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RESEARCH PAPER

A NUMERICAL APPROACH FOR FRACTIONAL ORDER RICCATI DIFFERENTIAL EQUATION USING B-SPLINE OPERATIONAL MATRIX

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Abstract

In this article, we develop an effective numerical method to achieve the numerical solutions of nonlinear fractional Riccati differential equations. We found the operational matrix within the linear B-spline functions. By this technique, the given problem converts to a system of algebraic equations. This technique is used to solve fractional Riccati differential equation. The obtained results are illustrated both applicability and validity of the suggested approach.

MSC 2010: Primary 34A08; Secondary 41A10, 41A35

Key Words and Phrases: fractional order Riccati equations, linear B-spline function, operational matrix

1. Introduction

The fractional calculus is an extension of the classical calculus and has a long history [20, 10, 1, 5, 11, 17]. Fractional calculus is hot topics which rapid development and implementations in various fields of engineering and science [20, 10, 1, 5, 11, 17]. As a result the fractional differential equations (FDEs) started to be used in describing of real world phenomena [1, 5]. For most of the FDEs, obtaining the analytical solutions are not easy, so today, it is natural that many authors have tried to solve fractional differential

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equations using different approximation and numerical techniques [8, 7, 9, 15, 19, 21].

We recall that in recent years, several methods based on orthogonal functions and wavelets have been used for solving FDE. Jafari et al. [9] solved fractional differential equations using Legendre wavelet.

Another alternative method is to divide the interval into a collection of subintervals and construct a (generally) different approximating polynomial on each subinterval. This is called B-spline function. We recall that "B" in B-spline stands for basis. Spline functions are instances of a piecewise polynomial function associated with a partition of an arbitrary interval. Approximation by functions of this type is called piecewise-polynomial approximation. principal applications of B-splines arise in computer-aided design, computer graphics, geometric modeling and many other different subjects [3]. We recall that several methods based on the orthogonal functions and the operational matrix of fractional derivatives (OMFD) were utilized for solving FDE [4, 15, 21].

As it known Lakestani et al. [12] suggested the OMFD and solved FDEs with the help of the powerful technique of B-splines collocation technique (BSCT). Also Li [14] solved FDE by using the BSCT. With the help of the B-spline functions, below we will generalize the operational matrix for fractional integration and multiplication. The core of this approach is to convert the linear FDEs into a system of algebraic equations. This transformation is possible by expanding the unspecified function within the linear B-spline functions. The speed of the computation increases. We utilize the operational matrix of integral to obtain the unknown coefficients which appear in this approach.

We focus on obtaining the numerical solution of Riccati equation with fractional order. One of most popular differential equation that was considered mostly in the literature, is Riccati's equation. There are several applications of this equation in algebraic geometry, theory of conformal mapping, physics and applied problems (see for example Refs. [13, 18] and the references therein). The fractional Riccati differential equation is the following:

$$D^{\alpha} f(x) = q_0(x) + q_1(x)f(x) + q_2(x)f^2(x), \qquad n-1 < \alpha \leq n, \tag{1.1}$$

and it is equipped with initial conditions

$$f^{(i)}(0) = d_i \qquad i = 0, 1, \ldots, n-1, \tag{1.2}$$

where $q_0(x), q_1(x)$ and $q_2(x)$ are given functions and $q_0(x), q_2(x) \neq 0$. If $q_0(x) = 0$, we have a fractional Bernoulli type equation; while if $q_2(x) = 0$, Riccati equation - a first order linear ordinary differential equation. Moreover, $D^{\alpha} f(x)$ denotes the Caputo fractional derivative namely [1, 17]:

$$D_x^{\alpha} f(x) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^x \dfrac{f(t)}{(x-t)^{1+\alpha-n}} dt, & n-1 < \alpha < n, \quad n \in I\!N, \\ \\ \dfrac{d^n}{dt^n} f(x), & \alpha = n. \end{cases} \tag{1.3}$$

Below the reader can find some basic properties of the Caputo derivative used in this manuscript, namely [1, 10, 17]

$$(i) \qquad D_x^{\alpha} C = 0, \qquad \text{(C is a constant)},$$

$$(ii) \qquad D_x^{\alpha} x^{\beta} = \begin{cases} 0 & \beta \in I\!N_0, \ \beta < \lceil \alpha \rceil, \\ \\ \dfrac{\Gamma(\beta+1)}{\Gamma(1+\beta-\alpha)} x^{\beta-\alpha}, & \\ & \beta \in I\!N_0, \beta \geq \lceil \alpha \rceil \text{ or } \beta \notin I\!N_0, \beta > \lfloor \alpha \rfloor, \end{cases}$$

$$(iii) \qquad I_x^{\alpha} D_x^{\alpha} f(x) = f(x) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{x^k}{k!}, \quad n-1 < \alpha \leq n. \tag{1.4}$$

The reader can find detailed explanations about the properties of the fractional operators in [1, 17]. Also I_x^{α} the fractional Riemann-Liouville integral, namely [17, 1]:

$$I_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad \alpha > 0. \tag{1.5}$$

Now we are ready to present the organization of our wok: In Section 2, the linear B-spline scaling functions is presented over $[0, 1]$. Also the operational matrix is computed for fractional integration and multiplication. The suggested approach is used to approximate the fractional Riccati differential equation in the next section. After that we applied the proposed technique to solve fractional Riccati differential equation in Section **??**. A conclusion part in Section 5 closed the manuscript.

2. **The Linear B-spline function**

For a given scaling function in $L^2(R)$ the scaling function can be utilized to expand it. We know that the scaling functions are defined R, therefore they can not be inside of the domain of the investigated issue. To bypass this aspect, in our manuscript we considered the B-spline scaling functions on $[0, 1]$.

We recall that [6]

$$N_m(x) = \frac{1}{(m-1)!} \sum_{k=0}^{m} \binom{m}{k} (-1)^k (x-k)_+^{m-1} \tag{2.1}$$

denotes the cardinal B-spline function of order $m \geq 2$ (degree m-1) for the knot sequence $\{\dots, -1, 0, 1, \dots\}$ and $supp[N_m(x)] = [0, m]$. Also let $N_1(x) = \chi_{[0,1]}(x)$.

Boor et al. [2, 6] define the explicit expression of $N_2(x)$ (the linear B-spline function) in the following form:

$$N_2(x) = \sum_{k=0}^{2} \binom{2}{k} (-1)^k (x-k)_+ = \begin{cases} x & x \in [0,1), \\ 2-x & x \in [1,2), \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

Suppose $N_{j,k}(x) = N_2(2^j x - k),\ j,k \in Z$ and

$$B_{j,k} = \text{supp}[N_{j,k}] = \text{close}\left\{x \ : \ N_{j,k} \neq 0\right\}.$$

By inspection we have that

$$B_{j,k} = [2^{-j}k, 2^{-j}(2+k)], \quad j,k \in Z. \tag{2.3}$$

The support of $N_{j,k}(x)$ can be outside of $[0,1]$, therefore we have to define $N_{j,k}(x)$ on $[0,1]$. Thus, we conclude that

$$\phi_{j,k} = N_{j,k}(x)\chi_{[0,1]}(x),\ j \in Z. \tag{2.4}$$

As a result we have

$$\begin{aligned} \phi_{j,k} &= \sum_{i=0}^{2} \binom{2}{i} (-1)^i (2^j x - (k+i))_+ \\ &= \begin{cases} 2^j x - k, & \frac{k}{2^j} \leq x < \frac{k+1}{2^j}, \\ 2 - (2^j x - k), & \frac{k+1}{2^j} \leq x < \frac{k+2}{2^j}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{2.5}$$

2.1. **Approximation of functions.** Let $f(x)$ is a defined function on $[0,1]$. It can be expanded by B-spline scaling functions for a fixed $j = J$, as

$$f(x) \approx \sum_{k=-1}^{2^j-1} c_k \phi_{j,k}(x) = C^T \Phi_J(x), \tag{2.6}$$

where

$$C = [c_{-1}, c_0, \dots, c_{2^j-1}]^T \tag{2.7}$$

$$\Phi_J(x) = [\phi_{j,-1}(x), \phi_{j,0}(x), \dots, \phi_{j,2^j-1}(x)]^T. \tag{2.8}$$

Then, c_k can be obtained by

$$c_k^T = \int_0^1 f(x)\widetilde{\phi}_{j,k}^T(x)dx, \quad k = -1, \dots, 2^j - 1, \tag{2.9}$$

where $\widetilde{\phi}_{j,k}$ is the vector of dual basis of Φ_J. By using the linear combinations of $\phi_{j,k}$, the $\widetilde{\phi}_{j,k}$ are obtained as:

$$\widetilde{\phi}_{j,k} = P^{-1}\Phi_J, \tag{2.10}$$

where

$$P = \int_0^1 \Phi_J(x)\Phi_J^T(x)dt = \frac{1}{2^{J-2}} \begin{bmatrix} \frac{1}{12} & \frac{1}{24} & & & \\ \frac{1}{24} & \frac{1}{6} & \frac{1}{24} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{24} & \frac{1}{6} & \frac{1}{24} \\ & & & \frac{1}{24} & \frac{1}{12} \end{bmatrix}, \tag{2.11}$$

where P is a symmetric $(2^J + 1) \times (2^J + 1)$ matrix. Replacing (2.10) in (2.9) leads

$$c_k^T = \left(\int_0^1 f(x)\phi_J^T(x)dx \right) P^{-1}. \tag{2.12}$$

2.2. **The operational matrix of fractional order integration.** The operational matrices of fractional order integration of the vector ϕ_J is approximated as:

$${}_0I_x^\alpha \Phi_J(x) \simeq I^\alpha \Phi_J(x), \tag{2.13}$$

where I^α is the $(2^J+1)\times(2^J+1)$ fractional operational matrix of integration for B-spline function. We obtain the matrix I^α as follows:

$$I^\alpha = \int_0^1 ({}_0I_x^\alpha \Phi_J(x))\widetilde{\Phi}_J^T(x)dx = \left(\int_0^1 ({}_0I_x^\alpha \Phi_J(x))\Phi_J^T(x)dx \right) P^{-1}, \tag{2.14}$$

where

$$E = \int_0^1 ({}_0I_x^\alpha \Phi_J(x))\Phi_J^T(x)dx. \tag{2.15}$$

In (2.15), E is a $(2^J + 1) \times (2^J + 1)$ matrix given by

$$E = \begin{bmatrix} \int_0^1 ({}_0I_x^\alpha \phi_{j,-1}(x))\phi_{j,-1}^T(x)dx & \dots & \int_0^1 ({}_0I_x^\alpha \phi_{j,-1}(x))\phi_{j,2^j-1}^T(x)dx \\ \vdots & \ddots & \vdots \\ \int_0^1 ({}_0I_x^\alpha \phi_{j,2^j-1}(x))\phi_{j,-1}^T(x)dx & \dots & \int_0^1 ({}_0I_x^\alpha \phi_{j,2^j-1}(x))\phi_{j,2^j-1}^T(x)dx \end{bmatrix}. \tag{2.16}$$

And we have ${}_0I_x^\alpha \Phi_J(x)$ as follows:

$$
\begin{aligned}
{}_0I_x^{\alpha}\phi_J(x) &= {}_0I_x^{\alpha}\left(\sum_{i=0}^{2}\binom{2}{i}(-1)^i(2^Jx-(k+i))_+\right)\\
&= \frac{2^{J\alpha}}{\Gamma(2+\alpha)}\sum_{i=0}^{2}\binom{2}{i}(-1)^i(2^Jx-(k+i))_+^{2+\alpha} \qquad (2.17)\\
&= \begin{cases} 0, & x \le \dfrac{k}{2^J},\\[2mm] \dfrac{2^{J\alpha}(2^Jx-k)^{\alpha+1}}{\Gamma(\alpha+2)}, & \dfrac{k}{2^J} \le x < \dfrac{k+1}{2^J},\\[2mm] \dfrac{2^{J\alpha}((2^Jx-k)^{\alpha+1}-2(2^Jx-(k+1))^{\alpha+1})}{\Gamma(\alpha+2)}, & \dfrac{k+1}{2^J} \le x < \dfrac{k+2}{2^J},\\[2mm] \dfrac{2^{J\alpha}((2^Jx-k)^{\alpha+1}-2(2^Jx-(k+1))^{\alpha+1}+(2^Jx-(k+2))^{\alpha+1})}{\Gamma(\alpha+2)}, & x \ge \dfrac{k+2}{2^J}. \end{cases}
\end{aligned}
$$

2.3. **The operational matrix of product.** For the product $\hat{C}$, the operational matrices by using linear B-spline function is given by

$$C^T\Phi_J(x)\Phi_J(x)^T \simeq \Phi_J(x)^T\hat{C}, \qquad (2.18)$$

where $\hat{C}$ is an $(2^J+1)\times(2^J+1)$ matrix. Since $c^T\Phi_J(x) = \sum_{i=-1}^{2^j-1} c_i\phi_{j,i}(x)$, we have

$$
\begin{aligned}
C^T\Phi_J(x)\Phi_J(x)^T &= c^T\Phi_J(x)[\phi_{j,-1}(x),\phi_{j,0}(x),\ldots,\phi_{j,2^j-1}(x)] \qquad (2.19)\\
&= \left[\sum_{i=-1}^{2^j-1} c_i\phi_{j,i}(x)\phi_{j,-1}(x), \sum_{i=-1}^{2^j-1} c_i\phi_{j,i}\phi_{j,0}(x),\ldots,\sum_{i=-1}^{2^j-1} c_i\phi_{j,i}\phi_{j,2^j-1}(x)\right].
\end{aligned}
$$

Now, we approximate the product of $\phi_{j,i}(x)\phi_{j,k}(x)$ in terms of $\phi_{j,i}$ for $i,k = -1,\ldots,2^j-1$ as

$$\phi_{j,i}(x)\phi_{j,k}(x) \approx e_{k,i}^T\Phi_J,\ \ i,k = -1,0,\ldots,m. \qquad (2.20)$$

that

$$e_{k,i} = [e_{-1}^{k,i}, e_0^{k,i},\ldots,e_{2^j-1}^{k,i}]^T \qquad (2.21)$$

by (2.6) we have

$$
\begin{aligned}
e_{k,i} &= P^{-1}\int_0^1 \phi_{j,i}(x)\phi_{j,k}(x)\Phi_J(x)dx \\
&= P^{-1}\begin{bmatrix} \int_0^1 \phi_{j,i}(x)\phi_{j,k}(x)\phi_{j,-1}(x)dx \\ \int_0^1 \phi_{j,i}(x)\phi_{j,k}(x)\phi_{j,0}(x)dx \\ \vdots \\ \int_0^1 \phi_{j,i}(x)\phi_{j,k}(x)\phi_{j,2^j-1}(x)dx \end{bmatrix}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\sum_{i=-1}^{2^j-1} c_i\phi_{j,i}(x)\phi_{j,k}(x) &\approx \sum_{i=-1}^{2^j-1} c_i \sum_{n=-1}^{2^j-1} e_n^{k,i}\phi_{j,n} \\
&= \sum_{n=-1}^{2^j-1} \phi_{j,n} \sum_{i=-1}^{2^j-1} c_i e_n^{k,i} \\
&= \Phi_J^T \begin{bmatrix} \sum_{i=-1}^{2^j-1} c_i e_{-1}^{k,i} \\ \sum_{i=-1}^{2^j-1} c_i e_0^{k,i} \\ \vdots \\ \sum_{i=-1}^{2^j-1} c_i e_{2^j-1}^{k,i} \end{bmatrix} \\
&= \Phi_J^T[e_{k,-1}, e_{k,0}, \ldots, e_{k,2^j-1}]c \\
&= \Phi_J^T \tilde{C}_{k+2}c, \qquad (2.22)
\end{aligned}
$$

where $\tilde{C}_{k+2}(k = -1, 0, \ldots, 2^j - 1)$ is a $(2^J + 1) \times (2^J + 1)$ matrix which has vectors $e_{k,i}(i = -1, 0, \ldots, 2^j-1)$ for each column. Therefore the operational matrix of product $\hat{C} = \tilde{C}_{k+2}c$ is obtained.

3. The operational matrix form of the fractional order Riccati equation

Below, the fractional Riccati differential equation (1.1) with initial conditions (1.2) is considered.

We expand the fractional derivative in Eq.(1.1) by linear B-spline function Φ_J as follows:

$$D^\alpha f(x) \approx C^T\Phi_J(x). \qquad (3.1)$$

Using (1.4) and (2.14), $f(x)$ can be expanded as:

$$f(x) \approx C^T I^\alpha \Phi_J(x). \qquad (3.2)$$

Also using (2.6) we approximate functions $q_0(x), q_1(x), q_2(x)$ using the linear B-spline basis as:

$$q_0(x) \approx Q_0^T \Phi_J(x), \quad q_1(x) \approx Q_1^T \Phi_J(x), \quad q_2(x) \approx Q_2^T \Phi_J(x). \tag{3.3}$$

Now, by substituting (3.1)-(3.3) into (1.1), leads

$$\begin{aligned} C^T \Phi_J(x) &= Q_0^T \Phi_J(x) + Q_1^T \Phi_J(x)(\Phi_J(x)^T (I^\alpha)^T C) \\ &+ Q_2^T \Phi_J(x)(C^T I^\alpha \Phi_J(x) \Phi_J(x)^T (I^\alpha)^T C). \end{aligned} \tag{3.4}$$

Then, from (2.18) we have

$$Q_1^T \Phi_J(x) \Phi_J(x)^T \approx \Phi_J(x)^T \hat{Q}_1, \tag{3.5}$$

$$C^T I^\alpha \Phi_J(x) \Phi_J(x)^T \approx \Phi_J(x)^T \hat{C}_\alpha, \tag{3.6}$$

where $C_\alpha = C^T I^\alpha$, Now substituting Eqs. (3.5) and (3.6) in Eq. (3.4) we obtain:

$$C^T \Phi_J(x) = Q_0^T \Phi_J(x) + \Phi_J(x)^T \hat{Q}_1 C_\alpha^T + (\Phi_J(x)^T \hat{Q}_2 \hat{C}_\alpha C_\alpha^T) \tag{3.7}$$

or

$$(C^T - Q_0^T - C_\alpha \hat{Q}_1{}^T - C_\alpha \hat{C}_\alpha{}^T \hat{Q}_2{}^T) \Phi_J(x) = 0. \tag{3.8}$$

Finally, by using the independent property of B-spline functions, we obtain:

$$(C^T - Q_0^T - C_\alpha \hat{Q}_1{}^T - C_\alpha \hat{C}_\alpha{}^T \hat{Q}_2{}^T) = 0. \tag{3.9}$$

The vector C can be obtained by solving the above system. Consequently the approximate value of $f(x)$ can be determined by substituting C in (3.2).

4. **Illustrative examples**

Below we use the presented approach in order to solve several FDEs.

Example 4.1. We recall the equation discussed in [16, 9], namely

$$D^\alpha f(x) = -f^2(x) + 1, \qquad 0 < \alpha \leq 1, \tag{4.1}$$

equipped with initial conditions

$$f(0) = 0. \tag{4.2}$$

The accuracy solution for $\alpha = 1$, is

$$f(x) = \frac{e^{2x} - 1}{e^{2x} + 1}. \tag{4.3}$$

Eq. (4.1) is studied by Jafari et al. [9] by using the Legendre wavelets. Odibat and Momani solved it using the decomposition method, [16]. Here we apply the linear B-spline function to solve it. Fig. 1 shows exact solution and the approximation solutions of $f(x)$ for $J = 6$ and different values of $\alpha = 1, 0.95, 0.9, 0.85$. Definitely, as α approaches 1, the approximate values of $f(x)$ will converge to the exact solutions. Both the numerical results $f(x)$ and the exact solution for $\alpha = 1$ and different values of J can be seen in Fig. 2. By the comparison Fig. 2 (using B-spline method) and Fig. 3 (Legendre wavelet method [9]), by inspection we understand that the present approximations are more efficient.

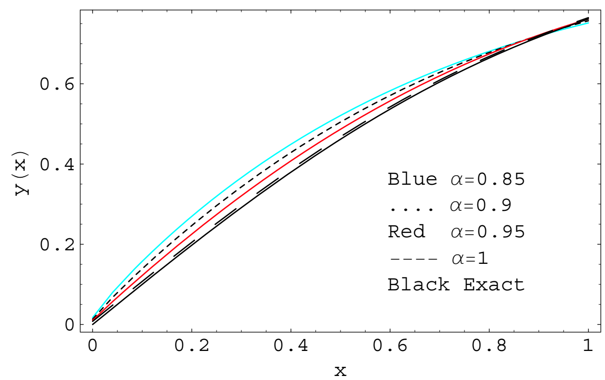

FIGURE 1. Exact solution and numerical results of Eq. (4.1) using linear B-spline function when $\alpha = 1, 0.95, 0.9, 0.85$.

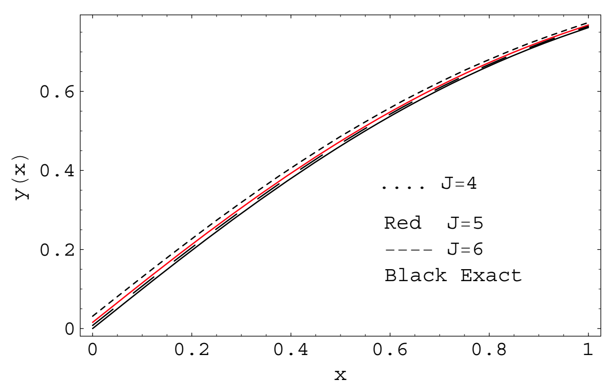

FIGURE 2. Exact solution and numerical results of Eq. (4.1) using linear B-spline function for $J = 4, 5, 6$ when $\alpha = 1$.

EXAMPLE 4.2. We demonstrate the accuracy of the presented numerical scheme by considering the fractional Riccati differential equation

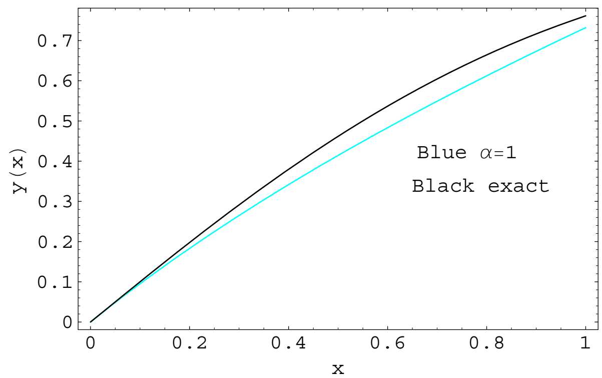

FIGURE 3. Exact solution and numerical results of Eq. (4.1) using Legendre wavelet solution when $\alpha = 1$.

possessing the initial value given in [16, 9] by

$$D^{\alpha} f(x) = 2f(x) - f^2(x) + 1, \qquad 0 < \alpha \leq 1, \tag{4.4}$$

$$f(0) = 0, \tag{4.5}$$

and admitting exact solution when $\alpha = 1$

$$f(x) = 1 + \sqrt{2}\tanh(\sqrt{2}x + \frac{1}{2}\log(\frac{\sqrt{2}-1}{\sqrt{2}+1})). \tag{4.6}$$

Fig. 4 shows the exact solution and the approximate solution of Eq. (4.4) for different values of α when $J = 6$ using the linear B-spline function. We see that as α approaches 1, the approximate solutions will converge to the exact solution. The numerical results $f(x)$ together with the exact solution for $J = 4, 5, 6$ and $\alpha = 1$ are plotted in Fig. 5. By comparing the Fig. 5 (using B-spline method) and the Fig. 6 (Legendre wavelet method [9]), we can see that the presented numerical scheme is more efficient.

5. **Conclusion**

In this work we proposed an accurate and efficient approach based on the linear B-spline function for solving the fractional type Riccati differential equation. B-spline operational matrices of fractional integration and multiplication were calculated. We provided the general procedure of forming this matrix. Specific applications were presented to show the applicability and validity of the approach. *Mathematica* was used for computations.

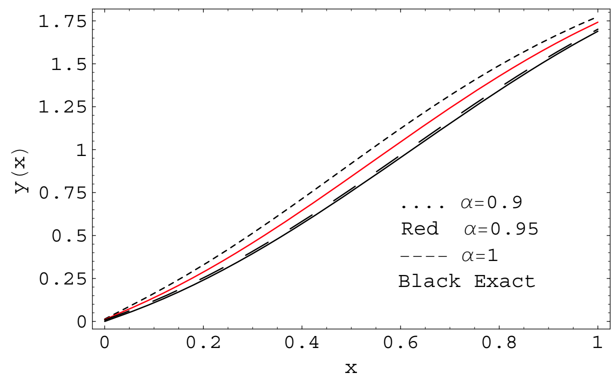

FIGURE 4. Solutions of Eq. (4.4) using linear B-spline function when $\alpha = 1, 0.95, 0.9$.

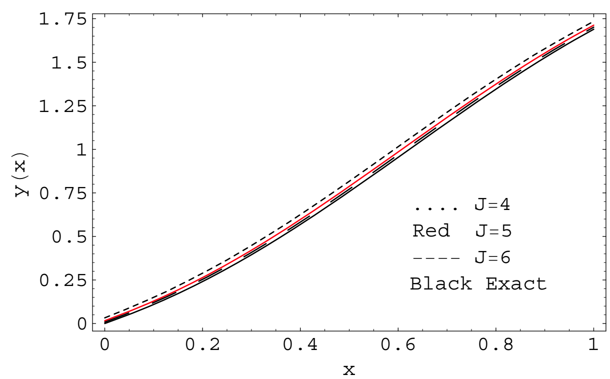

FIGURE 5. Solutions of Eq. (4.4) using linear B-spline function for $J = 4, 5, 6$ when $\alpha = 1$.

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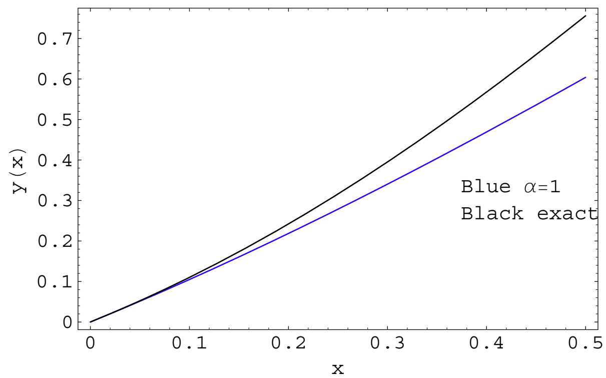

FIGURE 6. Exact solution and numerical results of Eq. (4.4) using Legendre wavelet solution when $\alpha = 1$.

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