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RESEARCH PAPER

BOGOLYUBOV-TYPE THEOREM WITH CONSTRAINTS GENERATED BY A FRACTIONAL CONTROL SYSTEM

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Abstract

We first study the existence results and properties of the solution set of a control system described by fractional differential equations with nonconvex control constraint. Then a problem of minimizing an integral functional over the solution set of the control system is considered. Along with the original minimizing problem, we also consider the problem of minimizing the integral functional whose integrand is the bipolar (with respect to the control variable) of the original integrand over the solution set of the same system but with the convexified control constraint. We prove that the relaxed problem has an optimal solution and obtain some relationships between these two minimizing problems. Finally, an example is given to illustrate the results.

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Key Words and Phrases: fractional differential equation, optimal control, Bogolyubov type theorem, relaxation property, nonconvex constraint

1. Introduction

Fractional calculus and differential equations have proved to be valuable tools in the modeling of many phenomena in engineering and sciences such as physics, mechanics, chemistry, economics and biology, etc., [4, 26]. As a consequence, there was an intensive development of the theory of differential equations of fractional order. One can see the monographs of Kilbas et al. [16] and Miller et al. [25], the papers [15, 5, 1, 7, 19, 17, 21, 35, 37] and the

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references therein. For control theory of fractional differential systems, we can refer to [20, 2, 28, 11, 3, 8, 39, 38, 36, 18] and the references therein.

In the article [6] dating back to 1930, Bogolyubov proved the following

THEOREM ([6, 30]). *Let $T = [0,1]$, $f : T \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Then for any function $x \in W_{\infty,1}(T, \mathbb{R}^n)$ there is a sequence of functions $x_k \in C^1(T, \mathbb{R}^n)$, $k \geq 1$, such that $x_k(0) = x(0)$, $x_k(1) = x(1)$, $x_k \to x$ uniformly on T as $k \to \infty$ and, moreover,*

$$\liminf_{k\to\infty} \int_T f(t, x_k(t), \dot{x}_k(t))dt \leq \int_T f^{**}(t, x(t), \dot{x}(t))dt.$$

Here $f^{**}(t, x, u)$ is the bipolar (second conjugate) of the function f with respect to the last argument and $W_{\infty,1}(T, \mathbb{R}^n)$ is the space of absolutely continuous functions from T to $\mathbb{R}^n$ whose derivatives are elements of $L^\infty(T, \mathbb{R}^n)$.

Since then this theorem has been extended in several directions by many authors including Macshane [22], Ekeland and Temam [12]. Among more recent generalizations are the works by Suslov [27], De Blasi, Pianigiani and Tolstonogov [9], Tolstonogov [30, 31], Timoshin and Tolstonogov [29].

In this paper we give an analogue of Bogolyubov's theorem with constraints induced by a fractional control system.

Let $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$ and $\overline{\mathbb{R}} = (-\infty, +\infty]$, $J = [0, b]$ ($b > 0$ a real number) with Lebesgue measure μ and σ-algebra Σ of μ-measurable subsets of J. For a function $l : J \times X \times Y \to \mathbb{R}$, we consider problem (P):

$$I(x, u) = \int_J l(t, x(t), u(t))dt \to \inf \tag{P}$$

on the solution set of a control system described by fractional differential equations of the following form

$$\begin{cases} {}^C D^\alpha x(t) = f(t, x(t)) + B(t)u(t), & \text{a.e. } t \in J,\ 0 < \alpha < 1, \\ x(0) = x_0, \end{cases} \tag{1.1}$$

with the mixed nonconvex constraint

$$u(t) \in U(t, x(t)) \text{ a.e. on } J. \tag{1.2}$$

Here ${}^C D^\alpha$ is the Caputo fractional derivative of order α, $f : J \times X \to X$ is a nonlinear function, $B : J \to L(Y, X)$ (the space of continuous linear operators acting from Y to X), $U : J \times X \to 2^Y$ is a multivalued map with closed values. The space Y models the control space.

Let $l_U : J \times X \times Y \to \overline{\mathbb{R}}$ be defined by

$$l_U(t, x, u) = \begin{cases} l(t, x, u), & u \in U(t, x), \\ +\infty, & u \notin U(t, x), \end{cases}$$

and $l_U^{**}(t, x, u)$ be the bipolar of the function $u \to l_U(t, x, u)$ [12].

Along with problem (P), we also consider its convex relaxation, i.e. the following problem (RP):

$$I^{**}(x,u) = \int_J l_U^{**}(t, x(t), u(t))dt \to \min \tag{RP}$$

on the solution set of control system (1.1) with the convexified control constraint

$$u(t) \in \overline{\text{co}}\, U(t, x(t)) \text{ a.e. on } J. \tag{1.3}$$

Here, $\overline{\text{co}}$ stands for the closed convex hull of a set.

Let us define what we mean by a solution of control system (1.1), (1.2) in this paper.

Definition 1.1 (cf. [5]). A pair of functions (x, u), $x \in C(J, X)$, $u \in L^1(J, Y)$ is a solution of control system (1.1), (1.2), if $x(0) = x_0$ and $u(t) \in U(t, x(t))$ for a.e. $t \in J$, such that

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \big(f(s, x(s)) + B(s)u(s) \big) ds. \tag{1.4}$$

A solution of control system (1.1), (1.3) is defined similarly. We denote by $\mathcal{R}_U$, $\mathcal{T}r_U$ ($\mathcal{R}_{\overline{\text{co}}U}$, $\mathcal{T}r_{\overline{\text{co}}U}$) the sets of all solutions, all admissible trajectories of control system (1.1), (1.2) (control system (1.1), (1.3)).

The main results (see Theorems 4.1 and 4.2 in Section 4) obtained in this paper are that: problem (RP) has at least one solution and for any solution $(x_*, u_*) \in \mathcal{R}_{\overline{\text{co}}U}$ of problem (RP), there exists a minimizing sequence $(x_n, u_n) \in \mathcal{R}_U$, $n \geq 1$, of problem (P) such that

$$x_n \to x_* \text{ in } C(J, X),$$

$$\int_J l(t, x_n(t), u_n(t))dt \to \int_J l_U^{**}(t, x_*(t), u_*(t))dt.$$

This property is usually called the relaxation ([12]) and the above two relations are an analogue of Bogolyubov's theorem with constraints generated by the solution sets of control systems (1.1), (1.2) and (1.1), (1.3). There are many papers dealing with the verification of the relaxation property for various classes of control systems, see, for instance, [9, 30, 23, 24, 32] and the references therein.

The rest of the paper is organized as follows: in Section 2 we present the notations, definitions and the preliminary results to be used in what follows, and the existence results of the control systems are given in Section 3. In Section 4 we prove our main results, in Section 5 we give an example to illustrate our results.

2. **Preliminaries**

The norm of the space X (or Y) will be denoted by $\|\cdot\|_X$ (or $\|\cdot\|_Y$). We denote by $C(J,X)$ the space of all continuous functions from J into X with the supnorm given by $\|x\|_C = \sup_{t\in J}\|x(t)\|_X$ for $x \in C(J,X)$. For any Banach space V, the symbol ω-V stands for V equipped with the weak $\sigma(V,V^*)$ topology. The same notation will be used for subsets of V. In all other cases we assume that V and its subsets are equipped with the strong (normed) topology.

Let us recall the following definitions from fractional differential theory. For more details, please see [16, 25].

Definition 2.1. The fractional integral of order α with the lower limit zero for a function f is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(s)}{(t-s)^{1-\alpha}}ds,\ t>0,\quad \alpha>0,$$

provided the right side is point-wise defined on $[0,\infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2. The Riemann-Liouville derivative of order α with the lower limit zero for a function f is defined as

$${}^L D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t \frac{f(s)}{(t-s)^{\alpha+1-n}}ds,\quad t>0,\ n-1<\alpha<n.$$

Definition 2.3. The Caputo derivative of order α with the lower limit zero for a function f is defined as

$${}^C D^\alpha f(t) = {}^L D^\alpha\Big(f(t) - \sum_{k=0}^{n-1}\frac{t^k}{k!}f^{(k)}(0)\Big),\quad t>0,\ n-1<\alpha<n.$$

Suppose V, Z are two Hausdorff topological spaces and $F: V \to 2^Z\backslash\{\emptyset\}$. We say that F is lower semicontinuous in the sense of Vietoris (l.s.c. for short) at a point $x_0 \in V$, if for any open set $W \subseteq Z$, $F(x_0)\cap W \neq \emptyset$, there is a neighborhood $O(x_0)$ of x_0 such that $F(x)\cap W \neq \emptyset$ for all $x \in O(x_0)$. F is said to be upper semicontinuous in the sense of Vietoris (u.s.c. for short) at a point $x_0 \in V$, if for any open set $W \subseteq Z$, $F(x_0) \subseteq W$, there is a neighborhood $O(x_0)$ of x_0 such that $F(x) \subseteq W$ for all $x \in O(x_0)$. For the properties of l.s.c and u.s.c, we can see the book [13].

Let (X,d) be a metric space. We recall that the Pompeiu-Hausdorff distance of the closed subsets A, $B \subseteq X$ is defined by

$$h(A,B)=\max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\}.$$

Let (X,d) be a separable metric space. We say that a multivalued map $F: J\to P_f(X)$ (all nonempty closed subsets of X) is measurable if $F^{-1}(E)=\{t\in J: F(t)\cap E\neq\emptyset\}\in\Sigma$ for every closed set $E\subseteq X$ (cf. [13]).

Besides the standard norm on $L^q(J,Y)$ (here Y is a separable, reflexive Banach space), $1<q<\infty$, we also consider the so-called weak norm,

$$\|u(\cdot)\|_\omega=\sup_{0\le t_1\le t_2\le b}\Big\|\int_{t_1}^{t_2} u(s)ds\Big\|_Y, \text{ for } u\in L^q(J,Y). \tag{2.1}$$

The space $L^q(J,Y)$ furnished with this norm will be denoted by $L^q_\omega(J,Y)$. The following result establishes a relation between convergence in ω-$L^q(J,Y)$ and convergence in $L^q_\omega(J,Y)$.

Lemma 2.1 (see [32]). *If a sequence $\{u_n\}_{n\ge1}\subseteq L^q(J,Y)$, is bounded and converges to u in $L^q_\omega(J,Y)$, then it converges to u in ω-$L^q(J,Y)$.*

3. Existence results of the control systems

In this section we deal with the existence results for the control systems. We assume the following assumptions on the data of our problems:

H(f): the function $f: J\times X\to X$ of Carathéodory type is such that

$$\|f(t,x)\|_X\le a_f(t)+c_f\|x\|_X$$

with $c_f\ge0$ and $a_f\in L^{\frac{1}{\beta}}(J,\mathbb{R}^+)$ (the number $\beta\in(0,\alpha)$).

H(B): $B\in L^\infty(J,L(Y,X))$ such that $\|B\|_{L^\infty(J,L(Y,X))}=K<+\infty$.

H(U): the multivalued map $U: J\times X\to 2^Y\backslash\{\emptyset\}$ with closed values is such that:
(1) for all $x\in X$, $t\to U(t,x)$ is measurable;
(2) $h(U(t,x),U(t,y))\le k_u\|x-y\|_X$ a.e. on J with $k_u>0$;
(3) for a.e. $t\in J$, $\sup\{\|v\|_Y: v\in U(t,x)\}\le a_u+c_u\|x\|_X$,
where $a_u>0$ and $c_u\ge0$.

H(L): $l: J\times X\times Y\to\mathbb{R}$ is a function such that:
(1) the function $t\to l(t,x,u)$ is measurable;
(2) for a.e. $t\in J$, $(x,u)\to l(t,x,u)$ is continuous;
(3) there exist functions k_1, k_2, $k_3\in L^1(J,\mathbb{R}^+)$ such that

$$|l(t,x,u)|\le k_1(t)+k_2(t)\|x\|_X+k_3(t)\|u\|_Y$$

for a.e. $t\in J$, $x\in X$ and $u\in Y$.

We begin with a prior estimation of the trajectory of the control systems.

LEMMA 3.1. *Suppose x is an admissible trajectory of control system (1.1), (1.3), i.e. $x \in \mathcal{T}r_{\overline{co}U}$, then there is a constant L such that*

$$\|x\|_C \leq L. \tag{3.1}$$

P r o o f. Let $x \in \mathcal{T}r_{\overline{co}U}$, from Definition 1.1, there exists a function $u(t) \in \overline{co}\,U(t, x(t))$ for a.e. $t \in J$, such that (1.4) holds. By H(f), H(B) and H(U)(3), we have

$$\begin{aligned}
\|x(t)\|_X &\leq \|x_0\|_X + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s, x(s)) + B(s)u(s)\|_X ds \\
&\leq \|x_0\|_X + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \big(a_f(s) + K a_u\big) ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \big(c_f + K c_u\big) \|x(s)\|_X ds \\
&\leq \|x_0\|_X + \frac{b^{\alpha-\beta}}{\Gamma(\alpha)} \Big(\frac{1-\beta}{\alpha-\beta}\Big)^{1-\beta} \|a_f\|_{L^{\frac{1}{\beta}}(J)} + \frac{K a_u b^\alpha}{\Gamma(\alpha+1)} \\
&\quad + \frac{c_f + K c_u}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|x(s)\|_X ds.
\end{aligned}$$

From the above inequality, using the well-known singular version Gronwall inequality (see Theorem 3.1, [10]), we can deduce that there exists a constant $L > 0$ such that $\|x\|_C \leq L$. □

Let $\mathrm{pr_L} : X \to X$ be the L-radial retraction, i.e.,

$$\mathrm{pr_L}(x) = \begin{cases} x, & \|x\|_X \leq L, \\ \frac{Lx}{\|x\|_X}, & \|x\|_X > L. \end{cases}$$

This map is Lipschitz continuous. We define $U_1(t, x) = U(t, \mathrm{pr_L}\, x)$. Obviously, U_1 satisfies H(U)(1) and H(U)(2). Moreover, by the properties of $\mathrm{pr_L}$, we have, for a.e. $t \in J$, all $x \in X$, and all $v \in U_1(t, x)$ the estimates

$$\|v\|_Y \leq a_u + c_u L, \text{ and } \|v\|_Y \leq a_u + c_u \|x\|_X.$$

Hence, Lemma 3.1 is still valid with $U(t, x)$ substituted by $U_1(t, x)$. Consequently, we assume without any loss of generality that, for a.e. $t \in J$ and all $x \in X$,

$$\sup\{\|v\|_Y : v \in U(t, x)\} \leq \varphi = a_u + c_u L, \text{ with } \varphi > 0 \tag{3.2}$$

and

$$\|f(t, x)\|_X \leq \lambda(t) = a_f(t) + c_f L, \text{ with } \lambda \in L^{\frac{1}{\beta}}(J, \mathbb{R}^+). \tag{3.3}$$

From H(f) and H(B), for any $x \in C(J, X)$ and $u \in L^{\frac{1}{\beta}}(J, Y)$, the function $t \to f(t, x(t)) + B(t)u(t)$ is an element of the space $L^{\frac{1}{\beta}}(J, X)$.

Now we consider the operator $\mathcal{A} : C(J, X) \times L^{\frac{1}{\beta}}(J, Y) \to L^{\frac{1}{\beta}}(J, X)$ defined by

$$\mathcal{A}(x, u)(t) = f(t, x(t)) + B(t)u(t). \tag{3.4}$$

Lemma 3.2. *The map $(x, u) \to \mathcal{A}(x, u)$ is sequentially continuous from $C(J, X) \times \omega\text{-}L^{\frac{1}{\beta}}(J, Y)$ into $\omega\text{-}L^{\frac{1}{\beta}}(J, X)$.*

Proof. Suppose that $x_n \to x$ in $C(J, X)$ and $u_n \to u$ in $\omega\text{-}L^{\frac{1}{\beta}}(J, Y)$. Then from H(f) and H(B), we have the following facts

$$f(t, x_n(t)) \to f(t, x(t)) \text{ a.e. } t \in J \text{ in } X, \tag{3.5}$$

$$B(t)u_n(t) \to B(t)u(t) \text{ in } \omega\text{-}L^{\frac{1}{\beta}}(J, X). \tag{3.6}$$

From (3.3), (3.5) and Lebesgue's theorem on dominated convergence, we obtain

$$f(t, x_n(t)) \to f(t, x(t)) \text{ in } L^{\frac{1}{\beta}}(J, X).$$

This together with (3.6) implies

$$\mathcal{A}(x_n, u_n) \to \mathcal{A}(x, u) \text{ in } \omega\text{-}L^{\frac{1}{\beta}}(J, X).$$

The lemma is proved. □

Let φ, λ be defined in (3.2) and (3.3), we put

$$Y_L = \{u \in L^{\frac{1}{\beta}}(J, Y) : \|u(t)\|_Y \leq \varphi \text{ a.e. } t \in J\}, \tag{3.7}$$

$$X_L = \{h \in L^{\frac{1}{\beta}}(J, X) : \|h(t)\|_X \leq K\varphi + \lambda(t) \text{ a.e. } t \in J\}. \tag{3.8}$$

Now we define an operator $S : L^{\frac{1}{\beta}}(J, X) \to C(J, X)$ by:

$$S(h)(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} h(s) ds, \; h \in L^{\frac{1}{\beta}}(J, X). \tag{3.9}$$

The following lemma which concerns with the continuity property of the operator S is important in the rest of the paper.

Lemma 3.3. *The operator S is continuous from $\omega\text{-}X_L$ into $C(J, X)$.*

Proof. Consider the operator $H : L^{\frac{1}{\beta}}(J, X) \to C(J, X)$ defined by

$$H(h)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} h(s) ds.$$

We know that H is linear. From simple calculation,

$$\|H(h)\|_C \leq \frac{b^{\alpha - \beta}}{\Gamma(\alpha)} \Big(\frac{1 - \beta}{\alpha - \beta}\Big)^{1 - \beta} \|h\|_{L^{\frac{1}{\beta}}(J, X)}, \tag{3.10}$$

we get that the operator H is continuous from $L^{\frac{1}{\beta}}(J,X)$ into $C(J,X)$, hence H is also continuous from ω-$L^{\frac{1}{\beta}}(J,X)$ into ω-$C(J,X)$.

Firstly, in view of (3.10), we know that for any bounded subset B of the space $L^{\frac{1}{\beta}}(J,X)$, $\|H(h)(t)\|_X$ is uniformly bounded for any $h \in B$ and each $t \in J$.

Secondly, we show that H is equicontinuous on any bounded subset B of the space $L^{\frac{1}{\beta}}(J,X)$. Assume that $\|h\|_{L^{\frac{1}{\beta}}(J,X)} \le R$ for each $h \in B$. Let $0 \le t_1 < t_2 \le b$ and $h \in B$, we have

$$\begin{aligned}
&\|H(h)(t_2) - H(h)(t_1)\|_X \\
=&\Big\| \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds - \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds \Big\|_X \\
\le&\Big\| \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds \Big\|_X + \Big\| \int_0^{t_1} \frac{(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds \Big\|_X \\
\le&\frac{R}{\Gamma(\alpha)} \Big(\frac{1-\beta}{\alpha-\beta} \Big)^{1-\beta} (t_2-t_1)^{\alpha-\beta} \\
&\quad + \frac{R}{\Gamma(\alpha)} \Big(\int_0^{t_1} \big((t_1-s)^{\frac{\alpha-1}{1-\beta}} - (t_2-s)^{\frac{\alpha-1}{1-\beta}} \big) ds \Big)^{1-\beta} \\
\le&(1+2^{1-\beta}) \frac{R}{\Gamma(\alpha)} \Big(\frac{1-\beta}{\alpha-\beta} \Big)^{1-\beta} (t_2-t_1)^{\alpha-\beta}.
\end{aligned}$$

This implies that H is equicontinuous on B. Since $X = \mathbb{R}^n$ is finite dimensional, it follows from Ascoli-Arzelá theorem that $H(B)$ is relatively compact in $C(J,X)$.

Since X_L is a metrizable convex compact set of ω-$L^{\frac{1}{\beta}}(J,X)$, it suffices to show that the map $h \to S(h)$ is sequentially continuous. Now let $h_n \in X_L$, $n \ge 1$, and assume

$$h_n \to h \text{ in } \omega\text{-}L^{\frac{1}{\beta}}(J,X).$$

We have $h \in X_L$,

$$H(h_n) \to H(h) \text{ in } \omega\text{-}C(J,X), \tag{3.11}$$

and there is a subsequence h_{n_k}, $k \ge 1$, of the sequence h_n, $n \ge 1$, such that

$$H(h_{n_k}) \to z \text{ in } C(J,X) \text{ with some } z \in C(J,X). \tag{3.12}$$

From (3.11) and (3.12), we obtain that $z = H(h)$ and

$$H(h_n) \to H(h) \text{ in } C(J,X).$$

Now it is obviously that

$$S(h_n) = x_0 + H(h_n) \to x_0 + H(h) = S(h) \text{ in } C(J,X),$$

when $h_n \to h$ in ω-X_L. This completes the proof of the lemma. □

Now we are in a position to present the existence results of control systems (1.1), (1.2) and (1.1), (1.3).

THEOREM 3.1. *The set $\mathcal{R}_U$ is nonempty and the set $\mathcal{R}_{\overline{co}U}$ is a compact subset of the space $C(J,X)\times\omega\text{-}L^{\frac{1}{\beta}}(J,Y)$.*

P r o o f. Let $\Lambda = S(X_L)$, from Lemma 3.3, we have Λ is a compact subset of $C(J,X)$. It follows from (3.2), (3.3) and (3.8) that $\mathcal{T}r_U\subseteq\mathcal{T}r_{\overline{co}U}\subseteq\Lambda$. Let $\overline{U}: C(J,X)\to 2^{L^{\frac{1}{\beta}}(J,Y)}$ defined by

$$\overline{U}(x) = \{h : J\to Y \text{ measurable} : h(t)\in U(t,x(t)) \text{ a.e.}\},\ \ x\in C(J,X). \tag{3.13}$$

By the hypotheses H(U)(1) and H(U)(2), we have that, for any continuous function $x : J\to X$, the map $t\to U(t,x(t))$ is measurable (Proposition 2.7.9 [13]) and has closed values. Therefore it has measurable selections (Theorem 2.2.1 [13]) and the operator $\overline{U}$ is well defined. It is clear that the values of $\overline{U}$ are closed and decomposable subsets of $L^{\frac{1}{\beta}}(J,Y)$.

We claim that the map $x\to\overline{U}(x)$ is l.s.c. Let $x_*\in C(J,X)$, $h_*\in\overline{U}(x_*)$ and let $\{x_n\}_{n\geq 1}\subseteq C(J,X)$ be a sequence converging to x_*. It follows the Lemma 3.2 in [40] that there is a sequence $h_n\in\overline{U}(x_n)$, $n\geq 1$, such that

$$\|h_*(t)-h_n(t)\|_Y\leq d_Y(h_*(t),U(t,x_n(t)))+\frac{1}{n},\ \text{ a.e. } t\in J. \tag{3.14}$$

Since the map $y\to U(t,y)$ is h-continuous for a.e. $t\in J$ (H(U)(2)), then for a.e. $t\in J$, the map $y\to U(t,y)$ is l.s.c. (Proposition 1.2.66 [13]). Hence by Proposition 1.2.26 in [13], the function $y\to d_Y(h_*(t),U(t,y))$ is u.s.c. for a.e. $t\in J$. Then it follows from (3.14) that, for a.e. $t\in J$,

$$\begin{aligned}\lim_{n\to\infty}\|h_*(t)-h_n(t)\|_Y\leq&\limsup_{n\to\infty} d_Y(h_*(t),U(t,x_n(t)))\\ \leq& d_Y(h_*(t),U(t,x_*(t)))=0.\end{aligned}$$

This together with (3.2) implies that $h_n\to h_*$ in $L^{\frac{1}{\beta}}(J,Y)$. Therefore the map $x\to\overline{U}(x)$ is l.s.c. By Proposition 2.2 in [33], there is a continuous function $m:\Lambda\to L^{\frac{1}{\beta}}(J,Y)$ such that

$$m(x)\in\overline{U}(x),\quad\text{for all}\quad x\in\Lambda. \tag{3.15}$$

Consider the map $\mathcal{P}: L^{\frac{1}{\beta}}(J,X)\to L^{\frac{1}{\beta}}(J,Y)$ defined by $\mathcal{P}(f)=m(S(f))$. Thanks to Lemma 3.3 and the continuity of m, the map $\mathcal{P}$ is continuous from $\omega\text{-}X_L$ into $L^{\frac{1}{\beta}}(J,Y)$. Then by Lemma 3.2, we deduce that the map $f\to\mathcal{A}(S(f),\mathcal{P}(f))$ is continuous from $\omega\text{-}X_L$ into $\omega\text{-}L^{\frac{1}{\beta}}(J,X)$. It follows from (3.2), (3.3), (3.4) and (3.8) that $\mathcal{A}(S(f),\mathcal{P}(f))\in X_L$ for every $f\in X_L$. Therefore, the map $f\to\mathcal{A}(S(f),\mathcal{P}(f))$ is continuous from $\omega\text{-}X_L$ into $\omega\text{-}X_L$.

Since ω-X_L is a convex metrizable compact set in ω-$L^{\frac{1}{\beta}}(J,X)$, Schauder's fixed point theorem implies that this map has a fixed point $f_* \in X_L$, i.e. $f_* = \mathcal{A}(S(f_*),\mathcal{P}(f_*))$. Let $u_* = \mathcal{P}(f_*)$ and $x_* = S(f_*)$, then we have $u_* = m(x_*)$ and $f_* = \mathcal{A}(x_*,u_*)$. That is to say, we have

$$x_*(t) = S(f_*)(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\big(f(s,x_*(s)) + B(s)u_*(s)\big)ds$$

and

$$u_*(t) \in U(t,x_*(t)) \text{ a.e. } t \in J.$$

These imply that (x_*,u_*) is a solution of control system (1.1), (1.2). Hence $\mathcal{R}_U$ is nonempty.

Since $\mathcal{R}_{\overline{\text{co}}U} \subseteq \Lambda \times Y_L$, Λ is compact in $C(J,X)$ and Y_L is metrizable convex compact in ω-$L^{\frac{1}{\beta}}(J,Y)$, we know that $\mathcal{R}_{\overline{\text{co}}U}$ is relatively compact in $C(J,X) \times \omega$-$L^{\frac{1}{\beta}}(J,Y)$. Hence to complete the proof of this theorem, it is sufficient to prove that $\mathcal{R}_{\overline{\text{co}}U}$ is sequentially closed in $C(J,X) \times \omega$-$L^{\frac{1}{\beta}}(J,Y)$.

Let $\{(x_n,u_n)\}_{n\geq 1} \subseteq \mathcal{R}_{\overline{\text{co}}U}$ be a sequence converging to (x,u) in $C(J,X)\times$ ω-$L^{\frac{1}{\beta}}(J,Y)$. Denote

$$g_n(t) = f(t,x_n(t)) + B(t)u_n(t),$$

$$g(t) = f(t,x(t)) + B(t)u(t).$$

According to Lemma 3.2, we have $g_n \to g$ in ω-$L^{\frac{1}{\beta}}(J,X)$. Since $g_n \in X_L$, $g \in X_L$ and $x_n = S(g_n)$, $n \geq 1$, Lemma 3.3 implies that

$$x = S(g).$$

Hence, to prove that $(x,u) \in \mathcal{R}_{\overline{\text{co}}U}$, we only need to verify that $u(t) \in \overline{\text{co}}\,U(t,x(t))$ a.e. $t \in J$.

Since $u_n \to u$ in ω-$L^{\frac{1}{\beta}}(J,Y)$, by Mazur's theorem, we have

$$u(t) \in \bigcap_{n=1}^{\infty} \overline{\text{co}}\,\big(\bigcup_{k=n}^{\infty} u_k(t)\big), \quad \text{a.e. } t \in J. \tag{3.16}$$

By H(U)(2) and the fact that $h(\overline{\text{co}}\,A, \overline{\text{co}}\,B) \leq h(A,B)$ for sets A,B, the map $x \to \overline{\text{co}}\,U(t,x)$ is h-continuous a.e. $t \in J$. Then from Proposition 1.2.86 in [13], the map $x \to \overline{\text{co}}\,U(t,x)$ has property Q a.e. $t \in J$. Therefore we have

$$\bigcap_{n=1}^{\infty} \overline{\text{co}}\,\Big(\bigcup_{k=n}^{\infty} \overline{\text{co}}\,U(t,x_k(t))\Big) \subseteq \overline{\text{co}}\,U(t,x(t)), \quad \text{a.e. } t \in J. \tag{3.17}$$

By virtue of (3.16) and (3.17), we obtain that $u(t) \in \overline{\text{co}}\,U(t,x(t))$ a.e. $t \in J$. This means that $\mathcal{R}_{\overline{\text{co}}U}$ is compact in $C(J,X) \times \omega$-$L^{\frac{1}{\beta}}(J,Y)$. The proof is complete. □

4. Analogue of Bogolyubov's theorem

In this section, we shall prove the Bogolyubov's type theorem. To get our results, we shall need another assumption on the nonlinear function $f : J \times X \to X$: for a.e. $t \in J$ and any x_1, $x_2 \in X$, there is a function $L_f(t) \in L^\infty(J, \mathbb{R}^+)$ such that

$$\|f(t, x_1) - f(t, x_2)\|_X \le L_f(t)\|x_1 - x_2\|_X. \tag{4.1}$$

Let $\widetilde{Y} = Y \times \mathbb{R}$ and its elements be denoted as $\tilde{u} = (u, \xi)$, $u \in Y$, $\xi \in \mathbb{R}$. We equip $\widetilde{Y}$ with the norm $\|\tilde{u}\|_{\widetilde{Y}} = \max\{\|u\|_Y, |\xi|\}$. It is obvious that $\widetilde{Y}$ is a separable reflexive Banach space. From (2.1), we infer that the norm on space $L^q_w(J, \widetilde{Y})$ is

$$\|\tilde{u}(\cdot)\|_\omega = \sup_{0\le t_1\le t_2\le b} \Big\{ \max\Big(\Big\| \int_{t_1}^{t_2} u(s)ds\Big\|_Y, \Big| \int_{t_1}^{t_2} \xi(s)ds\Big|\Big)\Big\}.$$

Consider a multivalued map $F : J \times X \to 2^{\widetilde{Y}}$ defined as

$$F(t, x) = \{(u, \xi) \in \widetilde{Y} : u \in U(t, x),\ \xi = l(t, x, u)\}. \tag{4.2}$$

Denote by $\operatorname{dom} l^{**}_U(t, x)$ the effective domain of the function $u \to l^{**}_U(t, x, u)$, i.e.,

$$\operatorname{dom} l^{**}_U(t, x) = \{u \in Y : l^{**}_U(t, x, u) < +\infty\}.$$

Lemma 4.1. *The multivalued map $F : J \times X \to 2^{\widetilde{Y}}$ has bounded, closed values and satisfies:*

(1) the map $t \to F(t, x)$ is measurable for all $x \in X$;
(2) for a.e. $t \in J$, $x \to F(t, x)$ is Hausdorff continuous;
(3) for any $(u, \xi) \in F(t, x)$,

$$\|u\|_Y \le a_u + c_u\|x\|_X,\ |\xi| \le k_1(t) + k_2(t)\|x\|_X + k_3(t)(a_u + c_u\|x\|_X).$$

One can refer to Lemma 3.3 in [30] (or Lemma 3.1 in [32]) for the proof of this lemma.

Lemma 4.2. *For a.e. $t \in J$, we have that:*

(1) $\operatorname{dom} l^{**}_U(t, x) = \overline{\text{co}}\, U(t, x)$;
(2) for any $u \in \operatorname{dom} l^{**}_U(t, x)$,

$$l^{**}_U(t, x, u) = \min\{a \in \mathbb{R} : (u, a) \in \overline{\text{co}}\, F(t, x)\}; \tag{4.3}$$

(3) for any $\epsilon > 0$, there is a closed set $J_\epsilon \subseteq J$, $\mu(J\backslash J_\epsilon) \le \epsilon$, such that the function $(t, x, u) \to l^{**}_U(t, x, u)$ is lower semicontinuous on $J_\epsilon \times X \times Y$.

For the proof of this lemma, please see Lemma 3.4 in [30].

Now we are in a position to obtain our main results.

THEOREM 4.1. *For any* $(x_*, u_*) \in \mathcal{R}_{\overline{\text{co}}U}$, *there exists a sequence* $(x_n, u_n) \in \mathcal{R}_U$, $n \geq 1$, *such that*

$$x_n \to x_* \text{ in } C(J, X), \tag{4.4}$$

$$u_n \to u_* \text{ in } L_\omega^{\frac{1}{\beta}}(J, Y) \text{ and } \omega\text{-}L^{\frac{1}{\beta}}(J, Y), \tag{4.5}$$

$$\lim_{n\to\infty} \sup_{0\leq t_1\leq t_2\leq b} \Big| \int_{t_1}^{t_2} \big(l_U^{**}(s, x_*(s), u_*(s)) - l(s, x_n(s), u_n(s)) \big) ds \Big| = 0. \tag{4.6}$$

P r o o f. From Lemma 4.2, we know that the function $t \to l_U^{**}(t, x_*(t), u_*(t))$ is measurable and the function $\tilde{u}_*(t) = (u_*(t), l_U^{**}(t, x_*(t), u_*(t)))$ is a measurable selection of the map $t \to \overline{\text{co}}\, F(t, x_*(t))$. By Lemma 4.1, we can apply Theorem 2.2 of [34] and obtain that, for each $n \geq 1$, there exists a measurable selector $\tilde{v}_n(t)$ of the map $t \to F(t, x_*(t))$ such that

$$\sup_{0\leq t_1\leq t_2\leq b} \Big\| \int_{t_1}^{t_2} \big(\tilde{u}_*(s) - \tilde{v}_n(s)\big) ds \Big\|_{\tilde{Y}} \leq \frac{1}{n}. \tag{4.7}$$

By the definition of F (see (4.2)), we have $\tilde{v}_n(t) = (v_n(t), l(t, x_*(t), v_n(t)))$ and $v_n(t) \in U(t, x_*(t))$. Then (4.7) implies that

$$\sup_{0\leq t_1\leq t_2\leq b} \Big\| \int_{t_1}^{t_2} \big(u_*(s) - v_n(s)\big) ds \Big\|_Y \leq \frac{1}{n}, \tag{4.8}$$

$$\sup_{0\leq t_1\leq t_2\leq b} \Big| \int_{t_1}^{t_2} \big(l_U^{**}(s, x_*(s), u_*(s)) - l(s, x_*(s), v_n(s)) \big) ds \Big| \leq \frac{1}{n}. \tag{4.9}$$

Let us fix an $n \geq 1$. From H(U)(2), we infer that, for any $x \in X$, a.e. $t \in J$, there is a $y \in U(t, x)$ such that

$$\|v_n(t) - y\|_Y < d_Y(v_n(t), U(t, x)) + \frac{1}{n} \leq k_u \|x_*(t) - x\|_X + \frac{1}{n}. \tag{4.10}$$

Now we define two auxiliary multivalued maps V_n, $U_n : J \times X \to 2^Y$ as:

$$V_n(t, x) = \{y \in Y : y \text{ satisfies } (4.10)\}, \tag{4.11}$$

$$U_n(t, x) = V_n(t, x) \cap U(t, x). \tag{4.12}$$

It follows from (4.10) that the maps V_n, U_n are well defined for a.e. $t \in J$ and $x \in X$ and the values of V_n are open. From H(U)(1), H(U)(2), Scorza-Dragoni theorem for multivalued maps (Proposition 2.7.16 [13]) and Luzin's theorem, we have that, for any $\epsilon > 0$, there exists a closed set $J_\epsilon \subseteq J$ with $\mu(J \backslash J_\epsilon) \leq \epsilon$, such that the restriction of $U(t, x)$ to $J_\epsilon \times X$ is continuous and the restriction of the function $v_n(t)$ to J_ϵ is continuous. Then by (4.10) and (4.11), the graph of the map V_n restricted to $J_\epsilon \times X$ is open in $J_\epsilon \times X \times Y$.

Therefore Proposition 1.2.47 in [13] implies that the restriction of U_n to $J_\epsilon \times X$ is Vietoris lower semicontinuous. Then the restriction of the map $\overline{U}_n(t,x) = \overline{U_n(t,x)}$ (the closure is taken in Y) to $J_\epsilon \times X$ is also Vietoris lower semicontinuous.

Now consider system (1.1) with the control constraints

$$u(t) \in \overline{U}_n(t, x(t)) \text{ a.e. on } t \in J. \tag{4.13}$$

From Remark 4.1 in [30], we have that the map $\overline{U}$ defined by (3.13) is also well defined and l.s.c. when $U(t,x(t))$ there was replaced by $\overline{U}_n(t,x(t))$. Then repeating the proof of Theorem 3.1, we get that there exists a solution (x_n, u_n) of control system (1.1), (4.13). According to the definition of $\overline{U}_n$, it is clear that $\overline{U}_n(t,x) \subseteq U(t,x)$. Hence we have $(x_n, u_n) \in \mathcal{R}_U$ and

$$\|v_n(t) - u_n(t)\|_Y \leq k_u \|x_*(t) - x_n(t)\|_X + \frac{1}{n}, \text{ a.e. } t \in J. \tag{4.14}$$

Since $(x_*, u_*) \in \mathcal{R}_{\overline{\text{co}}U}$, $(x_n, u_n) \in \mathcal{R}_U \subseteq \mathcal{R}_{\overline{\text{co}}U}$, $n \geq 1$, we have

$$x_*(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \big(f(s, x_*(s)) + B(s) u_*(s) \big) ds, \tag{4.15}$$

$$x_n(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \big(f(s, x_n(s)) + B(s) u_n(s) \big) ds. \tag{4.16}$$

By Theorem 3.1, we can assume without loss of generality that the sequence $(x_n, u_n) \to (\bar{x}, \bar{u}) \in \mathcal{R}_{\overline{\text{co}}U}$ in $C(J,X) \times \omega\text{-}L^{\frac{1}{\beta}}(J,Y)$. Subtracting (4.16) from (4.15), we have

$$\begin{aligned} &\|x_*(t) - x_n(t)\|_X \\ \leq &\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \big\| f(s, x_*(s)) - f(s, x_n(s)) \big\|_X ds \quad \text{denoted as } I_1 \\ &+ \frac{1}{\Gamma(\alpha)} \Big\| \int_0^t (t-s)^{\alpha-1} \big(B(s) u_*(s) - B(s) u_n(s) \big) ds \Big\|_X. \quad \text{denoted as } I_2 \end{aligned} \tag{4.17}$$

From (4.1), we have

$$I_1 \leq \frac{\|L_f\|_{L^\infty(J)}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \big\| x_*(s) - x_n(s) \big\|_X ds.$$

And we have

$$\begin{aligned} I_2 \leq &\frac{1}{\Gamma(\alpha)} \Big\| \int_0^t (t-s)^{\alpha-1} \big(B(s) u_*(s) - B(s) v_n(s) \big) ds \Big\|_X \quad \text{denoted as } II_1 \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \big\| B(s) v_n(s) - B(s) u_n(s) \big\|_X ds. \quad \text{denoted as } II_2 \end{aligned}$$

Lemma 2.1 and (4.8) imply that $v_n \to u_*$ in ω-$L^{\frac{1}{\beta}}(J,Y)$. Hence $B(t)v_n(t) \to B(t)u_*(t)$ in ω-$L^{\frac{1}{\beta}}(J,X)$ and from the proof of Lemma 3.3, we obtain

$$II_1 \to 0 \ \text{ in } \ C(J,X). \tag{4.18}$$

By H(B) and (4.14), we get

$$\begin{aligned} II_2 &\le \frac{K}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\big\|v_n(s)-u_n(s)\big\|_Y ds \\ &\le \frac{K}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big(k_u\|x_*(s)-x_n(s)\|_X+\frac{1}{n}\Big)ds \\ &\le \frac{Kb^\alpha}{n\alpha\Gamma(\alpha)}+\frac{Kk_u}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\big\|x_*(s)-x_n(s)\big\|_X ds. \end{aligned}$$

Combining the estimations of I_1, I_2, II_2 with (4.17), we have

$$\begin{aligned} &\|x_*(t)-x_n(t)\|_X \\ \le &\frac{Kk_u+\|L_f\|_{L^\infty(J)}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\big\|x_*(s)-x_n(s)\big\|_X ds \\ &+\frac{Kb^\alpha}{n\alpha\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha)}\Big\|\int_0^t (t-s)^{\alpha-1}\big(B(s)u_*(s)-B(s)v_n(s)\big)ds\Big\|_X. \end{aligned}$$

Note (4.18) and let $n\to\infty$ in the above inequality, we obtain

$$\|x_*(t)-\bar{x}(t)\|_X \le \frac{Kk_u+\|L_f\|_{L^\infty(J)}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\big\|x_*(s)-\bar{x}(s)\big\|_X ds.$$

Hence $x_n \to \bar{x} = x_*$ in $C(J,X)$. Then by (4.14) and (4.8), we get $u_n \to u_*$ in $L_\omega^{\frac{1}{\beta}}(J,Y)$ and ω-$L^{\frac{1}{\beta}}(J,Y)$, i.e., (4.4), (4.5) hold.

Lemma 3.1 implies that $\|x_*(t)\|_X \le L$, $\|x_n(t)\|_X \le L$ for $n \ge 1$ and all $t \in J$. From H(U)(3), we have $\|v_n(t)\|_Y \le M$, $\|u_n(t)\|_Y \le M$ for $n \ge 1$ and a.e. $t \in J$ (here $M = a_u + c_u L$). Hence from H(L)(3), we have

$$|l(t,x_n(t),u_n(t))| \le k_1(t)+k_2(t)L+k_3(t)M, \ \text{a.e. } t\in J, \tag{4.19}$$

$$|l(t,x_*(t),v_n(t))| \le k_1(t)+k_2(t)L+k_3(t)M, \ \text{a.e. } t\in J. \tag{4.20}$$

Since $x_n \to x_*$ in $C(J,X)$, then from (4.14), we have

$$\|v_n(t)-u_n(t)\|_Y \to 0, \ \text{a.e. } t\in J.$$

By H(L)(2), for a.e. $t \in J$, the function $(x,u) \to l(t,x,u)$ is uniform continuous on the compact set $\{x\in\mathbb{R}^n : \|x\|_X \le L\}\times\{u\in\mathbb{R}^m : \|u\|_Y \le a_u + c_u L\}$. Therefore we obtain

$$|l(t,x_*(t),v_n(t)) - l(t,x_n(t),u_n(t))| \to 0, \ \ \text{a.e. } \ t\in J.$$

Now according to (4.19) and (4.20), we obtain

$$\|l(t, x_*(t), v_n(t)) - l(t, x_n(t), u_n(t))\|_{L^1(J)} \to 0, \text{ as } n \to \infty. \tag{4.21}$$

From (4.21) and (4.9), we conclude that (4.6) holds. This is the end of the proof. □

Finally, we prove the existence results of problem (RP) and give some relationships between problem (RP) and problem (P).

THEOREM 4.2. *Problem (RP) has a solution and*

$$\min_{(x,u)\in\mathcal{R}_{\overline{co}U}} I^{**}(x,u) = \inf_{(x,u)\in\mathcal{R}_U} I(x,u). \tag{4.22}$$

For any solution (x_*, u_*) of problem (RP), there is a minimizing sequence $(x_n, u_n) \in \mathcal{R}_U$, $n \geq 1$, of problem (P) such that (4.4), (4.5) and (4.6) hold.

Conversely, if $(x_n, u_n) \in \mathcal{R}_U$, $n \geq 1$, is a minimizing sequence of problem (P), then there exists a subsequence (x_{n_k}, u_{n_k}), $k \geq 1$, of the sequence (x_n, u_n), $n \geq 1$, and a solution (x_*, u_*) of problem (RP) such that $x_{n_k} \to x_*$ in $C(J, X)$, $u_{n_k} \to u_*$ in ω-$L^{\frac{1}{\beta}}(J, Y)$ and (4.6) holds when (x_n, u_n) are replaced by (x_{n_k}, u_{n_k}).

P r o o f. From H(L)(3), H(U)(3) and the definition of the function l_U, we have

$$l_U(t, x, u) \geq \psi(t), \text{ for } x \in B_L, \ u \in U(t, x), \text{ a.e. } t \in J,$$

here $\psi(t) = -k_1(t) - k_2(t)L - k_3(t)(a_u + c_u L) \in L^1(J, \mathbb{R})$ and $B_L = \{x \in X : \|x\|_X \leq L\}$. This inequality and the bipolar properties [12] directly imply

$$l_U(t, x, u) \geq l_U^{**}(t, x, u) \geq \psi(t), \text{ for } x \in B_L, \ u \in Y, \text{ a.e. } t \in J. \tag{4.23}$$

Define a function $q : J \times X \times Y \to \overline{\mathbb{R}}$ according to the rule

$$q(t, x, u) = \begin{cases} l_U^{**}(t, x, u), & x \in B_L, \ u \in Y, \\ +\infty, & \text{otherwise.} \end{cases} \tag{4.24}$$

Since B_L is closed, the function $q(t, x, u)$ satisfies the property of $l_U^{**}(t, x, u)$ established in Lemma 4.2 item (3). Hence there exists an increasing sequence of closed sets $J_k \subseteq J$, $k \geq 1$, with $\mu(J \setminus \cup_{k=1}^{\infty} J_k) = 0$, such that $(t, x, u) \to q(t, x, u)$ is lower semicontinuous on $J_k \times X \times Y$. Since the set $\cup_{k=1}^{\infty} J_k$ is Borel, the function $(t, x, u) \to q(t, x, u)$ is Borel on $\cup_{k=1}^{\infty} J_k \times X \times Y$. Now, without loss of generality, we can suppose that the function $q(t, x, u)$ is Borel and hence is $\Sigma \otimes \mathcal{B}_{X\times Y}$ measurable, since we can change the values of q on the Borel set $(J \setminus \cup_{k=1}^{\infty} J_k) \times X \times Y$, for example, set $q(t, x, u) = 0$ when $t \in J \setminus \cup_{k=1}^{\infty} J_k$, $x \in B_L$ and $u \in Y$.

In virtue of (4.23), (4.24) and the bipolar properties, for a.e. $t \in J$, $(x,u) \to q(t,x,u)$ is lower semicontinuous, $q(t,x,u)$ is convex in the variable u and

$$q(t,x,u) \geq \psi(t), \text{ for } x \in X, \ u \in Y.$$

Therefore, by results obtained in [14], we know that the integral functional

$$I_q(x,u) = \int_J q(t,x(t),u(t))dt$$

is sequential lower semicontinuous on $C(J,X) \times \omega\text{-}L^{\frac{1}{\beta}}(J,Y)$. From Lemma 3.1 and (4.24), we have for any $(x,u) \in \mathcal{R}_{\overline{\text{co}}U}$, $x(t) \in B_L$, $t \in J$ and hence

$$\int_J l_U^{**}(t,x(t),u(t))dt = \int_J q(t,x(t),u(t))dt. \tag{4.25}$$

Because the set $\mathcal{R}_{\overline{\text{co}}U}$ is compact in $C(J,X) \times \omega\text{-}L^{\frac{1}{\beta}}(J,Y)$, we deduce from (4.25) that problem (RP) has a solution $(x_*,u_*) \in \mathcal{R}_{\overline{\text{co}}U}$.

Now let (x_*,u_*) be any solution of problem (RP), it is clear that

$$I^{**}(x_*,u_*) = \min_{(x,u)\in\mathcal{R}_{\overline{\text{co}}U}} I^{**}(x,u) \leq \inf_{(x,u)\in\mathcal{R}_U} I(x,u).$$

Applying Theorem 4.1, we obtain a sequence $(x_n,u_n) \in \mathcal{R}_U$, $n \geq 1$, converging to (x_*,u_*) such that (4.4), (4.5) and (4.6) are true. Therefore we have

$$I^{**}(x_*,u_*) = \lim_{n\to\infty} I(x_n,u_n) \geq \inf_{(x,u)\in\mathcal{R}_U} I(x,u).$$

So we have (4.22) and a minimizing sequence $(x_n,u_n) \in \mathcal{R}_U$, $n \geq 1$, of problem (P) such that (4.4), (4.5) and (4.6) hold.

Next we prove the converse part of the theorem.

Suppose $(x_n,u_n) \in \mathcal{R}_U$, $n \geq 1$, is a minimizing sequence of problem (P). According to Theorem 3.1, we assume, without loss of generality, that $(x_n,u_n) \to (x_*,u_*) \in \mathcal{R}_{\overline{\text{co}}U}$ in $C(J,X) \times \omega\text{-}L^{\frac{1}{\beta}}(J,Y)$. Then by (4.22) we have

$$\min_{(x,u)\in\mathcal{R}_{\overline{\text{co}}U}} I^{**}(x,u) = \lim_{n\to\infty} \int_J l(t,x_n(t),u_n(t))dt. \tag{4.26}$$

From the definitions of l_U and l_U^{**}, the sequential lower semicontinuous of I^{**} on $\mathcal{R}_{\overline{\text{co}}U}$, we obtain

$$\begin{aligned}\min_{(x,u)\in\mathcal{R}_{\overline{\text{co}}U}} I^{**}(x,u) &\leq \int_J l_U^{**}(t,x_*(t),u_*(t))dt \\ \leq \liminf_{n\to\infty} \int_J l_U^{**}(t,x_n(t),u_n(t))dt &\leq \lim_{n\to\infty} \int_J l(t,x_n(t),u_n(t))dt. \end{aligned} \tag{4.27}$$

This and (4.26) imply that $(x_*,u_*) \in \mathcal{R}_{\overline{\text{co}}U}$ is a solution of problem (RP).

From H(U)(3), H(L)(3) and Lemma 3.1, we have, for a.e. $t \in J$, $n \geq 1$,

$$|l(t,x_n(t),u_n(t))| \leq k_1(t) + k_2(t)L + k_3(t)(a_u + c_u L) \in L^1(J,\mathbb{R}).$$

Then there exists a subsequence $l(t, x_{n_k}(t), u_{n_k}(t))$, $k \geq 1$, of the sequence $l(t, x_n(t), u_n(t))$, $n \geq 1$, which converges to a function $\rho \in L^1(J, \mathbb{R})$ in ω-$L^1(J, \mathbb{R})$. Since $\big(u_{n_k}(t), l(t, x_{n_k}(t), u_{n_k}(t))\big) \in F(t, x_{n_k}(t))$ a.e. $t \in J$ and $u_{n_k} \to u_*$ in ω-$L^{\frac{1}{\beta}}(J, Y)$, Mazur Lemma and item (2) of Lemma 4.1 (cf. the proof of (3.16), (3.17)) imply that

$$(u_*(t), \rho(t)) \in \overline{\mathrm{co}}\, F(t, x_*(t)) \text{ a.e. } t \in J.$$

From item (2) of Lemma 4.2, we get

$$l_U^{**}(t, x_*(t), u_*(t)) \leq \rho(t) \text{ a.e. } t \in J.$$

Therefore we have, for all $t \in J$,

$$\int_0^t l_U^{**}(s, x_*(s), u_*(s))ds \leq \int_0^t \rho(s)ds = \lim_{k\to\infty} \int_0^t l(s, x_{n_k}(s), u_{n_k}(s))ds. \tag{4.28}$$

Recall that we have

$$\int_J l_U^{**}(t, x_*(t), u_*(t))dt = \lim_{k\to\infty} \int_J l(t, x_{n_k}(t), u_{n_k}(t))dt.$$

So it follows from (4.28) that, for a.e. $t \in J$,

$$l_U^{**}(t, x_*(t), u_*(t)) = \rho(t).$$

That is to say we have the sequence $l(t, x_{n_k}(t), u_{n_k}(t)) \to l_U^{**}(t, x_*(t), u_*(t))$ in ω-$L^1(J, \mathbb{R})$. From H(L)(3) it follows that $|l(t, x_{n_k}(t), u_{n_k}(t))| \leq k_1(t) + k_2(t)L + k_3(t)(a_u + c_u L)$. These facts immediately imply that

$$\lim_{k\to\infty} \sup_{0\leq t_1\leq t_2\leq b} \left| \int_{t_1}^{t_2} \big(l_U^{**}(s, x_*(s), u_*(s)) - l(s, x_{n_k}(s), u_{n_k}(s))\big)ds \right| = 0.$$

Therefore relation (4.6) holds for the subsequence (x_{n_k}, u_{n_k}), $k \geq 1$. This is the end of the proof. □

5. **An example**

In this section, an example is given to illustrate our abstract results obtained in the previous section.

Let $X = \mathbb{R}^2$, $Y = \mathbb{R}$, $\alpha = \frac{3}{4}$, $\beta = \frac{1}{2}$ and $b = 1$. We equip the space $\mathbb{R}^d$ $(d \geq 1)$ with the norm $\|x\| = \max\{|x_1|, |x_2|, \cdots, |x_d|\}$. We consider the following minimizing problem:

$$\int_0^1 l(t, x(t), u(t))dt \to \inf \tag{5.1}$$

on the solutions (x, u) of the fractional control system

$$^cD^{\frac{3}{4}}x(t) = f(t, x(t)) + B(t)u(t),$$
$$x(0) = x_0, \;\; u(t) \in U(t, x(t)) \;\text{ a.e. } t \in [0, 1],$$

where $x_0 \in \mathbb{R}^2$ is given and

$$f(t,x(t)) = \begin{pmatrix} 3t^2+5+\sin x_1(t)+\frac{|x_2(t)|}{1+|x_2(t)|} \\ x_1(t)+\arctan x_2(t) \end{pmatrix}, \ B(t) = \begin{pmatrix} 3t^3+5 \\ e^{-t}+5t \end{pmatrix},$$

$$U(t,x(t)) = \Big[-\frac{1}{(1+t)^2}|x_1(t)| - \frac{2}{3}, -\frac{1}{2}\Big] \bigcup \Big[\frac{1}{4}, 1+5t\sin^2 x_2(t)\Big],$$

$$l(t,x(t),u(t)) = \varphi(t) + \Big|\sqrt{x_1^2(t)+x_2^2(t)} - \phi(t)\Big| + \cos u(t),$$

here φ, $\phi \in L^1([0,1],\mathbb{R})$ are given functions.

We make no hypothesis concerning convexity ($U(t,x)$ is not convex valued and $l(t,x,u)$ is not convex with the third variable u), and as a result there is in general no solution to problem (5.1). We therefore consider the corresponding relaxed problem:

$$\int_0^1 l_U^{**}(t,x(t),u(t))dt \to \inf \tag{5.2}$$

on the solutions (x,u) of the fractional control system

$$^cD^{\frac{3}{4}}x(t) = f(t,x(t)) + B(t)u(t),$$
$$x(0) = x_0, \ u(t) \in \overline{\text{co}}U(t,x(t)) \text{ a.e. } t \in [0,1].$$

The solutions of the relaxed problem thus appear as “generalized solutions” of the original problem.

Next, we show that the above problems satisfy the assumptions of Theorems 4.1 and 4.2.

Since we have

$$\|f(t,x)\| = \Bigg\|\begin{pmatrix} 3t^2+5+\sin x_1+\frac{|x_2|}{1+|x_2|} \\ x_1+\arctan x_2 \end{pmatrix}\Bigg\| \le 3t^2+5+\frac{\pi}{2}+2\|x\|,$$

$$\|f(t,x)-f(t,y)\| \le 2\|x-y\|,$$

$$\|B(t)\|_{L(Y,X)} \le 8, \ t \in [0,1],$$

$$U(t,x) = \Big[-\frac{1}{(1+t)^2}|x_1| - \frac{2}{3}, -\frac{1}{2}\Big] \bigcup \Big[\frac{1}{4}, 1+5t\sin^2 x_2\Big],$$

$$\sup\{\|v\|_Y : v \in U(t,x)\} \le 6 + \frac{2}{3} + \|x\|, \ t \in [0,1],$$

$$h(U(t,x),U(t,y)) \le \max\{10t, \frac{1}{(1+t)^2}\}\|x-y\|$$

and

$$\begin{aligned} |l(t,x,u)| &= \big|\varphi(t) + |\sqrt{x_1^2+x_2^2} - \phi(t)| + \cos u\big| \\ &\le |\varphi(t)| + |\phi(t)| + 2\|x\| + 1 + \|u\|. \end{aligned}$$

Therefore we know that H(f), H(B), H(U), H(L) and (4.1) are all satisfied and the conclusions of Theorems 4.1 and 4.2 can be applied for these problems.

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References

[1] Z.B. Bai, On positive solutions of a nonlocal fractional boundary value problem. *Nonlinear Anal.* **72**, No 2 (2010), 916–924; DOI: 10.1016/j.na.2009.07.033.

[2] K. Balachandran, J. Kokila, J.J. Trujillo, Relative controllability of fractional dynamical systems with multiple delays in control. *Comput. Math. Appl.* **64**, No 10 (2012), 3037–3045; DOI: 10.1016/j.camwa.2012.01.071.

[3] K. Balachandran, J.Y. Park, J.J. Trujillo, Controllability of nonlinear fractional dynamical systems. *Nonlinear Anal.* **75** (2012), 1919–1926; DOI: 10.1016/j.na.2011.09.042.

[4] D. Băleanu, J.A.T. Machado, A.C.J. Luo, *Fractional Dynamics and Control*. Springer, New York (2012).

[5] M. Benchohra, S. Hamani, S.K. Ntouyas, Boundary value problems for differential equations with fractional order. *Surv. Math. Appl.* **3** (2008), 1–12.

[6] N.N. Bogolyubov, Sur quelques méthodes nouvelles dans le calcul des variations. *Ann. Mat. Pura Appl.* **7** (1930), 249–271.

[7] A. Cernea, On the existence of solutions for fractional differential inclusions with anti-periodic boundary conditions. *J. Appl. Math. Comput.* **38** (2012), 133–143; DOI: 10.1007/s12190-010-0468-6.

[8] A. Debbouchea, D. Baleanu, Controllability of fractional evolution nonlocal impulsive quasilinear delay integro-differential systems. *Comput. Math. Appl.* **62** (2011), 1442–1450; DOI: 10.1016/j.camwa.2011.03.075.

[9] F.S. De Blasi, G. Pianigiani, A.A. Tolstonogov, A bogolyubov-type theorem with a nonconvex constraint in banach spaces. *SIAM J. Control Optim.* **43**, No 2 (2004), 466–476; DOI: 10.1137/S0363012903423156.

[10] J. Dixon, S. McKee, Weakly Singular Discrete Gronwall Inequalities. *ZAMM·Z. Angew. Math. Mech.* **68**, No 11 (1986), 535–544; DOI: 10.1002/zamm.19860661107.

[11] A. Dzieliński, W. Malesza, Point to point control of fractional differential linear control systems. *Adv. Differ. Equ.* **2011** (2011), 13; DOI: 10.1186/1687-1847-2011-13.

[12] I. Ekeland, R. Temam, *Convex Analysis and Variational Problems.* North-Holland, Amsterdam (1976).

[13] S.C. Hu, N.S. Papageorgiou, *Handbook of Multivalued Analysis: Volume I: Theory.* Kluwer Academic Publishers, Dordrecht-Boston-London (1997).

[14] A.D. Ioffe, On lower semicontinuity of integral functionals, I. *SIAM J. Control Optim.* **15**, No 4 (1977), 521–538; DOI: 10.1137/0315035.

[15] A.A. Kilbas, S.A. Marzan, Nonlinear differential equations with the Caputo fractional derivative in the space of continuously differentiable functions. *Differential Equations* **41** (2005), 84–89; DOI: 10.1007/s10625-005-0137-y.

[16] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations.* North-Holland Mathematics Studies, Vol. 204, Elsevier Science B.V, Amsterdam (2006).

[17] X.P. Li, F.L. Chen, X.Z. Li, Generalized anti-periodic boundary value problems of impulsive fractional differential equations. *Commun. Nonlinear Sci. Numer. Simul.* **18** (2013), 28–41; DOI: 10.1016/j.cnsns.2012.06.014.

[18] X.Y. Liu, X. Fu, Control systems described by a class of fractional semilinear evolution equations and their relaxation property. *Abstr. Appl. Anal.***2012** (2012), Article ID 850529, 20 p.; DOI: 10.1155/2012/850529.

[19] X.Y. Liu, Y.L. Liu, Fractional differential equations with fractional non-separated boundary conditions. *Electron. J. Diff. Equ.* **2013** (2013), # 25, 13 p..

[20] Z.H. Liu, X.W. Li, On the controllability of impulsive fractional evolution inclusions in Banach spaces. *J. Optim. Theory Appl.* **156** (2013), 167–182; DOI: 10.1007/s10957-012-0236-x.

[21] Z.H. Liu, J.H. Sun, Nonlinear boundary value problems of fractional differential systems. *Comput. Math. Appl.* **64**, No 4 (2012), 463–475; DOI: 10.1016/j.camwa.2011.12.020.

[22] E.J. MacShane, Existence theorems for Bolza problems in the calculus of variations. *Duke Math. J.* **7**, No 1 (1940), 28–61.

[23] S. Migórski, Existence and relaxation results for nonlinear evolution inclusions revisited. *J. Appl. Math. Stoch. Anal.* **8**, No 2 (1995), 143–149.

[24] S. Migórski, Existence and relaxation results for nonlinear second order evolution inclusions. *Discuss. Math. Differ. Incl. Control Optim.* **15** (1995), 129–148.

[25] K.S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Differential Equations.* John Wiley, New York (1993).

[26] J. Sabatier, O.P. Agrawal, J.A.T. Machado (Eds.), *Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering.* Springer, Dordrecht (2007).

[27] S.I. Suslov, The Bogolyubov theorem with a differential inclusion as constraint. *Siberian Math. J.* **35**, No 4 (1994), 802–814; DOI: 10.1007/BF02106624.

[28] Z.X. Tai, X.C. Wang, Controllability of fractional-order impulsive neutral functional infinite delay integrodifferential systems in Banach spaces. *Appl. Math. Lett.* **22** (2009), 1760–1765; DOI: 10.1016/j.aml.2009.06.017.

[29] S.A. Timoshin, A.A. Tolstonogov, Bogolyubov-type theorem with constraints induced by a control system with hysteresis effect. *Nonlinear Anal.* **75** (2012), 5884–5893; DOI: 10.1016/j.na.2012.05.028.

[30] A.A. Tolstonogov, Bogolyubov's theorem under constraints generated by a controlled second-order evolution system. *Izv. Math.* **67**, No 5 (2003), 1031–1060; DOI: 10.1070/IM2003v067n05ABEH000456.

[31] A.A. Tolstonogov, Bogolyubov's theorem under constraints generated by a lower semicontinuous differential inclusion. *Sb. Math.* **196**, No 2 (2005), 263–285; DOI: 10.1070/SM2005v196n02ABEH000880.

[32] A.A. Tolstonogov, Relaxation in non-convex control problems described by first-order evolution equations. *Mat. Sb.* **190**, No 11 (1999), 135–160; Engl. transl., *Sb. Math.* **190** (1999), 1689–1714; DOI: 10.1070/SM1999v190n11ABEH000441.

[33] A.A. Tolstonogov, D.A. Tolstonogov, Lp-continuous extreme selectors of multifunctions with decomposable values: Existence theorems. *Set-Valued Anal.* **4** (1996), 173–203; DOI: 10.1007/BF00425964.

[34] A.A. Tolstonogov, D.A. Tolstonogov, Lp-continuous extreme selectors of multifunctions with decomposable values: Relaxation theorems. *Set-Valued Anal.* **4** (1996), 237–269; DOI: 10.1007/BF00419367.

[35] G.T. Wang, B. Ahmad, L.H. Zhang, Impulsive anti-periodic boundary value problem for nonlinear differential equations of fractional order. *Nonlinear Anal.* **74**, No 3 (2011), 792–804; DOI: 10.1016/j.na.2010.09.030.

[36] J.R. Wang, M. Fečkan, Y. Zhou, Relaxed controls for nonlinear fractional impulsive evolution equations. *J. Optim. Theory Appl.* **156**, No 1 (2013), 13–32; DOI: 10.1007/s10957-012-0170-y.

[37] J.R. Wang, L.L. Lv, Y. Zhou, Boundary value problems for fractional differential equations involving Caputo derivative in Banach spaces. *J. Appl. Math. Comput.* **38** (2012), 209–224; DOI: 10.1007/s12190-011-0474-3.

[38] J.R. Wang, Y. Zhou, Existence and controllability results for fractional semilinear differential inclusions. *Nonlinear Anal.-Real World Appl.* **12**, No 6 (2011), 3642–3653; DOI: 10.1016/j.nonrwa.2011.06.021.

[39] J. Wei, The controllability of fractional control systems with control delay. *Comput. Math. Appl.* **64**, No 10 (2012), 3153–3159; DOI: 10.1016/j.camwa.2012.02.065.

[40] Q.J. Zhu, On the solution set of differential inclusions in Banach space. *J. Differ. Equat.* **93**, No 2 (1991), 213–237; DOI: 10.1016/0022-0396(91)90011-W.

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