$. Then*

$$y_p^{(n)}(t) = \begin{cases} y_{p-n}(t), & p \geq n, \\ a_1 y_p^{(n-1)}(t) + ... + a_{m-1} y_{p+m-2}^{(n-1)}(t) + a_m y_{p+\alpha-1}^{(n-1)}(t), & p+1 \leq n. \end{cases}$$

P r o o f. If $n \leq p$, then

$$y_p^{(n)}(t) = \sum_{k=0}^{\infty} (a_1 D^{\alpha-1} + \cdots + a_{m-1} D^{\alpha-(m-1)} + a_m)^k \frac{t^{\alpha k+p-n}}{\Gamma(\alpha k+p+1-n)},$$

i.e. $y_p^{(n)}(t) = y_{p-n}(t)$ and the first part of theorem is proved.

If $p+1 \leq n < \alpha$, then making use of the previous equality, one has

$$\begin{aligned} y_p^{(n)}(t) &= y_0^{(n-p)}(t) = \Big[\sum_{k=1}^{\infty} (a_1 D^{\alpha-1} + \cdots + a_m)^k \frac{t^{\alpha(k-1)+\alpha-1}}{\Gamma(\alpha\, k)}\Big]^{(n-p-1)} \\ &= (a_1 D^{\alpha-1} + \cdots + a_m) \Bigg[\sum_{k=0}^{\infty} (a_1 D^{\alpha-1} + \cdots + a_m)^k \\ &\qquad \times \frac{t^{\alpha k+\alpha-1}}{\Gamma(\alpha k + (\alpha-1)+1)}\Bigg]^{(n-p-1)} \\ &= (a_1 D^{\alpha-1} + \cdots + a_m) y_{\alpha-1}^{(n-p-1)}(t) \\ &= (a_1 D^{\alpha-1} + \cdots + a_m) y_{\alpha-1+p}^{(n-1)}(t) \\ &= \left[\sum_{k=0}^{\infty} (a_1 D^{\alpha-1} + \cdots + a_m)^k \frac{(a_1 D^{\alpha-1} + \cdots + a_m) t^{\alpha k+\alpha-1+p}}{\Gamma(\alpha k + (\alpha-1)+p+1)}\right]^{(n-1)} \\ &= a_1 y_p^{(n-1)}(t) + a_2 y_{p+1}^{(n-1)}(t) + \cdots + a_{m-1} y_{p+m-2}^{(n-1)}(t) + a_m y_{p+\alpha-1}^{(n-1)}(t). \end{aligned}$$

□

COROLLARY **3.1**. *Let n be integer and $1 \leq n \leq m-1$. Then*

$$y_0^{(n)}(t) - a_1 y_0^{(n-1)}(t) - a_2 y_1^{(n-1)}(t) - ... - a_{m-1} y_{m-2}^{(n-1)}(t) - a_m y_{\alpha-1}^{(n-1)}(t) = 0.$$

P r o o f. The result holds by taking $p = 0$ in Theorem **3.1**. □

COROLLARY **3.2**. *Let n be integer and $1 \leq n \leq m-1$. Then*

$$y_0^{(n)}(0) - a_1 y_0^{(n-1)}(0) - a_2 y_0^{(n-2)}(0) - ... - a_{n-1} y_0^{(1)}(0) - a_n = 0.$$

P r o o f. Obviously if $p > 0$, then $y_p(0) = 0$ and $y_0(0) = 1$. Therefore, if $p \geq n$, from Theorem **3.1** we obtain $y_p^{(n)}(0) = \delta_{p,n}$ (the Kronecker delta function). Using this and first part of Theorem **3.1** we have, from Corollary **3.1**,

$$0 = y_0^{(n)}(0) - a_1 y_0^{(n-1)}(0) - a_2 y_1^{(n-1)}(0) - \ldots - a_{m-1} y_{m-2}^{(n-1)}(0) - a_m y_{\alpha-1}^{(n-1)}(0) =$$

$$y_0^{(n)}(0) - a_1 y_0^{(n-1)}(0) - a_2 y_0^{(n-2)}(0) - a_3 y_0^{(n-3)}(0) - \cdots - a_{n-1} y_0^{(1)}(0) - a_n.$$

□

Now, we are in a position to obtain the expression of the inverse of the fundamental matrix at $t = 0$.

THEOREM **3.2**. *The inverse of the fundamental matrix at* $t = 0$ *is given by the following expression*

$$Y^{-1}(0) = A := \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ -a_1 & 1 & 0 & \ldots & 0 \\ -a_2 & -a_1 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ -a_{m-1} & -a_{m-2} & -a_{m-3} & \ldots & 1 \end{pmatrix}.$$

P r o o f. As it was stated in Theorem **3.1**, all above the diagonal elements of matrix $Y(0)$ are zero, i.e. $y_s^{(n)}(0) = 0$ if $s > n$. Moreover it has the following form with the diagonal elements $y_s^{(s)}(0) = 1$:

$$Y(0) = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ y_0^{(1)}(0) & 1 & 0 & \ldots & 0 \\ y_0^{(2)}(0) & y_0^{(1)}(0) & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ y_0^{(n-1)}(0) & y_0^{(n-2)}(0) & y_0^{(m-3)}(0) & \ldots & 1 \end{pmatrix}.$$

Let us denote by Y_i the row of matrix $Y(0)$ with number i and by A_j the column of matrix A with number j, i.e.

$$Y_i = (y_0^{(i-1)}(0), \ldots, y_0^{(1)}(0), 1, 0, \ldots, 0),$$

$$A_j = (0, \ldots, 0, 1, a_1, \ldots, a_{m-j})^T.$$

Then $Y(0) \cdot A = (Y_i \cdot A_j)_{i,j=\overline{1,m}}$.

Note that the last $m - i$ elements of Y_i are zero, and the first $j - 1$ elements of A_j are zero. Therefore, if $i = j$, then $Y_i \cdot A_j = 1$ and if $i < j$, then $Y_i \cdot A_j = 0$. Finally, if $i > j$, then

$$Y_i \cdot A_j = 1 \cdot y_0^{(i-j)}(0) - a_1 \cdot y_0^{(i-j-1)}(0) - \ldots - 1 \cdot a_{i-j},$$

and if we use Corollary **3.2** with $n = i - j$, then we obtain $Y_i \cdot A_j = 0$. Thus $Y_i \cdot A_j = \delta_{i,j}$, which implies that $Y(0) \cdot A$ is the identity matrix. □

EXAMPLE **3.1**. Let $3 < \alpha \leq 4$. Consider the Cauchy problem

$$D^{\alpha} y(t) - a_1 D^{\alpha-1} y(t) - a_2 D^{\alpha-2} y(t) - a_3 D^{\alpha-3} y(t) - a_4 y(t) = 0,$$
$$y^{(j)}(0) = b_j, \quad j = 0, 1, 2, 3.$$

According to (3.1), the solution of this problem has the form

$$y(t) = (Y^{-1}(0) \cdot b, y_F(t)), \tag{3.3}$$

where

$$Y^{-1}(0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -a_1 & 1 & 0 & 0 \\ -a_2 & -a_1 & 1 & 0 \\ -a_3 & -a_2 & -a_1 & 1 \end{pmatrix},$$

$$b = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix}, \quad y_F(t) = \begin{pmatrix} y_0 & y_1 & y_2 & y_3 \end{pmatrix},$$

and

$$y_s(t) = \sum_{k=0}^{\infty} (a_1 D^{\alpha-1} + a_2 D^{\alpha-2} + a_3 D^{\alpha-3} + a_4)^k \frac{t^{\alpha k+s}}{\Gamma(\alpha k + s + 1)}, \quad s = 0, 1, 2, 3.$$

The initial conditions give us

$$y(t) = b_0 y_0(t) + (b_1 - a_1 b_0) y_1(t)$$
$$+ (b_2 - a_2 b_0 - a_1 b_1) y_2(t) + (b_3 - a_3 b_0 - a_2 b_1 - a_1 b_2) y_3(t).$$

In particular, if $a_1 = a_2 = a_3 = 0, a_4 \neq 0$ and $b_0 = 1, b_1 = b_2 = b_3 = 0$, then the solution follows the expression

$$y(t) = b_0\, y_0(t) = b_0 \sum_{k=0}^{\infty} a_4^k \frac{t^{\alpha k}}{\Gamma(\alpha k + 1)} = b_0\, E_{\alpha,1}(a_4 t^{\alpha}).$$

Next consider the case when $a_1 \neq 0, a_2 = a_3 = a_4 = 0$, and $b_3 = 1, b_0 = b_1 = b_2 = 0$.

In this case one has

$$y(t) = (b_3 - a_3 b_0 - a_2 b_1 - a_1 b_2) y_3(t) = y_3(t) = \sum_{k=0}^{\infty} a_1^k D^{(\alpha-1)k} \frac{t^{\alpha k+3}}{\Gamma(\alpha k + 4)}$$
$$= \sum_{k=0}^{\infty} a_1^k \frac{t^{\alpha k+3-(\alpha-1)k}}{\Gamma(\alpha k + 4 - (\alpha-1)k)} = \sum_{k=0}^{\infty} a_1^k \frac{t^{k+3}}{(k+3)!} = a_1^{-3} \sum_{k=3}^{\infty} a_1^k \frac{t^k}{k!}$$
$$= a_1^{-3} [e^{a_1 t} - 1 - a_1 t - a_1^2 \frac{t^2}{2}].$$

Thus the solution of the Cauchy problem

$$D^{\alpha}y(t) - a_1 D^{\alpha-1}y(t) = 0,$$
$$y^{(j)}(0) = 0, \;\; j = 0, 1, 2, \;\;\; y^{(3)}(0) = 1,$$

has the form

$$y(t) = a_1^{-3}[e^{a_1 t} - 1 - a_1 t - a_1^2 \frac{t^2}{2}].$$

Example **3.2**. Let m be a positive integer and $m - 1 < \alpha \leq m$. Consider the Cauchy problem

$$D^{\alpha}y(t) - \lambda y(t) = 0,$$
$$y^{(j)}(0) = b_j, \;\; j = 0, 1, \cdots, m-1.$$

Obviously, for this equation $Y^{-1}(0) = E$, i.e. the identity matrix and the fundamental system of solutions was found in Example **2.1**. Therefore according to (3.1) the solution of the Cauchy problem has the form

$$y(t) = \sum_{s=0}^{m-1} b_s t^s E_{\alpha, s+1}(\lambda t^{\alpha}),$$

i.e. we have the known result from [17].

Example **3.3**. Consider the Cauchy problem (1.2), (4.1), with coefficients $a_j = -1$, $j = 1, 2, ..., m-1$. Then the solution of the problem has the expression:

$$y(t) = \sum_{s=0}^{m-1} \left(\sum_{i=0}^{s} b_s \right) t^s E_{(1,...,m-1,\alpha),s+1}(a_1 t, ..., a_{m-1} t^{m-1}, a_m t^{\alpha}).$$

4. Non-homogeneous equations

Let $f(t)$ be an arbitrary continuous function in $[0, T)$. In the present section we consider a non-homogeneous equation

$$D^{\alpha}y(t) - a_1 D^{\alpha-1}y(t) - ... - a_{m-1} D^{\alpha-(m-1)}y(t) - a_m y(t) = f(t),$$
$$t \in (0, T), \tag{4.1}$$

and the Cauchy problem (1.2),(4.1).

We look for solutions $y \in C^m([0, T))$.

Again, as in the homogenous case, if $m = 1$, i.e. $0 < \alpha \leq 1$, then we have equation $D^{\alpha}y(t) - a_1 y(t) = f(t)$ and $y \in C^1([0, T))$.

If we consider the initial data

$$y^{(n)}(0) = 0, \;\; n = 0, ..., m-1, \tag{4.2}$$

with equation (4.1), then, as it was noted above, this Cauchy problem has a unique solution. We denote this solution by $y_f(t)$. Let $\tilde{y}(t)$ be the unique solution of problem (1.1), (1.2), which has the form (3.1). Then, because of the linearity, function $y_f(t)+\tilde{y}(t)$ will be the unique solution of problem (4.1), (1.2). Thus, to solve the Cauchy problem (4.1), (1.2) it is sufficient to find $y_f(t)$.

Let $y_{\alpha-1}(t)$ be the function defined in (3.2) and $L_0 := \frac{d}{dt}D^{\alpha-1}$. We first study some properties of $y_{\alpha-1}(t)$.

LEMMA **4.1**. *The function $y_{\alpha-1}(t)$ is the solution of the Cauchy problem:*

$$(L_0 - L_2)y(t) = 0,\ t > 0,$$
$$y^{(j)}(0) = 0,\ j = 0, 1, ..., m-2,\quad y^{(\alpha-1)}(0) = 1.$$

P r o o f. Consider the system of functions

$$f_{\alpha-1,k}(t) = \frac{t^{\alpha k+\alpha-1}}{\Gamma(\alpha k+\alpha)},\ \ k = 0, 1, 2, ...$$

By a direct calculation, we can verify that $D^{\alpha-1}f_{\alpha-1,0}(t) = 1$ and therefore $L_0 f_{\alpha-1,0}(t) = 0$. Hence in the same way as in Section **2**, one can show that this system is 0-normed with respect to L_0 and satisfies conditions (i) and (ii) from Section **1**. Therefore,

$$y_{\alpha-1}(t) = \sum_{k=0}^{\infty} L_2^k f_{\alpha-1,k}(t)$$

is a solution of equation $(L_0 - L_2)y(t) = 0, t > 0$. Now it is not difficult to show that $y_{\alpha-1}(t)$ satisfies the Cauchy conditions. □

THEOREM **4.1**. *The unique solution of the Cauchy problem (4.1), (4.2) has the form*

$$y_f(t) = \int_0^t f(\tau)y_{\alpha-1}(t-\tau)d\tau. \tag{4.3}$$

P r o o f. Since $f(t)$ is a continuous function in $[0, T)$, using the Cauchy conditions for $y_{\alpha-1}(t)$ one obtains

$$\frac{d^j}{dt^j}y_f(t) = \int_0^t f(\tau)\frac{d^j}{dt^j}y_{\alpha-1}(t-\tau)d\tau,\ \ j = 1, ..., m-1,\ t \in [0, T).$$

Therefore $y_f(t)$ satisfies the Cauchy conditions (4.2). On the other hand (see the proof of Theorem **2.1**)

$$D^{\alpha-j}y_f(t) = \int_0^t f(\tau)(D^{\alpha-j}y_{\alpha-1})(t-\tau)d\tau,\ \ j=1,...,m-1,\ \ t\in[0,T). \tag{4.4}$$

Function $F(t) := \frac{d^{m-1}}{dt^{m-1}}y_f(t)$ is absolutely continuous in $[0,T)$ and $F(0)=0$. Therefore (see [28], p.40)

$$I^{m-\alpha}\frac{d}{dt}F(t) = \frac{d}{dt}I^{m-\alpha}F(t).$$

Making use of this equality, we apply the operator $\frac{d}{dt}$ to (4.4) with $j=1$. Since $D^{\alpha-1}y_{\alpha-1}(0)=1$, we deduce

$$L_1y_f(t) = D^\alpha y_f(t) = f(t) + \int_0^t f(\tau)(L_0y_{\alpha-1})(t-\tau)d\tau.$$

Hence, due to Lemma **4.1**, we conclude that

$$(L_1-L_2)y_f(t) = f(t) + \int_0^t f(\tau)((L_0-L_2)y_{\alpha-1})(t-\tau)d\tau = f(t).$$

□

Remark **4.1**. As in the case of function $y_s(t)$, the solution of the non-homogeneous equation can be written in terms of multivariate Mittag-Leffler function. Indeed,

$$y_{\alpha-1}(t) = \sum_{k=0}^{\infty} L_2^k f_{\alpha-1,k}(t) =$$

$$\sum_{i_1+...+i_{m-1}+i_m=k} C_{k,i_1...i_m} a_1^{i_1}...a_{m-1}^{i_{m-1}}a_m^{i_m}\frac{t^{\alpha-1}t^{i_1}...t^{(m-1)i_{m-1}}t^{\alpha i_m}}{\Gamma(s+1+i_1+...+(m-1)i_{m-1}+\alpha i_m)}$$

$$= t^{\alpha-1}E_{(1,...,m-1,\alpha),s+1}(a_1t,...,a_{m-1}t^{m-1},a_mt^\alpha).$$

Therefore

$$y_f(t) = \int_0^t (t-\tau)^{\alpha-1}$$

$$\times E_{(1,...,m-1,\alpha),s+1}\left(a_1(t-\tau),...,a_{m-1}(t-\tau)^{m-1},a_m(t-\tau)^\alpha\right) f(\tau)d\tau.$$

Now, let us see that if $f(t)$ is a real analytic function in $(0,T)$, i.e.

$$f(t) = \sum_{n=0}^{\infty} f^{(n)}(0)\frac{t^n}{n!},\ \ t\in(0,T), \tag{4.5}$$

then function $y_f(t)$ has a particularly simple form.

Lemma **4.2**. *If $f(t)=t^n/n!$, for n a non-negative integer, then $y_f(t) = y_{\alpha+n}(t)$.*

P r o o f. Consider the system of functions

$$f_{\alpha+n,k}(t)=\frac{t^{\alpha k+\alpha+n}}{\Gamma(\alpha k+\alpha+n+1)},\ \ k=0,1,2,...$$

By a direct calculation, we can verify that $L_1 f_{\alpha+n,0}(t)=f(t)$. Therefore in the same way as in Section **2**, one can show that this system is f-normed with respect to L_1 and, from Section **1**, it satisfies conditions (i) and (ii). Hence

$$y_{\alpha+n}(t)=\sum_{k=0}^{\infty}L_2^k f_{\alpha+n,k}(t)$$

is a solution of equation $(L_1-L_2)y(t)=f(t), t\in(0,T)$.

Obviously $y_{\alpha+n}(t)$ satisfies the Cauchy conditions (4.2). □

EXAMPLE **4.1**. Let m be a positive integer and $m-1<\alpha\le m$. Consider the Cauchy problem

$$D^{\alpha}y(t)-y(t)=\frac{t^n}{n!},$$
$$y^{(j)}(0)=0,\ \ j=0,1,...,m-1.$$

According to Lemma **4.2** the solution of this problem has the form

$$y_f(t)=y_{\alpha+n}(t)=\sum_{k=0}^{\infty}\frac{t^{\alpha k+\alpha+n}}{\Gamma(\alpha k+\alpha+n)}=t^{\alpha+n}E_{\alpha,\alpha+n}(t^{\alpha}).$$

The following statement is an easy corollary of Lemma **4.2**.

THEOREM **4.2**. *Let $f(t)$ be a real analytic function in $(0,T)$, i.e. $f(t)$ has the form (4.5). Then*

$$y_f(t)=\sum_{n=0}^{\infty}f^{(n)}(0)y_{\alpha+n}(t).$$

REMARK **4.2**. Since

$$y_{\alpha+n}(t)=\int_0^t\frac{(t-\tau)^n}{n!}y_{\alpha-1}(\tau)d\tau,$$

then it is satisfied that

$$y_f(t)=\sum_{n=0}^{\infty}f^{(n)}(0)y_{\alpha+n}(t)$$
$$=\int_0^t\sum_{n=0}^{\infty}f^{(n)}(0)\frac{(t-\tau)^n}{n!}y_{\alpha-1}(\tau)d\tau=\int_0^t f(\tau)y_{\alpha-1}(t-\tau)d\tau,$$

i.e. $y_f(t)$ follows the form (4.3).

5. Conclusion

Some 30 years ago a mathematician from Uzbekistan, B.A. Bondarenko, introduced the *Operator Algorithms method* to solve partial differential equations (see [4]). Recently, in 2012, V.V. Karachik [14] adopted this method to solve ordinary differential equations and the authors of [30] used the same method for solving some fractional differential equations.

The main purpose of this paper is to show that by using the Bondarenko method one can construct the fundamental solutions of more general fractional differential equations (1.1) (in fact, we may apply this method for the general linear differential equation with constant coefficients and the Caputo derivatives considered in [20]). As it was shown in Introduction, this method is very simple, a solution of the equation has the form (1.3), and to use this method, unlike to other methods, we do not need to introduce and explore many new notions (for example, in the modified Mikusinski method, see [20] and [24], there are introduced new spaces C_α and with the operations of the Laplace convolution it is obtained a commutative ring, then this ring is extended to the quotient field). We also note that in the Bondarenko method the solution of the Cauchy problem has a particularly simple form (see formula (3.1) and Theorem 3.2).

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References

[1] Yu.I. Babenko, *Heat and Mass Transfer*. Chemia, Leningrad (1986) (In Russian).

[2] R.L. Bagley, On the fractional order initial value problem and its engineering applications. In: *Fractional Calculus and Its Applications* (Ed. K. Nishimoto), College of Engineering, Nihon University, Tokyo (1990), 12–20.

[3] H. Beyer, S. Kempfle, Definition of physically consistent damping laws with fractional derivatives. *ZAMM* **75** (1995), 623–635.

[4] B.A. Bondarenko, *Operator Algorithms in Differential Equations.* "FAN" Publishers, Tashkent (1984) (In Russian).
[5] R. Caponetto, G. Dongola, L. Fortuna, I. Petraš, *Fractional Order Systems: Modeling and Control Applications.* Ser. on Nonlinear Science **72**, World Scientific, Singapore (2010).
[6] J.Fa Cheng, Y.M. Chu, Solution to the linear fractional differential equation using Adomian decomposition method. *Mathematical Problems in Engineering* **2011** (2011), Article ID 587068, 14 p.
[7] K. Diethelm, *The Analysis of Fractional Differential Equations.* Lecture Notes in Mathematics, Springer, Berlin-Heidelberg (2010).
[8] M.M. Dzerbashyan, A.B. Nersesyan, Fractional derivatives and the Cauchy problem for differential equations of fractional order. *Izv. Akad. Nauk Armyan. SSR, Ser. Mat.* **3** (1968), 3–29 (In Russian).
[9] R. Hilfer, *Applications of Fractional Calculus in Physics.* World Scientific, Singapore (2000).
[10] R. Hilfer, Fractional time evolution. In: *Applications of Fractional Calculus in Physics* (Ed. R. Hilfer), World Scientific, Singapore (2000), 87–130.
[11] R. Hilfer, Experimental evidence for fractional time evolution in glass forming materials, *Chem. Phys.* **284** (2002), 399–408.
[12] R. Hilfer, Y. Luchko, Z. Tomovski, Operational method for the solution of fractional differential equations with generalized Riemann-Liouville fractional derivatives. *Fract. Calc. Appl. Anal* **12**, No 3 (2009), 299–318; at http://www.math.bas.bg/∼fcaa.
[13] Y. Hu, Y. Luo, Z. Lu, Analytical solution of the linear fractional differential equation by Adomian decomposition method. *J. of Computational and Applied Mathematics* **215** (2008), 220–229.
[14] V.V. Karachik, Method for constructing solutions of linear ordinary differential equations with constant coefficients. *Computational Mathematics and Mathematical Physics* **52**, No 2 (2012), 219–234; DOI: 10.1134/S0965542512020108.
[15] A.A. Kilbas, New trends on fractional integral and differential equations. *Kazan. Gos. Univ. Uchen. Zap. Ser. Fiz.-Mat. Nauki* **147**, No 1 (2005), 72–106 (In Russian).
[16] A.A. Kilbas and S.A. Marzan, Cauchy problem for differential equation with Caputo derivative. *Fract. Calc. Appl. Anal.* **7**, No 3 (2004), 297–321; at http://www.math.bas.bg/∼fcaa.
[17] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations.* Elsevier, North Holland (2006).

[18] M.H. Kim, G.C. Ri, Hyong-Chol O, Operational method for solving multi-term fractional differential equations with the generalized fractional derivatives. *Fract. Calc. Appl. Anal.* **17**, No 1 (2014), 79–95; DOI: 10.2478/s13540-014-0156-6; http://www.degruyter.com/view/j/fca.2014.17.issue-1/issue-files/fca.2014.17.issue-1.xml.

[19] Yu. Luchko, Operational method in fractional calculus. *Fract. Calc. Appl. Anal.* **2**, No 4 (1999), 463–488; http://www.math.bas.bg/∼fcaa.

[20] Yu. Luchko, R. Gorenflo, An operational method for solving fractional differential equations. *Acta Mathematica Vietnamica* **24** (1999), 207–234.

[21] Yu. Luchko, H.M. Srivastava, The exact solution of certain differential equations of fractional order by using operational calculus. *Comput. Math. Appl.* **29** (1995), 73–85.

[22] Yu. Luchko, S.B. Yakubovich, An operational method for solving some classes of integro-differential equations. *Differential Equations* **30** (1994), 247–256.

[23] C.A. Monje, Y. Chen, B.M. Vinagre, D. Xue, V. Feliu, *Fractional–Order Systems and Controls: Fundamentals and Applications*. Ser. Advances in Industrial Control, Springer, London (2010).

[24] I. Podlubny, *Fractional Differential Equations*. Academic Press, San Diego (1999).

[25] A.V. Pskhu, Initial-value problem for a linear ordinary differential equation of noninteger order. *Sbornik: Mathematics* **202** (2011), 571–582.

[26] M. Rivero, L. Rodriguez-Germa, J.J. Trujillo, Linear fractional differential equations with variable coefficients. *Appl. Math. Letters* **21** (2008), 892–897.

[27] J. Sabatier, O.P. Agrawal, J.A. Tenreiro Machado, *Advances in Fractional Calculus*. Ser. Theoretical Developments and Applications in Phys. and Eng., Springer, Berlin (2007).

[28] S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional Integrals and Derivatives, Theory and Applications*. Gordon and Breach, Switzerland (1993).

[29] D. Shantanu, *Functional Fractional Calculus for System Identification and Controls*. Springer, Berlin (2008).

[30] K.M. Shinaliyev, B.Kh. Turmetov, S.R. Umarov, A fractional operator algorithm method for construction of solutions of fractional order differential equations. *Fract. Calc. Appl. Anal.* **15**, No 2 (2012), 267–281; DOI:10.2478/s13540-012-0020-5; http://www.degruyter.com/view/j/fca.2012.15.issue-2/issue-files/fca.2012.15.issue-2.xml.

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