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RESEARCH PAPER

NEW RESULTS FROM OLD INVESTIGATION: A NOTE ON FRACTIONAL m-DIMENSIONAL DIFFERENTIAL OPERATORS. THE FRACTIONAL LAPLACIAN

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Abstract

The non local fractional Laplacian plays a relevant role when modeling the dynamics of many processes through complex media. From 1933 to 1949, within the framework of potential theory, the Hungarian mathematician Marcel Riesz discovered the well known Riesz potential operators, a generalization of the Riemann–Liouville fractional integral to dimension higher than one. The scope of this note is to highlight that in the above mentioned works, Riesz also gave the necessary tools to introduce several new definitions of the generalized coupled fractional Laplacian which can be applied to much wider domains of functions than those given in the literature, which are based in both the theory of fractional power of operators or in certain hyper-singular integrals. Moreover, we will introduce the corresponding fractional hyperbolic differential operator also called fractional Lorentzian Laplacian.

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Key Words and Phrases: n-dimensional fractional operators, fractional Laplacian, fractional Lorentzian Laplacian, Riesz potencial operators, fractional spatial derivatives, Riemann–Liouville operators

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1. Introduction

As it is well known the fractional models play an important role in many applied areas of sciences and engendering to describe the dynamics of processes through strongly complex media with power law non-locality. We consider that a medium is strongly complex if the conventional models do not work enough well to shape the dynamics of classical process (for example, a diffusion process, a advection-diffusion process or a wave process, etc.) on it. Fractals media are included in such kind of complex media. Nowadays, the applied research groups are very interesting to find a suitable definition of fractional Laplacian for isotropic and anisotropic media. Some papers have been written on m-dimensional spatial fractional coupled operators and the so called generalized fractional vector calculus (see, for example, [18], [21], [12], [22], [14], [4], [23], [24], [7], and [27]) with the objective to be used from both theoretical and applied point of view (see, for example, [19], [15], [9], [16], [20], [29], [11], [10], [1], [25], [6], [26], [3], [5], [2], [28], and [29]). This important issue is far to be enough well solved.

The definitions given in the literature, which are essentially based in the following techniques:

- The theory of fractional power of operators. What mean work on a very strong domain of functions.
- By the ambiguous following property:

  $$\mathcal{F}((-\Delta)^{\frac{\alpha}{2}} f) = |w|^{\alpha} \mathcal{F}(f)(w)$$

  where $\mathcal{F}$ denote the classical Fourier transform. What mean a ambiguous definition, because it is obvious that many explicit operators verify the property this property.
- By certain hyper-singular integral. What mean work on a very strong domain of functions.

Then, we can get as a first conclusion that such definitions are ambiguous or too restrictive; moreover in many cases they can not used in real models. Therefore, to extended the known explicit definitions of such operators so they work on a wider set of functions will help to can review the known fractional model which use these kind of operators and also they will produce novel and more accuracy numerical tools to can estimate the solutions of such models.

In this note we try to highlight some important aspects, not so much well known, related to such operator, which can be found in the Riesz paper on the potential theory, titled "L'intégrale de Riemann–Liouville et le problème de Cauchy", published in 1949 [18], where he introduced and studied two of his famous m-dimensional Riesz's potential integrals. Those operators have had during a large time a really very important influence in the potential theory, specially early the numerical methods could be used in the applied sciences. The first remarkable fact that one can get from the title of this paper is that Riesz got his inspiration to introduce such m-dimensional fractional operators from the Riemann–Liouville fractional one-dimensional integrals.

Finally, we will show that the m-dimensional integral operator given by Riesz, their properties and the well known techniques used to give the wider definition for many of the known one-dimensional fractional differential operators will permit us to get a more suitable explicit definitions of the differential fractional m-dimensional coupled Laplacian. Moreover, it will be introduced the corresponding fractional hyperbolic differential operator also called fractional Lorentzian Laplacian.

In Section **2** we will explain the technique to give the wider definition of fractional derivative operators for the case one-dimensional. The new definitions of the fractional coupled fractional Laplacian are given in Section **3**, and the last Section **4** is dedicated to introduction and justification of the corresponding definitions of the Lorentizian Laplacian.

2. Some one-dimensional fractional differential operators

First of all, we must remark that is well known that there exist many one-dimensional fractional integral and derivative operators. For instance, the called Riemann–Liouville, Caputo, Grünwald-Letnikov, Hadamard, Marchaud, Erdélyi-Kober, Riesz-Feller, etc. (see, for example, [8], [13], [21], [17], [19], [16], and [27]). In the most of these cases the fractional integral operators are defined for a enough wide set of functions and any real (or complex) order $n-1<\alpha<=n$ and also they verify three basic properties. Then, it is defined the corresponding left inverse of such operators, or its fractional derivative of order α, using such properties, because it is well known that the direct way replacing α in the integral definition by $-\alpha$ is not the more suitable technique to get the more general definition. As an example, we will illustrate bellow the mentioned technique with the left-sided fractional operator of order α:

Let f a suitable good function (for instance, $f \in L_1(a,b)$ a measurable Lebesgue), $0<\alpha<1$ and $[a,b] \subset \mathbb{R}$. Then the Riemann–Liouville left-side operator of order α is defined by

$$I_{a+}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad (x>a) \tag{2.1}$$

which verifies among others, the following basic properties:

1. $I^0_{a+}f(x) = \lim\limits_{\alpha\to 0} I^\alpha_{a+}f(x) = f(x)$ (2.2)

2. $I^\alpha_{a+}I^\beta_{a+}f(x) = I^{\alpha+\beta}_{a+}f(x)$ (2.3)

3. $DI^{\alpha+1}_{a+}f(x) = I^\alpha_{a+}f(x), \quad (0 < \alpha \leq 1).$ (2.4)

We remark here that (2.3) is the semigroup property for the studied operator and (2.4) is, in some sense, an application of the fundamental theorem of the classical integral calculus.

Now we could define the left inverse of the operator (2.1) as follow:

$$I^{-\alpha}_{a+}f(x) := \frac{1}{\Gamma(-\alpha)}\int_a^x (x-t)^{-\alpha-1}f(t)dt, \quad (x > a) \tag{2.5}$$

which obviously is a hyper-singular operator and then it is convergence only for functions f with enough strong restriction. Then, using the above properties (2.2)-(2.4) we could write the above definition as follow:

$$I^{-\alpha}_{a+}f(x) := DII^{-\alpha}_{a+}f(x) = DI^{1-\alpha}_{a+}f(x). \tag{2.6}$$

Therefore, taking in account (2.6) we can improve the definition (2.5) as follow:

$$D^\alpha_{a+}f(x) := DI^{1-\alpha}_{a+}f(x). \tag{2.7}$$

The above explanation justifies why this is the usual definition one find in the literature as the left-side fractional Riemann–Liouville derivative of order α.

The above technique can be extended easily to the more general case $n-1 < \alpha \leq n$. See, for example, left and right Riemann–Liouville and Liouville fractional integrals, which are given as follows:

$$I^\alpha_{a+}f(x) := \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \quad (x > a) \tag{2.8}$$

$$I^\alpha_{b-}f(x) := \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, \quad (x < b) \tag{2.9}$$

$$I^\alpha_{+}f(x) := \frac{1}{\Gamma(\alpha)}\int_{-\infty}^x (x-t)^{\alpha-1}f(t)dt, \quad (x \in \mathbb{R}) \tag{2.10}$$

$$I^\alpha_{-}f(x) := \frac{1}{\Gamma(\alpha)}\int_x^\infty (t-x)^{\alpha-1}f(t)dt, \quad (x \in \mathbb{R}) \tag{2.11}$$

and their respective fractional derivatives, given by:

$$D^{\alpha}_{a+}f)(x) := D^n(I^{n-\alpha}_{a+}f)(x) \quad (x > a) \tag{2.12}$$

$$(D^{\alpha}_{b-}f)(x) := (-D)^n(I^{n-\alpha}_{b-}f)(x) \quad (x < b) \tag{2.13}$$

$$(D^{\alpha}_{+}f)(x) := D^n(I^{n-\alpha}_{+}f)(x) \quad (x \in \mathbb{R}) \tag{2.14}$$

$$(D^{\alpha}_{-}f)(x) := (-D)^n(I^{n-\alpha}_{-}f)(x) \quad (x \in \mathbb{R}) \tag{2.15}$$

Bellow it can be found a piece of the paper by Riesz [18] where it is possible to check that he knew perfectly the Riemann–Liouville integral operators and the three properties above mentioned:

CHAPITRE I.

Introduction.(1)

C'est la notion de la *partie finie* de certaines intégrales due à M. Hadamard qui forme le point de départ des recherches dont j'aurai l'honneur de vous entretenir dans cette conférence. On connaît les applications brillantes à la théorie des équations aux dérivées partielles que M. Hadamard a données de la notion qu'il a créée(2). Chez lui il s'agit surtout d'équations à coefficients variables, tandis que la méthode que nous allons développer ne s'applique qu'à des équations à coefficients constants(3). En revanche, elle permet de donner pour ces équations une solution du problème de Cauchy qui est la même pour les dimensions impaires et paires.

Considérons d'abord l'intégrale de Riemann-Liouville

$$I^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x f(t)(x-t)^{\alpha-1}\,dt.\,(4)$$

Elle converge pour $\alpha > 0$ et satisfait aux relations fondamentales

$$I^{\alpha}(I^{\beta}) = I^{\alpha+\beta}, \qquad \frac{d}{dx}(I^{\alpha+1}) = I^{\alpha}. \tag{1}$$

Fig. 2.1: Pieces of paper by Riesz (1949): Chap. I and Introduction.

In [18] Riesz introduced two different generalizations of the Riemann–Liouville one-dimensional integral operators to the m-dimensional case, which are well known as Riesz's potential operators. His main objective was to give an explicit expression to the solution of certain elliptic and hyperbolic Cauchy problem type involving the classical Laplacian.

3. Riesz Potentials and coupled fractional m-dimensional Laplacian

In this section we are interesting to show that the basic results included in the second chapter of the mentioned Riesz's paper will permit us to introduce new definitions of m-dimensional fractional Laplacian more suitable that such others introduced in the early literature.

3.1. Elliptical m-dimensional Riesz potential

Here our main interest is the m-dimensional Riesz's potential operator corresponding to the elliptic case, which is given by:

$$ {}^{R}\mathbb{I}^{\alpha} f(x) := \frac{1}{\gamma_m(\alpha)} \int_{\mathbb{R}^m} \frac{f(y)}{|x-y|^{m-\alpha}} dy, \tag{3.1} $$

where $0 < \alpha < m$, f is a suitable function, for example, locally integrable function on $\mathbb{R}^m$ or with decaying sufficiently rapidly at infinity, and with the following normalized constant

$$ \gamma_m(\alpha) = \frac{\pi^{\frac{m}{2}} 2^{\alpha} \Gamma(\frac{\alpha}{2})}{\Gamma(\frac{m-\alpha}{2})}, \tag{3.2} $$

in order to verify the identity ${}^{R}\mathbb{I}^{\alpha}(e^{ix_1}) = e^{ix_1}$ and $x = (x_1, x_2, ..., x_m)$. The operator (3.1) verifies the following three properties, which are similar to such mentioned above for the Riemann–Liouville fractional operator:

1. $${}^{R}\mathbb{I}^{0} f(x) = \lim_{\alpha \to 0} {}^{R}\mathbb{I}^{\alpha} f(x) = f(x), \quad (0 < \alpha), \tag{3.3}$$

2. $${}^{R}\mathbb{I}^{\alpha} \left({}^{R}\mathbb{I}^{\beta} f \right)(x) = {}^{R}\mathbb{I}^{\alpha+\beta} f(x), \quad (0 < \alpha, \beta, \alpha+\beta < m), \tag{3.4}$$

3. $$-\Delta {}^{R}\mathbb{I}^{\alpha+2} f(x) = {}^{R}\mathbb{I}^{\alpha} f(x), \tag{3.5}$$

where Δ is the classical Laplacian, and $0 \leqq \alpha, \alpha+2 \leqq m$.

Also, the following property is well known, which is a particular case of (3.5):

4. $$-\Delta {}^{R}\mathbb{I}^{2} f(x) = f(x). \tag{3.6}$$

This property informs us about that the $-\Delta$ is the left inverse operator of ${}^{R}\mathbb{I}^{2}$ for suitable functions. Then the operator (3.1) is a suitable candidate to give a definition of the couple fractional Laplacian.

Therefore, several authors, as Samko-Kilbas-Marichev (1987; 1993) [21], have realized in an explicit form the corresponding to (3.1) fractional Laplacian given by the hypersingular integral ${}^{R}\mathbb{I}^{-\alpha}$ for $0 < \alpha < m$. The expression obtained by them is the following:

$$((-\Delta)^{\frac{\alpha}{2}} f)(x) = \frac{1}{d_m} \int_{\mathbb{R}^m} \frac{(\Delta_t^l f)(x)}{|t|^{\alpha+m}} dt, \quad (l > \alpha), \tag{3.7}$$

where $(\Delta_t^l f)(x) = (E - \tau_t)^l f(x) = \sum_{k=0}^{l} (-1)^k \begin{pmatrix} l \\ k \end{pmatrix} f(x - kt)$ and

$d_m(l,\alpha) = \frac{2^{-\alpha}\pi^{1+\frac{m}{2}} A_l(\alpha)}{\Gamma(1+\frac{\alpha}{2})\Gamma(\frac{\alpha+m}{2}) sin(\frac{\alpha\pi}{2})}$ with $A_l(\alpha) = \sum_{k=0}^{l} (-1)^{k-1} \begin{pmatrix} l \\ k \end{pmatrix} k^\alpha$.

On the other hand, the following properties are well known:

5. $\mathcal{F}(\Delta f)(w) = |w|^2 \mathcal{F}(f)(w),$ (3.8)

6. $\mathcal{F}({}^{R}\mathbb{I}^\alpha f)(w) = |w|^{-\alpha} \mathcal{F}(f)(w),$ (3.9)

where $\mathcal{F}$ is the classical coupled m-dimensional Fourier transform. However, we can not use (3.7) to give the classical Laplacian or the Riesz's potential (3.1) a biunivocal form as follow:

$$\Delta f(x) = \mathcal{F}^{-1}(|w|^2 \mathcal{F}(f))(x) \tag{3.10}$$

and

$$({}^{R}\mathbb{I}^\alpha f)(x) = \mathcal{F}^{-1}(|w|^{-\alpha} \mathcal{F}(f))(x). \tag{3.11}$$

Therefore, the following definition, used by many authors in the literature to define the inverse of the Riesz's operator (1.17), is ambiguous:

$$(-\Delta)^{\frac{\alpha}{2}} f(x) = {}^{R}\mathbb{I}^{-\alpha} f(x) = \mathcal{F}^{-1}(((|w|^\alpha \mathcal{F}(f))(x). \tag{3.12}$$

We will show in Subsection 3.2 that we can introduce several fractional inverse operator of (3.1) which verify (3.12).

3.2. New definitions of the fractional Riesz Laplacian

Following the Riesz potential (3.1) and the properties (3.3)-(3.5), and the technique used to define the Riemann–Liouville fractional derivative, for suitable functions $f(x)$, ($x \in \mathbb{R}^m$), and $n \in \mathbb{N}$ such that whether $m > 1$ then $n = -[-\alpha]$, but whether $m = 1$ then $n = -[\alpha]$ and $\{\alpha\} = \alpha - [\alpha]$, where $[\bullet]$ is used to note the integer part of the argument.

We can introduce the following explicit definitions of the inverse operators on the left of the Riesz m-dimensional elliptic potential (3.1):

1. $$\left((-\Delta)^{\alpha} f\right)(x) := (-\Delta)^{n}(\mathbb{I}^{2n-2\alpha} f)(x), \tag{3.13}$$
2. $$\left(\Delta^{\alpha} f\right)(x) := -(-\Delta)^{n}(\mathbb{I}^{2n-2\alpha} f)(x), \tag{3.14}$$
3. $$\left({}^{C}\Delta^{\alpha} f\right)(x) := -(-1)^{n}(\mathbb{I}^{2n-2\alpha} \Delta^{n} f)(x). \tag{3.15}$$

Any of these new operators verify the Fourier property requested in (3.12):

$$\mathcal{F}((-\Delta)^{\alpha} f)(w) = \mathcal{F}(-\Delta^{\alpha} f)(k) = \mathcal{F}(-{}^{C}\Delta^{\alpha} f)(k) = |w|^{2\alpha}\mathcal{F}(f)(w). \tag{3.16}$$

We consider that the new definitions given by (3.13)-(3.15):

- Will permit to review the many theoretical results in the framework of the PDE theory presented in the literature by many authors with the objective to proof some of them under weaker hypothesis. See, for example, the above mentioned references by Luis Caffarelli, Juan L. Vázquez, Xavier Cabré or Luis Silvestre, among other many papers on this topic.
- From the point of view of the real applications such definitions could open a new way to arrange many investigations on isotropic spaces stopped from long time ago. And following the same techniques that in the classical case on anisotropic spaces.
- Moreover, such definition will permit to develop more suitable and accuracy numerical estimations to get the solutions of many applied fractional models in n-dimensional spaces involving the fractional Laplacian.

3.3. Some pieces of Chapter II of the paper by Riesz

CHAPITRE II.

Fig. 3.1: Pieces of paper by Riesz (1949): Index Chap. II.

CHAPITRE II.

Intégrale de Riemann-Liouville dans l'espace euclidien. Potentiels.

6. Définition et propriétés de l'intégrale I^α. Remarque. — Les généralisations de l'intégrale de R.-L.[1] dont nous allons nous occuper dans la suite se rapportent à un espace Ω_m (ou plus brièvement Ω) à un nombre quelconque m ($\geqq 1$) de dimensions, pourvu d'une métrique soit *euclidienne*, soit *lorentzienne*. Nous désignons les procédés d'intégration respectifs par *intégration elliptique* et *intégration hyperbolique*. C'est l'intégration hyperbolique qui forme la généralisation la plus directe de l'intégrale de R.-L. qu'on vient d'étudier. C'est encore uniquement l'intégration hyperbolique qui intervient dans l'application à l'équation des ondes qui est le but principal de ce travail. Cependant, ce procédé présentant sous certains rapports des difficultés considérables qui n'interviennent pas dans le procédé elliptique, il nous paraît utile de donner ici un aperçu sur le dernier procédé qui, tout en facilitant la lecture des chapitres suivants, n'y est nullement indispensable. Nous nous dispensons pourtant de faire précéder l'exposé de l'intégration dans plusieurs dimensions par un exposé concernant le cas linéaire ($m = 1$)[2], les difficultés dépendant très peu du nombre de dimensions, dans le cas actuel.

Dans ce chapitre on désigne la distance *euclidienne* de deux points P et Q par $r_{PQ} = r_{QP} = r$. Introduisons de la manière suivante un *potentiel généralisé* d'ordre α. Nous posons, pour α positif et pour des fonctions $f(P)$ dont on va préciser la nature plus loin,

$$I^\alpha f(P) = \frac{1}{H_m(\alpha)} \int_\Omega f(Q)\, r_{PQ}^{\alpha-m}\, dQ, \tag{1}$$

$$I^\alpha(I^\beta f(P)) = I^{\alpha+\beta} f(P) \tag{2}$$

et

$$\Delta I^{\alpha+2} f(P) = -I^\alpha f(P) \tag{3}$$

soient satisfaites, Δ désignant l'opérateur de Laplace

$$\Delta = \sum_{k=1}^{m} \frac{\partial^2}{\partial x_k^2},$$

où $x_1, x_2, \ldots, x_m$ sont les coordonnées cartésiennes du point P.[1]

Fig. 3.2: Pieces of paper by Riesz (1949): Chap. II a)

10. Formule de Green pour l'espace entier. Prolongement analytique. — Les considérations qui précèdent s'étendent immédiatement au cas où l'on remplace les couches définies moyennant des fonctions de point représentant la densité spatiale par des couches définies moyennant des fonctions additives d'ensemble, généralisation embrassant entre autres choses les simples couches. (Avec quelques précautions on peut même étendre les résultats à des intégrales définies par des doubles couches.) Ici nous abordons un autre problème, celui des indices zéro et négatifs, où, au lieu de généraliser les couches en question, il faut les assujettir à de nouvelles restrictions.

En effet, il est clair que tout comme dans l'espace de dimension 1, on pourra dans des conditions supplémentaires étendre le procédé à ces derniers indices. L'extension se fera encore par prolongement analytique; pour l'indice zéro on peut y arriver par un passage à la limite. En supposant que $f(Q)$ soit continu au point P, on trouve en copiant les raisonnements qui s'attachaient à la formule (7) du n° **3**

$$I^0 f(P) = \lim_{\alpha \to +0} I^\alpha f(P) = f(P), \tag{16}$$

formule qui, rapprochée de la formule $\Delta I^{\alpha+2} = - I^\alpha$, exprime d'une façon un peu rude la formule de Poisson ($\lim_{\varepsilon \to +0} \Delta I^{2+\varepsilon} f(P) = - \lim_{\varepsilon \to +0} I^\varepsilon f(P) = - f(P)$).

Il reste les indices négatifs. Supposons d'abord que la fonction $f(P)$ admette des dérivées continues de tout ordre $\leqq 2p$ et de plus que $f(P)$ et ses dérivées se comportent à l'infini de façon que les intégrales qui interviendront dans la suite soient absolument convergentes et que les intégrations par parties qu'on exécutera soient légitimes. En appliquant p fois la formule de Green

$$\int_\Omega U(Q) \, \Delta \, V(Q) \, dQ = \int_\Omega V(Q) \, \Delta \, U(Q) \, dQ,$$

on trouve moyennant les formules (1) et (12), d'abord pour α positif,

$$I^\alpha f(P) = (-1)^p I^{\alpha+2p} \Delta^p f(P). \tag{17}$$

Par cette formule on a le prolongement analytique de $I^\alpha f(P)$ pour toute valeur de $\alpha > -2p$.

Si l'on admet en particulier que $\Delta^p f(Q)$ soit continu pour $Q = P$ (hypothèse qui n'est pas nécessaire pour la validité de la dernière formule) on trouve comme plus haut en se servant du prolongement par passage à la limite

Fig. 3.3: Pieces of paper by Riesz (1949): Chap. II b)

3.4. Definitions of the fractional Laplacian for anisotropic media

It is well known that the models involving the classical Laplacian are suitable only for isotropic media, also the models that involve the fractional Laplacian defined in Subsection 2.2 are suitable for isotropic media. However, they can be used on anisotropic media using same technique that in the classical case, that is through a tensor that characterize such kind of media.

Although this is not the main objective of this note we must remark that the classical Laplacian could be defined as the inner product of two gradient operators on a suitable space of functions. Following such guide, many authors have given the following definition based in a natural definition of fractional gradient, which is directly suitable to be used on anisotropic media:

$$(\Delta)^{\overline{\alpha}}U(x,y,z) := (\nabla^{\overline{\alpha}}.\nabla\overline{\alpha}U)(x,y,z), \tag{3.17}$$

where $(\nabla^{\overline{\alpha}}U)(x,y,z) = (D_x^{\alpha}U, D_y^{\beta}U, D_z^{\gamma}U)$, $\overline{\alpha} = (\alpha,\beta,\gamma)$, $0 < \alpha,\beta,\gamma \leqq 1$, and the components of the fractional gradient $\nabla^{\overline{\alpha}}$ are any suitable one-dimensional spatial fractional derivative of order between 0 and 1, for example, they could be the Caputo or Riemann–Liouville fractional derivative, etc.

On the other hand, other possible definition could be the following one:

$$(\Delta^{\overline{\alpha}}U)(x,y,z) = (D_x^{\alpha}U + D_y^{\beta}U + D_z^{\gamma}U), \quad (1 < \alpha,\beta,\gamma \leqq 2). \tag{3.18}$$

4. Riesz hyperbolic potentials and fractional m-dimensional coupled hyperbolic differential operator

Here we are interesting just in the Riesz potential operator corresponding to the hyperbolic case introduced in Chapter III of the mentioned paper by Riesz [18].

4.1. Hyperbolic m-dimensional Riesz potential

The Riesz's potential operator corresponding to the hyperbolic case, is given by:

$${}^{R}\mathcal{I}^{\alpha}f(x) := \frac{1}{H_m(\alpha)}\int_{D_s^x}\frac{f(y)}{|x-y|^{m-\alpha}}dy, \tag{4.1}$$

where $\alpha > m-2$, $m > 1$, D_s^x is given by certain cone (see bellow Fig. 4.1), f is a suitable function, and with the normalized constant

$$H_m(\alpha) = \pi^{\frac{m-2}{2}}2^{\alpha-1}\Gamma(\frac{\alpha}{2})\Gamma(\frac{\alpha+2-m}{2}), \tag{4.2}$$

in order to verify the identity ${}^{R}II^{\alpha}(e^{x_1}) = e^{x_1}$ and $x = (x_1, x_2, ..., x_m)$. The operator (4.1) verifies the following three properties, which are similar to such mentioned above for the Riemann–Liouville fractional operator:

1. $${}^{R}\mathcal{I}^{0} f(x) = \lim_{\alpha \to 0} {}^{R}\mathcal{I}^{\alpha} f(x) = f(x), \quad (\alpha > 0), \tag{4.3}$$

2. $${}^{R}\mathcal{I}^{\alpha R}\mathcal{I}^{\beta} f(x) =^{R} \mathcal{I}_{a+}^{\alpha+\beta} f(x), \quad (\alpha, \beta > m-2), \tag{4.4}$$

3. $$\Box\ {}^{R}\mathcal{I}^{\alpha+2} f(x) =^{R} \mathcal{I}^{\alpha} f(x), \quad (\alpha > m-2), \tag{4.5}$$

where

$$\Box = \left(\frac{\delta}{\delta x_1} - \frac{\delta}{\delta x_2} - \frac{\delta}{\delta x_3} - ...\frac{\delta}{\delta x_m} \right). \tag{4.6}$$

4.2. New definitions of the fractional hyperbolic operator or Lorentzian Laplacian

Following the above three properties proved by Riesz, the corresponding analytic prolongation of the Riesz hyperbolic integral (see [18]) and the technique used in the elliptic case, we can easily introduce an explicit definition of the fractional Riesz hyperbolic differential operator or Lorentzian Laplacian as the corresponding left-side inverse to the mentioned hyperbolic Riesz Integral (4.1), for suitable functions $f(x)$, ($x \in \mathbb{R}^m$, $\alpha > 0$, $m > 1$, and $n, l \in \mathbb{N}$ such that $n = -[-\alpha]$; $l = m + n - 2$, are given by:

1. $$\left(\Box^{\alpha} f\right)(x) := \Box^{l} ({}^{R}\mathcal{I}^{l-\alpha} f)(x), \tag{4.7}$$

2. $$\left({}^{c}\Box^{\alpha} f\right)(x) := \left({}^{R}\mathcal{I}^{l-\alpha} \Box^{l} f\right)(x). \tag{4.8}$$

4.3. Some pieces of Chapter III of the paper by Riesz

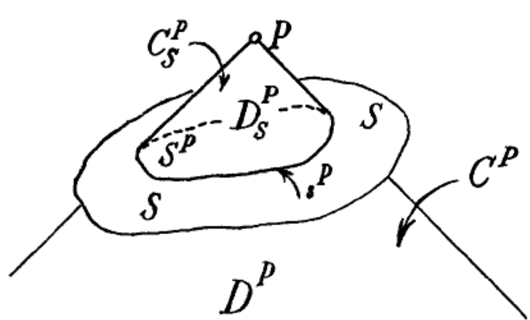

Fig. 4.1: Pieces of paper by Riesz (1949): Cone used in hyperbolic case

CHAPITRE III.

Fig. 4.2: Pieces of paper by Riesz (1949): Chap. III a)

CHAPITRE III.

Intégrale de Riemann-Liouville dans l'espace lorentzien.

I. Définition de l'intégrale.

14. La forme métrique lorentzienne et l'opérateur des ondes. Remarque. — J'ai insisté sur ces questions élémentaires bien familières à vous tous, parce qu'elles nous permettront de bien saisir la différence qu'on rencontre dans la solution du problème de Cauchy de l'équation des ondes, posé pour un nombre pair ou pour un nombre impair de variables.

Considérons dans l'espace à m dimensions les points P et Q aux coordonnées respectives $x_1, x_2, \ldots, x_m$ et $\xi_1, \xi_2, \ldots, \xi_m$. Nous introduisons la distance lorentzienne de ces points

$$r_{PQ} = \sqrt{(x_1 - \xi_1)^2 - (x_2 - \xi_2)^2 - \cdots - (x_m - \xi_m)^2},$$

en supposant que l'expression qui figure sous le signe racine carrée soit $\geqq 0$. En considérant le point P comme fixe et le point Q comme variable, $r^2_{PQ} = 0$ définit la nappe du cône de lumière au sommet P, $r^2_{PQ} > 0$ définit son intérieur, $x_1 - \xi_1 > 0$ le cône rétrograde et $x_1 - \xi_1 < 0$ le cône direct. C'est le cône rétrograde que nous allons considérer en général, en le désignant par D^P. Soit encore S une surface (c'est-à-dire une variété à $m-1$ dimensions) qui, suivant la terminologie de M. Hadamard, ait une *orientation d'espace.* Cela veut dire qu'on a pour tout déplacement infinitésimal sur la surface $dx_1^2 - dx_2^2 - \cdots - dx_m^2 < 0$. On admet aussi que la surface soit assez régulière pour que les dérivations qui interviendront plus tard puissent être effectuées. Le domaine limité par la nappe rétrograde et la surface S sera désigné par D_S^P.

Cela étant, nous posons

$$I^\alpha f(P) = \frac{1}{H_m(\alpha)} \int_{D_S^P} f(Q)\, r_{PQ}^{\alpha - m}\, dQ, \tag{2}$$

avec

$$H_m(\alpha) = \pi^{\frac{m-2}{2}}\, 2^{\alpha-1}\, \Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{\alpha + 2 - m}{2}\right). \tag{3}$$

Cette intégrale converge pour $\alpha > m-2$ et l'on vérifie facilement qu'elle satisfait aux relations fondamentales

$$I^\alpha I^\beta = I^{\alpha+\beta}, \qquad \Delta I^{\alpha+2} = I^\alpha, \tag{4}$$

où l'on a désigné par Δ l'opérateur des ondes

$$\Delta = \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \cdots - \frac{\partial^2}{\partial x_m^2}.$$

Fig. 4.3: Pieces of paper by Riesz (1949): Chap. III b)

Pour les indices $\alpha \leqq m - 2$ l'intégrale (2) se définit au moyen de prolongement analytique par rapport à α, bien entendu dans des conditions de régularité convenables portant sur la fonction f et sur la surface S. *On trouve en particulier* $I^0 f(P) = f(P)$. Remarquons que ce fait, qui est d'une importance capitale pour la suite, n'est nullement évident. En effet, I^0 est défini ici comme prolongement analytique de I^α qui ne converge que pour $\alpha > m - 2$, et le prolongement en question n'existe que dans certaines conditions de dérivabilité[1]. Des remarques analogues s'appliquent aux relations $I^{-2k} = \Delta^k$.

Fig. 4.4: Pieces of paper by Riesz (1949): Chap. III c)

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