Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME 18, NUMBER 6 (2015)
(Print) ISSN 1311-0454
(Electronic) ISSN 1314-2224

RESEARCH PAPER

NONEXISTENCE OF SOLUTIONS OF SOME NON-LINEAR NON-LOCAL EVOLUTION SYSTEMS ON THE HEISENBERG GROUP

Mahmoud Boutefnouchet [1], Mokhtar Kirane [2]

Abstract

We present non-existence results for systems of non-local in space hyperbolic equations, for systems of non-local in space parabolic equations, and for systems of non-local in space hyperbolic equations with linear damping terms. Our method of proof is based on the test function method with a help of a convexity inequality recently proved in [2].

MSC 2010: 35A01, 35A23, 35R11, 35R45

Key Words and Phrases: nonlinear hyperbolic systems; nonlinear parabolic systems; nonlocal operators on the Heisenberg group; blowing-up solutions

1. Introduction

In this paper we are concerned with nonexistence of global in time solutions of systems of nonlinear non-local in space hyperbolic equations, of systems of nonlinear non-local in space parabolic equations, as well of systems of nonlinear non-local in space hyperbolic equations with linear posed in the Heisenberg group. This work is an extension to systems of the recent work [2].

NOTATION. $_{T} = \mathbb{R}^{2N+1} \times (0, T)$, $\quad = \mathbb{R}^{2N+1} \times (0, +\infty)$.

We adopt the notation $X \lesssim Y$ to denote the estimate $X \leq CY$ for some positive constant C.

pp. 1336–1349 , DOI: 10.1515/fca-2015-0077
DE GRUYTER

We start with

1.1. **The system of non-linear non-local hyperbolic equations on the Heisenberg group.**

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} + (-\Delta_{\mathbb{H}})^{\frac{\alpha}{2}}(u^{2m}) = |v|^p, \\ &\frac{\partial^2 v}{\partial t^2} + (-\Delta_{\mathbb{H}})^{\frac{\alpha}{2}}(v^{2l}) = |u|^q, \end{aligned} \tag{1.1}$$

where $1 < p, q$ are real numbers and $0 < \alpha \leq 2$, supplemented with the initial data

$$\begin{aligned} u(x, y, \tau, 0) &= u_0(x, y, \tau), \qquad \frac{\partial u}{\partial t}(x, y, \tau, 0) = u_1(x, y, \tau), \\ v(x, y, \tau, 0) &= v_0(x, y, \tau), \qquad \frac{\partial v}{\partial t}(x, y, \tau, 0) = v_1(x, y, \tau), \end{aligned} \tag{1.2}$$

for which we consider weak solutions defined as follows.

Definition 1.1. A couple of locally integrable functions (u, v), $u \in L^q_{loc}(\mathcal{Q}_{\mathcal{T}})$, $v \in L^p_{loc}(\mathcal{Q}_{\mathcal{T}})$ is called a weak solution of the system of differential equations (1.1) in $\mathbb{R}^{2N+1,1}_+$ subject to the initial data $u_0, u_1, v_0, v_1 \in L^1_{loc}(\mathbb{R}^{2N+1})$ if the equalities

$$\begin{aligned} &\int_{\mathcal{Q}_{\mathcal{T}}} \left(u\frac{\partial^2 \varphi}{\partial t^2} + u^{2m}(-\Delta_{\mathbb{H}})^{\frac{\alpha}{2}}(\varphi) - |v|^p \varphi \right) d\eta dt \\ &= -\int_{\mathbb{R}^{2N+1}} u_0(\eta)\frac{\partial \varphi}{\partial t}(\eta, 0)\, d\eta + \int_{\mathbb{R}^{2N+1}} u_1(\eta)\varphi(\eta, 0)\, d\eta \end{aligned} \tag{1.3}$$

and

$$\begin{aligned} &\int_{\mathcal{Q}_{\mathcal{T}}} \left(v\frac{\partial^2 \varphi}{\partial t^2} + v^{2l}(-\Delta_{\mathbb{H}})^{\frac{\alpha}{2}}(\varphi) - |u|^q \varphi \right) d\eta dt \\ &= -\int_{\mathbb{R}^{2N+1}} v_0\frac{\partial \varphi}{\partial t}(\eta, 0)\, d\eta + \int_{\mathbb{R}^{2N+1}} v_1(\eta)\varphi(\eta, 0)\, d\eta \end{aligned} \tag{1.4}$$

are satisfied for any regular function φ belonging to the homogeneous Sobolev space $H^\alpha(\mathbb{R}^{2N+1})$ for the variable x and to $C^2(0, +\infty)$ for the t variable, $\varphi(\eta, T) = 0$.

For the system (1.1), we will prove the following theorem.

Theorem 1.1. *Let* $m, l \in \mathbb{N}^*, 2l < p$, $2m < q\, Q = 2N + 2$,

$$\int_{\mathbb{R}^{2N+1}} (u_1(\eta) + v_1(\eta))\, d\eta > 0,$$

and

$$Q+2<\alpha p \min\left\{\frac{2(q+1)}{pq-1},\frac{2q+1}{pq-2l},\frac{4m+q}{pq-2m},\frac{2m+q}{pq-4ml}\right\}. \tag{1.5}$$

Then, system (1.1) *does not admit a nontrivial weak solution.*

Then, we will consider

1.2. **The system of non-linear non-local parabolic equations on the Heisenberg group.**

$$\begin{aligned}\frac{\partial u}{\partial t}+(-\Delta_{\mathbb{H}})^{\frac{\alpha}{2}}(u)=|v|^p,\\ \frac{\partial v}{\partial t}+(-\Delta_{\mathbb{H}})^{\frac{\alpha}{2}}(v)=|u|^q,\end{aligned} \tag{1.6}$$

where $1<p,q$ are real numbers and $0<\alpha\leq 2$, supplemented with initial data

$$u(x,y,\tau,0)=u_0(x,y,\tau),\quad v(x,y,\tau,0)=v_0(x,y,\tau). \tag{1.7}$$

Definition 1.2. A couple of locally integrable functions (u,v), $u\in L^q_{loc}(\mathcal{Q}_\mathcal{T})$, $v\in L^p_{loc}(\mathcal{Q}_\mathcal{T})$ is called a weak solution of the system of differential equations (1.6) in $\mathbb{R}_+^{2N+1,1}$ subject to the initial data $u_0,v_0\in L^1_{loc}(\mathbb{R}^{2N+1})$ if the equalities

$$\begin{aligned}&\int_{\mathcal{Q}_\mathcal{T}}\left(-u\frac{\partial\varphi}{\partial t}+u(-\Delta_{\mathbb{H}})^{\frac{\alpha}{2}}(\varphi)-|v|^p\varphi\right)d\eta dt\\ &=\int_{\mathbb{R}^{2N+1}}u_0(\eta)\varphi(\eta,0)\,d\eta\end{aligned} \tag{1.8}$$

and

$$\begin{aligned}&\int_{\mathcal{Q}_\mathcal{T}}\left(-v\frac{\partial\varphi}{\partial t}+v(-\Delta_{\mathbb{H}})^{\frac{\alpha}{2}}(\varphi)-|u|^q\varphi\right)d\eta dt\\ &=\int_{\mathbb{R}^{2N+1}}v_0\frac{\partial\varphi}{\partial t}(\eta,0)\,d\eta\end{aligned} \tag{1.9}$$

are satisfied for any φ belonging to the homogeneous Sobolev space $H^\alpha(\mathbb{R}^{2N+1})$ for the variable x and to $C^2(0,+\infty)$ for the t variable, $\varphi(\eta,T)=0$.

Our result concerning the system (1.6) is provided by the following theorem.

Theorem 1.2. *Let* $1<p,<q$, $u_0(\eta)\geq 0$, $v_0(\eta)\geq 0, u_0(\eta)+v_0(\eta)>0$, $Q=2N+2$, *and*

$$Q\leq\frac{\alpha}{pq-1}\max\{p,q\}. \tag{1.10}$$

Then, system (1.6) *does not admit a nontrivial weak solution.*

And finally, we present a nonexistence result for the system of non-linear and non-local in space hyperbolic equations with a linear damping in each equation.

1.3. **The system of non-linear non-local hyperbolic equations with a linear damping on the Heisenberg group.**

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} + (-\Delta_{\mathbb{H}})^{\frac{\alpha}{2}}(u^{2m}) + \frac{\partial u}{\partial t} = |v|^p, \\ &\frac{\partial^2 v}{\partial t^2} + (-\Delta_{\mathbb{H}})^{\frac{\alpha}{2}}(v^{2l}) + \frac{\partial v}{\partial t} = |u|^q, \end{aligned} \tag{1.11}$$

where $1 < p, q$ are real numbers and $0 < \alpha \leq 2$, supplemented with the initial data

$$\begin{aligned} u(x, y, \tau, 0) &= u_0(x, y, \tau), \qquad \frac{\partial u}{\partial t}(x, y, \tau, 0) = u_1(x, y, \tau), \\ v(x, y, \tau, 0) &= v_0(x, y, \tau), \qquad \frac{\partial v}{\partial t}(x, y, \tau, 0) = v_1(x, y, \tau), \end{aligned} \tag{1.12}$$

for which we consider weak solutions defined as follows.

Definition 1.3. A couple of locally integrable functions (u, v), $u \in L^q_{loc}(\mathcal{Q}_{\mathcal{T}})$, $v \in L^p_{loc}\mathcal{Q}_{\mathcal{T}})$ is called a weak solution of the system of differential equations (1.11) in $\mathbb{R}^{2N+1,1}_+$ subject to the initial data $u_0, u_1, v_0, v_1 \in L^1_{loc}(\mathbb{R}^{2N+1})$ if the equalities

$$\begin{aligned} &\int_{\mathcal{Q}_{\mathcal{T}}} \left(u\frac{\partial^2 \varphi}{\partial t^2} + u\frac{\partial \varphi}{\partial t} + u^{2m}(-\Delta_{\mathbb{H}})^{\frac{\alpha}{2}}(\varphi) - |v|^p \varphi \right) d\eta dt \\ &= -\int_{\mathbb{R}^{2N+1}} u_0 \frac{\partial \varphi}{\partial t}(\eta, 0)\, d\eta + \int_{\mathbb{R}^{2N+1}} (u_0(\eta) + u_1(\eta))\, \varphi(\eta, 0)\, d\eta \end{aligned} \tag{1.13}$$

and

$$\begin{aligned} &\int_{\mathcal{Q}_{\mathcal{T}}} \left(v\frac{\partial^2 \varphi}{\partial t^2} + v\frac{\partial \varphi}{\partial t} + v^{2l}(-\Delta_{\mathbb{H}})^{\frac{\alpha}{2}}(\varphi) - |u|^q \varphi \right) d\eta dt \\ &= -\int_{\mathbb{R}^{2N+1}} v_0 \frac{\partial \varphi}{\partial t}(\eta, 0)\, d\eta + + \int_{\mathbb{R}^{2N+1}} (v_0(\eta) + v_1(\eta))\, \varphi(\eta, 0)\, d\eta \end{aligned} \tag{1.14}$$

are satisfied for any φ belonging to the homogeneous Sobolev space $H^\alpha(\mathbb{R}^{2N+1})$ for the variable x and to $C^2(0, +\infty)$ for the t variable, $\varphi(\eta, T) = 0$.

For the system (1.11), we will prove the following theorem.

Theorem 1.3. *Let $m, l \in \mathbb{N}^*$, $l \leq m$, $pq > 4ml$, $Q = 2N + 2$,*

$$\int_{\mathbb{R}^{2N+1}} (u_1(\eta) + v_1(\eta))\, d\eta > 0,$$

and

$$Q + \alpha \leq \max\left\{\frac{\alpha p(q+1)}{pq-1}, \frac{\alpha p(q+1)}{pq-2l}, \frac{\alpha p(2m+q)}{pq-4ml}\right\}. \tag{1.15}$$

Then, system (1.11) *does not admit a nontrivial weak solution.*

Before we give the proofs of the theorems, let us dwell on the existing literature concerning systems of hyperbolic equations and systems of parabolic equations posed on the Euclidian space $\mathbb{R}^N, N \geq 1$. The wave equations $(W1)$ $u_{tt} - \Delta u = |u|^{p-1}u$ and $(W2)$ $u_{tt} - \Delta u = |u|^p$ received great attentions and are well documented (see [25], [9]). For $(W1)((W2))$, it took twenty years to establish that the critical exponent $p_w(N)$ is the positive root of $(N-1)p^2 - (N+1)p - 2 = 0$, when $N \geq 2$ is the space dimension ($p_w(1) = \infty$, see Sideris [24], Rammaha [23]). The critical exponent for the heat equation $u_{tt} - \Delta u = u^p$ was established by Fujita in his pioneering work [7]. The critical exponent for the Cauchy problem for the wave equation with linear damping $u_{tt} - \Delta u + u_t = |u|^p$ has mainly been established in [28], and completed by [32] and [16]; they showed that the critical exponent leads to blow-up.

Global or blowing-up solutions for the heat or the wave equation on nilpotent Lie groups has been studied in [6], [8], [22], [29], [30]. Wave equations on the Heisenberg group has been discussed in [20], [3] for example for the existence of local solutions in various spaces, and [33] also for global solution but under sub-linear non-linearities.

Various equations (elliptic, parabolic, hyperbolic, of Schrödinger type) with fractional powers of the Laplacian are now accepted as good models in many applied situations (see for example [1], [5], [11], [10], [13], [14], [15], [17], [26], [27]).

2. Preliminaries

For the reader's convenience, let us briefly recall the definition, the basic properties of the Heisenberg group, some facts about fractional powers of the sub-elliptic Laplacian on the Heisenberg group, and an important inequality proved in [2].

2.1. The Heisenberg group.

The Heisenberg group $\mathbb{H}$, whose points will be denoted by $\eta = (x, y, \tau)$, is the Lie group $(\mathbb{R}^{2N+1}, \circ)$ with the non-commutative group operation $\circ$ defined by

$$\eta \circ \tilde{\eta} = (x + \tilde{x}, y + \tilde{y}, \tau + \tilde{\tau} + 2(< x, \tilde{y} > - < \tilde{x}, y >)),$$

where $< \cdot, \cdot >$ is the usual inner product in $\mathbb{R}^N$. The Laplacian $\Delta_{\mathbb{H}}$ over $\mathbb{H}$ is obtained, from the vector fields $X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial \tau}$ and $Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial \tau}$, by

$$\Delta_{\mathbb{H}} = \sum_{i=1}^{N} (X_i^2 + Y_i^2). \tag{2.1}$$

Observe that the vector field $T = \frac{\partial}{\partial \tau}$ does not appear in (2.1)). This fact makes us presume a "loss of derivative" in the variable τ. The compensation comes from the relation

$$[X_i, Y_j] = -4T, \quad i, j \in \{1, 2, \cdots, N\}. \tag{2.2}$$

The relation (2.2)) proves that $\mathbb{H}$ is a nilpotent Lie group of order 2. Incidently, (2.2)) constitutes an abstract version of the canonical relations of commutation of Heisenberg between momentum and positions. Explicit computation gives the expression

$$\Delta_{\mathbb{H}} = \sum_{i=1}^{N} \left(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} + 4y_i \frac{\partial^2}{\partial x_i \partial \tau} - 4x_i \frac{\partial^2}{\partial y_i \partial \tau} + 4(x_i^2 + y_i^2) \frac{\partial^2}{\partial \tau^2} \right). \tag{2.3}$$

A natural group of dilatations on $\mathbb{H}$ is given by

$$\delta_\lambda(\eta) = (\lambda x, \lambda y, \lambda^2 \tau), \quad \lambda > 0,$$

whose Jacobian determinant is λ^Q, where

$$Q = 2N + 2 \tag{2.4}$$

is the homogeneous dimension of $\mathbb{H}$.

The operator $\Delta_{\mathbb{H}}$ is a degenerate elliptic operator. It is invariant with respect to the left translation of $\mathbb{H}$ and homogeneous with respect to the dilatations δ_λ. More precisely, we have

$$\begin{aligned} &\Delta_{\mathbb{H}}(u(\eta \circ \tilde{\eta})) = (\Delta_{\mathbb{H}} u)(\eta \circ \tilde{\eta}), \\ &\Delta_{\mathbb{H}}(u \circ \delta_\lambda) = \lambda^2 (\Delta_{\mathbb{H}} u) \circ \delta_\lambda, \quad \eta, \tilde{\eta} \in \mathbb{H}. \end{aligned} \tag{2.5}$$

The natural distance from η to the origin is

$$|\eta|_{\mathbb{H}} = \left(\tau^2 + \sum_{i=1}^{N} (x_i^2 + y_i^2) \right)^{1/4}. \tag{2.6}$$

2.2. **Fractional powers of sub-elliptic Laplacians.** Here, we collect some results on fractional powers of sub-Laplacian in the Heisenberg group. To begin with, let us characterize $(-\Delta_{\mathbb{H}})^s$ as the spectral resolution of $\Delta_{\mathbb{H}}$ in $L^2(\mathbb{H})$.

THEOREM 2.1. *The operator $-\Delta_{\mathbb{H}}$ is a positive self-adjoint operator with domain $W^{2,2}_{\mathbb{H}}(\mathbb{H})$. Denote now by $\{E(\lambda)\}$ the spectral resolution of $-\Delta_{\mathbb{H}}$ in $L^2(\mathbb{H})$.*

If $\alpha > 0$, then

$$(-\Delta_{\mathbb{H}})^{\frac{\alpha}{2}} = \int_0^{+\infty} \lambda^{\frac{\alpha}{2}}\, dE(\lambda),$$

with domain

$$W_{\mathbb{H}}^{\alpha,2}(\mathbb{H}) := \{v \in L^2(\mathbb{H});\ \int_0^{+\infty} \lambda^{\alpha}\, d < E(\lambda)v, v > < \infty\},$$

endowed with graph norm.

PROPOSITION 2.1. *Assume that the function φ is smooth and bounded. Then,*

$$\sigma\varphi^{\sigma-1}(-\Delta_{\mathbb{H}})^{\frac{\alpha}{2}}\varphi \geq (-\Delta_{\mathbb{H}})^{\frac{\alpha}{2}}\varphi^{\sigma}. \tag{2.7}$$

For the proof see [2].

In the Euclidian case, this inequality has been first established in [5] and then generalized in [11].

Proof of Theorem 1.1. Let (u, v) be a solution of (1.1)-(1.2) and φ be a smooth nonnegative function. Let us set

$$\mathcal{A}(r, k, \varphi) = \left(\int_{\mathcal{Q}} |(-\Delta_{\mathbb{H}})^{\frac{\alpha}{2}}(\varphi)|^{\frac{r}{r-2k}}\varphi^{\sigma-\frac{r}{r-2k}}\, d\eta dt\right)^{\frac{r-2k}{r}}, \tag{2.8}$$

for $r > 2k$, and

$$\mathcal{B}(r, \varphi) = \left(\int_{\mathcal{Q}} \varphi^{\sigma-2r'}\left|\frac{\partial\varphi}{\partial t}\right|^{2r'} + \varphi^{\sigma-r'}\left|\frac{\partial^2\varphi}{\partial t^2}\right|^{r'} d\eta dt\right)^{\frac{1}{r'}} \tag{2.9}$$

for $\sigma \gg 1, r + r' = rr'$. Making use of inequality (2.7)) with φ^{σ} in place of φ in (1.3) and (1.4), we obtain

$$\int_{\mathcal{Q}} |u|^q\varphi^{\sigma}\, d\eta dt + \int_{\mathbb{R}^{2N+1}} v_1(\eta)\varphi^{\sigma}(\eta, 0)\, d\eta$$

$$= \int_{\mathcal{Q}} \left(v\frac{\partial^2\varphi^{\sigma}}{\partial t^2} + v^{2l}(-\Delta_{\mathbb{H}})^{\frac{\alpha}{2}}(\varphi^{\sigma})\right) d\eta dt \tag{2.10}$$

$$\lesssim \left(\int_{\mathcal{Q}} |v|^p\varphi^{\sigma}\, d\eta dt\right)^{\frac{1}{p}} \mathcal{B}(p, \varphi) + \left(\int_{\mathcal{Q}} |v|^p\varphi^{\sigma}\, d\eta dt\right)^{\frac{2l}{p}} \mathcal{A}(p, l, \varphi),$$

and

$$\int_{\mathcal{Q}} |v|^p\varphi^{\sigma}\, d\eta dt + \int_{\mathbb{R}^{2N+1}} u_1(\eta)\varphi^{\sigma}(\eta, 0)\, d\eta$$

$$= \int_{\mathcal{Q}} \left(u \frac{\partial^2 \varphi^\sigma}{\partial t^2} + u^{2m} (-\Delta_{\mathbb{H}})^{\frac{\alpha}{2}} (\varphi^\sigma) \right) d\eta dt$$
$$\lesssim \left(\int_{\mathcal{Q}} |u|^q \varphi^\sigma \, d\eta dt \right)^{\frac{1}{q}} \mathcal{B}(q,\varphi) + \left(\int_{\mathcal{Q}} |u|^q \varphi^\sigma \, d\eta dt \right)^{\frac{2m}{q}} \mathcal{A}(q,m,\varphi). \tag{2.11}$$

for some positive constant C and where we have chosen φ such that $\varphi(\eta, T) = 0$.

Let us anticipate that

$$\int_{\mathbb{R}^{2N+1}} u_1(\eta) \varphi^\sigma(\eta, 0) \, d\eta > 0, \qquad \int_{\mathbb{R}^{2N+1}} v_1(\eta) \varphi^\sigma(\eta, 0) \, d\eta > 0$$

as it will be showed hereafter for a certain choice of the function φ. Setting

$$\mathcal{I} = \int_{\mathcal{Q}} |u|^q \varphi^\sigma \, d\eta dt, \qquad \mathcal{J} = \int_{\mathcal{Q}} |v|^p \varphi^\sigma \, d\eta dt,$$

we write (2.10) and(2.11) in the form

$$\begin{aligned} \mathcal{J} &\lesssim \mathcal{I}^{\frac{1}{q}} \mathcal{B}(q,\varphi) + \mathcal{I}^{\frac{2m}{q}} \mathcal{A}(q,m,\varphi), \\ \mathcal{I} &\lesssim \mathcal{J}^{\frac{1}{p}} \mathcal{B}(p,\varphi) + \mathcal{J}^{\frac{2l}{p}} \mathcal{A}(p,l,\varphi), \end{aligned} \tag{2.12}$$

which leads to

$$\begin{aligned} \mathcal{I} \lesssim\, & \mathcal{I}^{\frac{1}{pq}} \mathcal{B}^{\frac{1}{p}}(q,\varphi) \mathcal{B}(p,\varphi) + \mathcal{I}^{\frac{2m}{pq}} \mathcal{B}(p,\varphi) \mathcal{A}^{\frac{1}{p}}(q,m,\varphi) \\ & + \mathcal{I}^{\frac{2l}{pq}} \mathcal{A}(p,l,\varphi) \mathcal{B}^{\frac{2l}{p}}(q,\varphi) + \mathcal{I}^{\frac{4ml}{pq}} \mathcal{A}^{\frac{2l}{p}}(q,m,\varphi) \mathcal{A}(p,l,\varphi), \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} \mathcal{J} \lesssim\, & \mathcal{J}^{\frac{1}{pq}} \mathcal{B}^{\frac{1}{q}}(p,\varphi) \mathcal{B}(q,\varphi) + \mathcal{J}^{\frac{2l}{pq}} \mathcal{B}(q,\varphi) \mathcal{A}^{\frac{1}{q}}(p,l,\varphi) \\ & + \mathcal{J}^{\frac{2m}{pq}} \mathcal{A}(q,m,\varphi) \mathcal{B}^{\frac{2m}{q}}(p,\varphi) + \mathcal{J}^{\frac{4ml}{pq}} \mathcal{A}^{\frac{2m}{q}}(p,l,\varphi) \mathcal{A}(q,m,\varphi). \end{aligned} \tag{2.14}$$

Using the Young inequality with ε

$$ab \le \varepsilon a^r + C_\varepsilon b^s, \quad r + s = rs, \; a \ge 0, \; b \ge 0,$$

where ε and C_ε are two positive constants, we obtain the inequalities

$$\mathcal{J}^{\frac{1}{pq}} \mathcal{B}(q,\varphi) \mathcal{B}^{\frac{1}{q}}(p,\varphi) \le \varepsilon \mathcal{J} + C_\varepsilon \mathcal{B}^{\frac{pq}{pq-1}}(q,\varphi) \mathcal{B}^{\frac{p}{pq-1}}(p,\varphi),$$
$$\mathcal{J}^{\frac{2l}{pq}} \mathcal{B}(q,\varphi) \mathcal{A}^{\frac{1}{q}}(p,l,\varphi) \le \varepsilon \mathcal{J} + C_\varepsilon \mathcal{B}^{\frac{pq}{pq-2l}}(q,\varphi) \mathcal{A}^{\frac{p}{pq-2l}}(p,l,\varphi),$$
$$\mathcal{J}^{\frac{2m}{pq}} \mathcal{B}^{\frac{2m}{q}}(p,\varphi) \mathcal{A}(q,m,\varphi) \le \varepsilon \mathcal{J} + C_\varepsilon \mathcal{B}^{\frac{2mp}{pq-2m}}(p,\varphi) \mathcal{A}^{\frac{pq}{pq-2m}}(q,m,\varphi),$$
$$\mathcal{J}^{\frac{4ml}{pq}} \mathcal{A}(q,m,\varphi) \mathcal{A}^{\frac{2m}{q}}(p,l,\varphi) \le \varepsilon \mathcal{J} + C_\varepsilon \mathcal{A}^{\frac{pq}{pq-4ml}}(q,m,\varphi) \mathcal{A}^{\frac{2mp}{pq-4ml}}(p,l,\varphi).$$

Whereupon, choosing ε small enough,

$$\mathcal{J} \lesssim \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 + \mathcal{J}_4, \tag{2.15}$$

where

$$\begin{aligned}
\mathcal{J}_1 &= \mathcal{B}^{\frac{pq}{pq-1}}(q,\varphi)\mathcal{B}^{\frac{p}{pq-1}}(p,\varphi),\\
\mathcal{J}_2 &= \mathcal{B}^{\frac{pq}{pq-2l}}(q,\varphi)\mathcal{A}^{\frac{p}{pq-2l}}(p,l,\varphi),\\
\mathcal{J}_3 &= \mathcal{B}^{\frac{2mp}{pq-2m}}(p,\varphi)\mathcal{A}^{\frac{pq}{pq-2m}}(q,m,\varphi),\\
\mathcal{J}_4 &= \mathcal{A}^{\frac{pq}{pq-4ml}}(q,m,\varphi)\mathcal{A}^{\frac{2mp}{pq-4ml}}(p,l,\varphi).
\end{aligned}$$

Set

$$\varphi(\eta,t) = \Phi\left(\frac{\tau^2+|x|^4+|y|^4+t^{\frac{4}{\alpha}}}{R^4}\right), \tag{2.16}$$

where $R>0$, and $\Phi \in \mathcal{D}([0,+\infty[)$ is the standard cut-off function

$$\Phi(r) = \begin{cases} 1, & 0 \le r \le 1,\\ 0, & r \ge 2. \end{cases} \qquad 0 \le \Phi(r) \le 1, \tag{2.17}$$

Note that supp(φ) is a subset of

$$\Omega_R = \left\{\eta \in \mathbb{H};\ 0 \le \tau^2+|x|^4+|y|^4+t^{\frac{4}{\alpha}} \le 2R^4\right\}. \tag{2.18}$$

Note also that

$$\alpha\frac{\partial\varphi}{\partial t}(\eta,t) = 4t^{\frac{4}{\alpha}-1}R^{-2}\Phi'\left(\frac{\tau^2+|x|^4+|y|^4+t^{\frac{4}{\alpha}}}{R^4}\right) \implies \frac{\partial\varphi}{\partial t}(\eta,0)=0,$$

as required here above. Moreover, using the scaled variables

$$\tilde\tau = R^{-2}\tau, \quad \tilde x = R^{-1}x, \quad \tilde y = R^{-1}y, \quad \tilde t = R^{-\alpha}t, \tag{2.19}$$

we obtain the estimates

$$\begin{aligned}
&\mathcal{A}(r,k,\varphi) \lesssim R^{\delta_1}, \quad \delta_1 = -\alpha + \tfrac{r-2k}{r}(2+2N+\alpha),\\
&\mathcal{B}(r,\varphi) \lesssim R^{\delta_2}, \quad \delta_2 = -2\alpha + \tfrac{r-1}{r}(2+2N+\alpha).
\end{aligned} \tag{2.20}$$

Consequently,

$$\mathcal{J} \lesssim \sum_{i=1}^{i=4} R^{-\vartheta_i}, \tag{2.21}$$

where

$$\begin{aligned}
\vartheta_1 &= \tfrac{1}{pq-1}(2\alpha p(q+1) - (2+2N+\alpha)(pq-1)),\\
\vartheta_2 &= \tfrac{1}{pq-2l}(\alpha p(2q+1) - (2+2N+\alpha)(pq-2l)),\\
\vartheta_3 &= \tfrac{1}{pq-2m}\,(\alpha p(4m+q) - (pq-2m)(2+2N+\alpha))\\
\vartheta_4 &= \tfrac{1}{pq-4ml}(\alpha p(2m+q) - (pq-4ml)(2+2N+\alpha)).
\end{aligned} \tag{2.22}$$

Now, if

$$Q+2<\alpha p\min\left\{\frac{2(q+1)}{pq-1},\frac{2q+1}{pq-2l},\frac{4m+q}{pq-2m},\frac{2m+q}{pq-4ml}\right\},$$

which means $\min\{\vartheta_1,\ \vartheta_2,\ \vartheta_3,\ \vartheta_4\}>0$, then passing to the limit when $R\to\infty$ in (2.15), we obtain

$$\int_{\mathcal{Q}}|v|^p\,d\eta dt=0.$$

This is a contradiction.

In the limit case when

$$\vartheta_1=\vartheta_2=\vartheta_3=\vartheta_4=0,$$

we obtain

$$\int_{\mathcal{Q}}|v|^p\,d\eta dt\le C<\infty,$$

which leads to

$$\lim_{R\to\infty}\int_{\mathcal{C_R}}|v|^p\varphi^\sigma(\eta,t)\,d\eta dt=0,$$

where

$$\mathcal{C}_R=\left\{\eta\in\mathbb{H};\ R^4\le\tau^2+|x|^4+|y|^4+t^{\frac{4}{\alpha}}\le 2R^4\right\}.$$

Passing to the limit in (2.10) as $t\to\infty$, we obtain the contradiction

$$\int_{\mathcal{Q}}|u|^q\,d\eta dt=0.$$

□

Remark 2.1. It appears clear from the proof that we need the positive powers u^{2m} and v^{2l} in (1.1) in order to apply the convexity inequality (2.7).

Proof of Theorem 1.2. Assume to the contrary that the solution is global. Proceeding as in the proof of Theorem 1.1, we have the estimates

$$\int_{\mathcal{Q}}|v|^p\varphi^\sigma\,d\eta dt\le\left(\int_{\mathcal{Q}}|u|^q\varphi^\sigma\,d\eta dt\right)^{\frac{1}{q}}\left(\mathcal{C}(q,\varphi)+\mathcal{D}(q,\varphi)\right),\qquad(2.23)$$

$$\int_{\mathcal{Q}}|u|^q\varphi^\sigma\,d\eta dt\le\left(\int_{\mathcal{Q}}|v|^p\varphi^\sigma\,d\eta dt\right)^{\frac{1}{p}}\left(\mathcal{C}(p,\varphi)+\mathcal{D}(p,\varphi)\right),\qquad(2.24)$$

where we have set

$$\mathcal{C}(r,\varphi)=\left(\int_{\mathcal{Q}}\varphi^{(\sigma-1-\frac{\sigma}{r})r'}|\varphi_t|^{r'}\right)^{\frac{1}{r'}},$$

$$\mathcal{D}(r,\varphi)=\left(\int_{\mathcal{Q}}\varphi^{(\sigma-1-\frac{\sigma}{r})r'}|(-\Delta_{\mathbb{H}})^{\frac{\alpha}{2}}(\varphi)|^{r'}\right)^{\frac{1}{r'}}.$$

Setting

$$\mathcal{X} = \int_{\mathcal{Q}} |u|^q \varphi^\sigma \, d\eta dt, \quad \mathcal{Y} = \int_{\mathcal{Q}} |v|^p \varphi^\sigma \, d\eta dt,$$

and using (2.23) and (2.24), we obtain the estimates

$$\mathcal{X}^{1-\frac{1}{pq}} \lesssim \left(\mathcal{C}^{\frac{1}{p}}(q,\varphi) + \mathcal{D}^{\frac{1}{p}}(q,\varphi)\right)(\mathcal{C}(p,\varphi) + \mathcal{D}(p,\varphi)) \tag{2.25}$$

and

$$\mathcal{Y}^{1-\frac{1}{pq}} \lesssim \left(\mathcal{C}^{\frac{1}{q}}(p,\varphi) + \mathcal{D}^{\frac{1}{q}}(p,\varphi)\right)(\mathcal{C}(q,\varphi) + \mathcal{D}(q,\varphi)). \tag{2.26}$$

Now if we take the same test function as before and we use the same scaled variables, we obtain

$$\mathcal{X}^{1-\frac{1}{pq}} \lesssim R^{Q(1-\frac{1}{pq})-\frac{\alpha}{p}} \tag{2.27}$$

and

$$\mathcal{X}^{1-\frac{1}{pq}} \lesssim R^{Q(1-\frac{1}{pq})-\frac{\alpha}{q}}. \tag{2.28}$$

Consequently, if (1.10) is satisfied, repeating the same reasoning as in the previous proof, we obtain a contradiction. □

REMARK 2.2. As $u_0 \geq 0$ and $v_0 \geq 0$, relying on [4], $u \geq 0$ and $v \geq 0$, so we can safely use the convexity inequality (2.7).

Proof of Theorem 1.3. Proceeding as in the previous theorems, we obtain for

$$\mathcal{U} = \int_{\mathcal{Q}} |u|^q \varphi^\sigma \, d\eta dt, \quad \mathcal{V} = \int_{\mathcal{Q}} |v|^p \varphi^\sigma \, d\eta dt,$$

the estimates

$$\mathcal{V} \leq \mathcal{U}^{\frac{1}{q}} \mathcal{B}(q,\varphi) + \mathcal{U}^{\frac{2m}{q}} \mathcal{A}(q,m,\varphi) + \mathcal{U}^{\frac{1}{q}} \mathcal{C}(q,\varphi) \tag{2.29}$$

and

$$\mathcal{U} \leq \mathcal{V}^{\frac{1}{p}} \mathcal{B}(p,\varphi) + \mathcal{V}^{\frac{2l}{p}} \mathcal{A}(p,l,\varphi) + \mathcal{V}^{\frac{1}{p}} \mathcal{C}(p,\varphi). \tag{2.30}$$

Consequently,

$$\begin{aligned}
\mathcal{V} \leq\ & \mathcal{V}^{\frac{1}{pq}} \left(\mathcal{B}^{\frac{1}{q}}(p,\varphi) + \mathcal{C}^{\frac{1}{q}}(p,\varphi)\right)(\mathcal{B}(q,\varphi) + \mathcal{C}(q,\varphi)) \\
& + \mathcal{V}^{\frac{2l}{pq}} \mathcal{A}^{\frac{1}{q}}(p,l,\varphi)(\mathcal{B}(q,\varphi) + \mathcal{C}(q,\varphi)) \\
& + \mathcal{V}^{\frac{2m}{pq}} \left(\mathcal{B}^{\frac{2m}{q}}(p,\varphi) + \mathcal{C}^{\frac{2m}{q}}(p,\varphi)\right) \mathcal{A}(q,m,\varphi) \\
& + \mathcal{V}^{\frac{4ml}{pq}} \mathcal{A}^{\frac{2m}{q}}(p,l,\varphi) \mathcal{A}(q,m,\varphi).
\end{aligned}$$

Using the ε-Young inequality, we obtain

$$\begin{aligned}
\mathcal{V} \lesssim\ & \left(\mathcal{B}^{\frac{1}{q}}(p,\varphi) + \mathcal{C}^{\frac{1}{q}}(p,\varphi)\right)^{\frac{pq}{pq-1}} (\mathcal{B}(q,\varphi) + \mathcal{C}(q,\varphi))^{\frac{pq}{pq-1}} \\
& + \mathcal{A}^{\frac{p}{pq-2l}}(p,l,\varphi)(\mathcal{B}(q,\varphi) + \mathcal{C}(q,\varphi))^{\frac{pq}{pq-2l}} \\
& + \left(\mathcal{B}^{\frac{2m}{q}}(p,\varphi) + \mathcal{C}^{\frac{2m}{q}}(p,\varphi)\right)^{\frac{pq}{pq-2m}} \mathcal{A}^{\frac{pq}{pq-2m}}(q,m,\varphi) \\
& + \mathcal{A}^{\frac{2mp}{pq-4ml}}(p,l,\varphi) \mathcal{A}^{\frac{pq}{pq-4ml}}(q,m,\varphi).
\end{aligned} \tag{2.31}$$

Now, using the test function (2.16) with the choice (2.17) and the scaled variables (2.19), we obtain

$$\mathcal{V} \lesssim \sum_{i=1}^{4} R^{-\beta_i}, \tag{2.32}$$

where

$$\beta_1 = \tfrac{pq}{pq-1}\left(\tfrac{1}{q}(-\alpha + (Q+\alpha)\tfrac{1}{p'}) - \alpha + (Q+\alpha)\tfrac{1}{q'}\right)$$
$$\beta_2 = \tfrac{pq}{pq-2l}\left(\tfrac{1}{q}(-\alpha + (Q+\alpha)\tfrac{p-2l}{l}) - \alpha + (Q+\alpha)\tfrac{1}{q'}\right)$$
$$\beta_3 = \tfrac{pq}{pq-2m}\left(\tfrac{2m}{q}(-\alpha + (Q+\alpha)\tfrac{1}{p'}) - \alpha + (Q+\alpha)\tfrac{q-2m}{q}\right)$$
$$\beta_4 = \tfrac{p}{pq-4ml}\left(2m(-\alpha + (Q+\alpha)\tfrac{p-2l}{l}) - \alpha + (Q+\alpha)\tfrac{q-2m}{m}\right).$$

The choice

$$\beta_i \geq 0,\ \ i = 1, \ldots, 4,$$

which lead to a contradiction as in the previous proofs means

$$Q + \alpha \leq \max\left\{\frac{\alpha p(q+1)}{pq-1}, \frac{\alpha p(q+1)}{pq-2l}, \frac{\alpha p(2m+q)}{pq-4ml}\right\};$$

this is the requirement (1.15).

Acknowledgements

This publication was made possible by NPRP Grant NPRP 6-137-1-026 from the Qatar National Research Fund (a member of Qatar Foundation). The statements made herein are solely the responsibility of the authors.

References

[1] S. Abe, Anomalous diffusion in view of Einstein's 1905 theory of Brownian motion. *Physica A* **356**, No 2-4 (2006), 403–407.

[2] A. Alsaedi, B. Ahmad, M. Kirane, Blowing-up solutions of some nonlinear non-local evolution equations on the Heisenberg group. *Electron. J. Diff. Equ.* **2015** (2015), Paper No 227, 1–10.

[3] H. Bahouri, I. Gallagher, Paraproduit sur le groupe de Heisenber et applications. *Revista Mathematica Iberoamericana* **17** (2001), 69–105.

[4] C.-H. Chang, Der-C.E. Chang, P. Greiner, H.-P. Lee, The positivity of the heat kernel on Heisenberg group. *Anal. Appl.* **11** (2013), # 1350019.

[5] A. Cordoba, D. Cordoba, A maximum principle applied to quasi-geostrophic equations. *Comunn. Math. Phys.* **249** (2004), 511–528.

[6] A. Elhamidi, M. Kirane, Nonexistence results of solutions to systems of semilinear differential inequalities on the Heisenberg group. *Abstract and Applied Analysis* **2004**, No 2 (2004), 155–164.

[7] H. Fujita, On the blowing-up of solutions of the Cauchy problem for $u_t = \Delta u + u^{p+1}$. *J. Fac. Sci. Univ. Tokyo Sec. I* **13** (1966), 109–124.

[8] J.A. Goldstein, I. Kombe, Nonlinear degenerate parabolic equations on the Heisenberg group. *Int. J. Evol. Equ.* **1**, No 1 (2005), 1-22.

[9] F. John, Blow-up of solutions of nonlinear wave equations in three space dimensions. *Manuscripta Math.* **28** (1979), 235–268.

[10] A. Iomin, Fractional-time quantum dynamics. *Phys. Rev. E* **80** (2009), # 022103.

[11] N. Ju, The maximum principle and the global attractor for the dissipative 2-D quasi-geostrophic equations. *Comm. Pure Appl. Ana.* (2005), 161–181.

[12] T. Kato, Blow-up of solutions of some nonlinear hyperbolic equations in three space dimensions. *Comm. Pure Appl. Math.* **37** (1984), 443–455.

[13] N. Laskin, Fractional Schrödinger equation. *Physical Review E* **66** (2002), # 056108, 7 pages; available online: http://arxiv.org/abs/quant-ph/0206098.

[14] N. Laskin, Fractional quantum mechanics. *Physical Review E* **62** (2000), 3135–3145; available online: http://arxiv.org/abs/0811.1769.

[15] R. Klages, G. Radons, I.M. Sokolov (Eds.), *Anomalous Transport: Foundations and Applications*. Wiley-VCH, Weinheim (2008).

[16] M. Kirane, M. Qafsaoui, Fujita's exponent for a semilinear wave equation with linear damping. *Advanced Nonlinear Studies* **2**, No 1 (2002), 41–49.

[17] F. Mainardi, Fractional relaxation-oscillation and fractional diffusion-wave phenomena. *Chaos, Solitons, Fractals* **7** (1986), 1461–1477.

[18] R. Metzler, J. Klafter, The random walk's guide to anomalous diffusion: a fractional dynamics approach. *Physics Reports* **339** (2000), 1–77.

[19] E. Mitidieri, S.I. Pohozaev, A priori estimates and blow-up of solutions to nonlinear partial differential equations and inequalities. *Proc. of the Steklov Institute of Mathematics* **234** (2001), 1–383.

[20] D. Müller, E. M. Stein, L^p-estimates for the wave equation on the Heisenberg group. *Rev. Mat. Iberoamericana* **15**, No 2 (1999), 297-334.

[21] A.I. Nachman, The wave equation on the Heisenberg group. *Comm. Partial Differential Equations* **7**, No 6 (1982), 675-714.

[22] A. Pascucci, Semilinear equations on nilpotent Lie groups: global existence and blow-up of solutions. *Matematiche* **53** (1998), 345–357.

[23] M. Rammaha, Nonlinear wave equation in high dimensions. In: *Differential Equations and Applications*, Ohio Univ. Press (1989), 322–326.

[24] T. Sideris, Nonexitence of global solution to semilinear wave equatins in high dimensions. *J. Differential Equations* **52** (1984), 378–406.

[25] W.A. Strauss, *Nonlinear Wave Equations.* C.B.M.S Lecture Notes **73**, Amer. Math. Soc., Providence, RI (1989).
[26] V.E. Tarasov, *Fractional Dynamics.* Ser. Nonlinear Physical Science, Springer (2010).
[27] V.E. Tarasov, Fractional Heisenberg equation. *Phys. Lett. A* **372** (2008), 2984–2988.
[28] G. Todorova, B. Yordanov, Critical exponent for a nonlinear wave equation with damping. *J. Differential Equations* **174** (2001), 464–489.
[29] L. Véron, S.I. Pohozaev, Nonexistence results of solutions of semilinear differential inequalities on the Heisenberg group. *Manuscripta Math.* **102** (2000), 85–99.
[30] Q.S. Zhang, The critical exponent of a reaction diffusion equation on some Lie groups. *Math. Z.* **228** (1998), 51–72.
[31] Q.S. Zhang, Blow-up results for nonlinear parabolic equations on manifolds. *Duke Math. J.* **97** (1999), 515–539.
[32] Q.S. Zhang, A blow-up result for a nonlinear wave equation with damping: the critical case. *C. R. Acad. Sci. Paris Série I Math.* **333** (2001), 109–114.
[33] C. Zuily, Existence globale de solutions régulières pour l'équation des ondes non linéaires amortie sur le groupe de Heisenberg (In French) [Global existence of regular solutions for the damped nonlinear wave equation on the Heisenberg group]. *Indiana Uni. Math. J.* **62**, No 2 (1993), 323–360.

College of Arts and Sciences, Dept. of Mathematics, Statistics and Physics
University of Qatar
P.O. Box 2713, Doha, QATAR
e-mail: boutef@qu.edu.qa

Université de La Rochelle
Laboratoire de Mathématiques MIA, Pôle Sciences et Technologies
Avenue Michel Crépeau
17042 La Rochelle Cedex 1, FRANCE
e-mail: mkirane@univ-lr.fr *Received: April 1, 2015*

Please cite to this paper as published in:

Fract. Calc. Appl. Anal., Vol. 18, No 6 (2015), pp. 1336–1349,
DOI: 10.1515/fca-2015-0077